

Advanced Sampled-Data Control (038798)

lecture no. 12

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Content

Networked control

A flavor of robustification

Lebesgue sampling

Another flavor of event-triggered sampling

Intermittent redesign

Content

Networked control

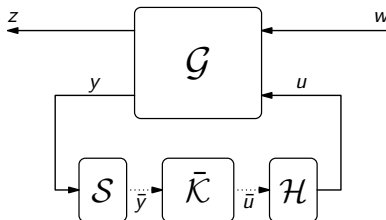
A flavor of robustification

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Classical setup



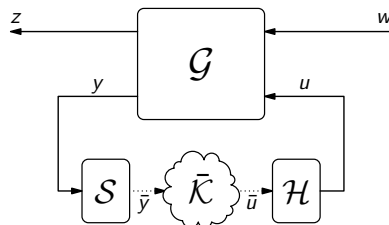
- regular sampling
- problem-independent $\mathcal{S}$
- problem-independent $\mathcal{H}$

$$t_i = ih$$

$$\bar{y}[i] = y(t_i)$$

$$u(t) = \bar{u}[i], \forall t \in [t_i, t_{i+1})$$

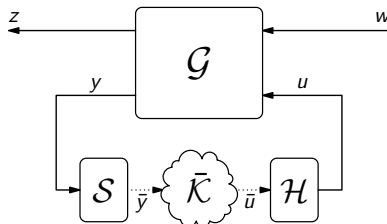
Second digital revolution



Trend to

- distribute information processing and
- use communication **networks** to exchange information

Leitmotif



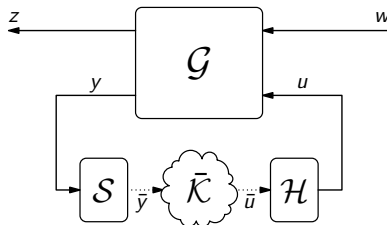
Computations

- powerful, reliable, cheap, ...

Information exchange

- precious (e.g. shared resources, power saving)
- unreliable / irregular (e.g. packet dropouts)

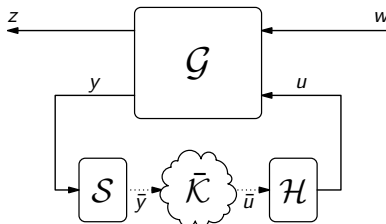
Emphases



Important to understand how to design controllers that

- maintain stability / performance under **intermittent** sampling
- insensitive to information losses and irregular delays
- not greedy for communication resources (e.g. rates, bandwidth)
- communication protocol friendly

Approaches



In line with mainstream control:

- robust embedding (by far the most prolific, quantity-wise)
- event-based sampling mechanisms (promising, but hard)
- sparsity constraints (natural, but normally also hard)

More exotic

- information-theoretic methods
(channel capacity, quantization mechanisms, encoding / decoding, et cetera)

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A flavor of robustification

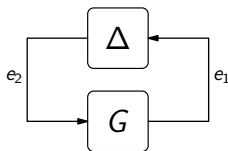
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Robust stability

One of key steps in robustness analysis is to separate uncertainty from the rest of system dynamics (plant, weights, et cetera) as



and then apply various robust stability conditions (the Small Gain Theorem, passivity, μ , IQC, etc).

For example, the Small Gain Theorem (SGT) states that

- if G and Δ are stable, the closed-loop system is stable if $\|G\|\|\Delta\| < 1$ (for any induced norm). Moreover, the
- closed-loop system is stable *for all* Δ such that $\|\Delta\| \leq \alpha$ *iff* $\|G\| < \frac{1}{\alpha}$.

A toy problem

Let $\dot{x}(t) = Ax(t) + Bu(t)$ and assume that the feedback law $u(t) = Fx(t)$ is satisfactory (and $A_F := A + BF$ is Hurwitz). Consider the sampled-data control law

$$u(t) = Fx(t_i), \quad \forall t \in [t_i, t_{i+1})$$

(i.e. with $\mathcal{S}_{\text{idl}}$ and $\mathcal{H}_{\text{ZOH}}$). Our goal is to

- find under what conditions on $\{t_i\}$ the closed-loop system is stable.

The closed-loop system in this case is

$$\dot{x}(t) = Ax(t) + BFx(t_i) = A_F x(t) + BF(x(t_i) - x(t)).$$

Rewriting closed-loop dynamics

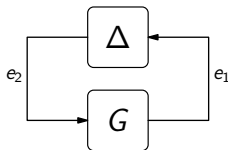
The trick is to rewrite it as

$$\dot{x}(t) = A_F x(t) - B \int_{t_i}^t F \dot{x}(s) ds.$$

Introduce two signals

$$e_1(t) := -F \dot{x}(t) \quad \text{and} \quad e_2(t) := - \int_{t_i}^t F \dot{x}(s) ds = \int_{t_i}^t e_1(s) ds$$

Then the closed-loop system can be cast as

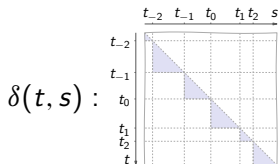


where $G(s) = -sF(sI - A_F)^{-1}B \in H_\infty$ and

$$\Delta : \dot{e}_2(t) = e_1(t), \quad e_2(t_i) = 0$$

$L_2(\mathbb{R}^+)$ -induced norm of Δ

The lifting of the operator Δ is the feedthrough part of the lifted integrator (but the lifting is over the nonuniform partition):

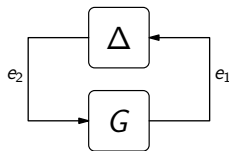


Lemma 1 from Lecture 11 then applies and the conclusion is that

— $\|\Delta\|_{L_2(\mathbb{R}^+) \rightarrow L_2(\mathbb{R}^+)}$ equals the $L_2[0, \bar{h}]$ -induced norm of an integrator, where $\bar{h} := \sup_i (t_{i+1} - t_i)$. Then, we know from Slide 41, Lecture 11, that

$$\|\Delta\|_{L_2(\mathbb{R}^+) \rightarrow L_2(\mathbb{R}^+)} = \frac{2}{\pi} \bar{h}.$$

Stability via the SGT



Thus, the closed-loop system is stable if

$$\|G\|_{\infty} < \frac{\pi}{2 \sup_i (t_{i+1} - t_i)},$$

where $G(s) = -sF(sI - A_F)^{-1}B$.

Example

Consider the first-order plant $\dot{x}(t) = x(t) + u(t)$ and the analog control law $u(t) = -fx(t)$ with $f > 1$. The sampled-data controller

$$u(t) = -fx(t_i), \quad \forall t \in [t_i, t_{i+1})$$

then stabilizes the closed-loop system if

$$\left\| \frac{sf}{s + (f - 1)} \right\|_{\infty} = f < \frac{\pi}{2 \sup_i (t_{i+1} - t_i)},$$

so we can guarantee the closed-loop stability for any sampling pattern such that

$$\sup_i (t_{i+1} - t_i) < \frac{\pi}{2f} < \frac{\pi}{2}.$$

General observations

Robustification methods also include

- Lyapunov analyses
- robustness analysis of discretized plant
- ...

Advantage:

- 😊 simplicity

Disadvantages:

- ☹ conservatism or non-practical complexity
- ☹ not design friendly (mainly an analysis tool)

Content

Networked control

A flavor of robustification

Lebesgue sampling

Another flavor of event-triggered sampling

Intermittent redesign

(Sloppy) setup

Consider the plant

$$\dot{x}(t) = u(t) + w(t)$$

where $w(t)$ is a unit intensity white noise process. The goal is to

- design control law for u that minimizes $\text{var}(x)$.

Continuous-time version of the problem is trivial: with

$$u(t) = -kx(t)$$

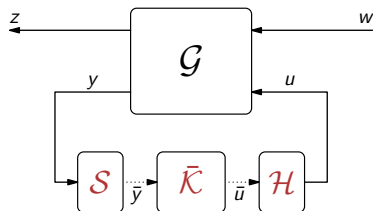
the closed-loop system is $\dot{x}(t) = -kx(t) + w(t)$, so that the variance of its output x , which equals the squared H_2 norm of the system $\frac{1}{s+k} : w \mapsto x$, is

$$\text{var}(x) = \frac{1}{2k} \xrightarrow{k \rightarrow \infty} 0.$$

But what about sampled-data control?

Standard problem

Can be reformulated as the Type IV problem



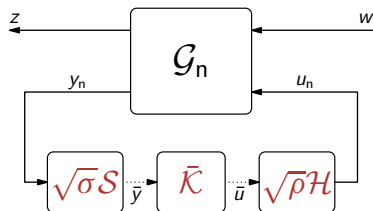
where

$$G(s) = \left[\begin{array}{c|ccc} 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\rho} \\ \hline 1 & 0 & \sqrt{\sigma} & 0 \end{array} \right]$$

and $\rho > 0$ and $\sigma > 0$ are very small (to render the problem nonsingular).

Normalization

To fit it to the setup studied in Lecture 9, D_{zu} and D_{yw} must be normalized:



where

$$G_n(s) = \left[\begin{array}{ccc|ccc} 0 & & & 1 & 0 & 1/\sqrt{\rho} \\ \hline 1 & & & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 1 \\ \hline 1/\sqrt{\sigma} & & & 0 & 1 & 0 \end{array} \right] .$$

CAREs

We need the stabilizing solutions of

$$\begin{aligned}A'X + XA + C_z' C_z - (XB_u + C_z' D_{zu})(B_u' X + D_{zu}' C_z) &= 0, \\ AY + YA' + B_w B_w' - (YC_y' + B_w D_{yw}') (C_y Y + D_{yw} B_w') &= 0\end{aligned}$$

In our case, the control Riccati is

$$0X + X0 + 1 - \frac{1}{\rho} X^2 = 0 \iff X = \sqrt{\rho}$$

and the filtering Riccati is

$$0Y + Y0 + 1 - \frac{1}{\sigma} Y^2 = 0 \iff Y = \sqrt{\sigma}$$

with the corresponding gains

$$F_u = -B_u' X - D_{zu}' C_z = -1 \quad \text{and} \quad L_y = -YC_y' - B_w D_{yw}' = -1.$$

Optimal performance

We know (Lecture 10, Slide 47, and Lecture 9, Slide 10) that

$$\begin{aligned}\|T_{zw}\|_2^2 &= \frac{1}{h} \int_0^h \int_0^t \|F_u e^{A(t-s)} L_y\|_F^2 ds dt + \text{tr}(C_z Y C_z') + \text{tr}(L_y' X L_y) \\ &= \frac{1}{h} \int_0^h \int_0^t 1 ds dt + \sqrt{\sigma} + \sqrt{\rho} = \frac{h}{2} + \sqrt{\rho} + \sqrt{\sigma}\end{aligned}$$

and it approaches

$$\|T_{zw}\|_2^2 = \frac{h}{2}$$

as $\rho \rightarrow 0$ and $\sigma \rightarrow 0$.

Optimal controller

We also know (Lecture 10, Slide 34) that the optimal controller comprises a static sampler with (mind scaling back by $1/\sqrt{\rho}$)

$$\psi_y(t) = -\frac{1}{\sqrt{\rho}} e^{(A+L_y C_y)t} L_y \mathbb{1}_{[0,h]}(t) = \frac{1}{\sqrt{\rho}} e^{-t/\sqrt{\rho}} \mathbb{1}_{[0,h]}(t) \xrightarrow{\rho \rightarrow 0} \delta(t),$$

a discrete system with

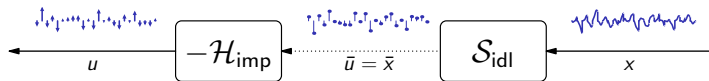
$$\begin{aligned} \bar{K}(z) &= z \left(zI - \begin{bmatrix} I & 0 \end{bmatrix} \exp \left(\begin{bmatrix} A + L_y C_y & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} h \right) \begin{bmatrix} I \\ I \end{bmatrix} \right)^{-1} \\ &= \frac{(\sqrt{\rho} - \sqrt{\sigma})z}{(\sqrt{\rho} - \sqrt{\sigma})z + \sqrt{\rho} e^{-h/\sqrt{\sigma}} - \sqrt{\sigma} e^{-h/\sqrt{\rho}}} \xrightarrow{\rho \rightarrow 0, \sigma \rightarrow 0} 1, \end{aligned}$$

and a static hold with (mind scaling back by $1/\sqrt{\sigma}$)

$$\phi_u(t) = \frac{1}{\sqrt{\sigma}} F_u e^{(A+B_u F_u)t} \mathbb{1}_{[0,h]}(t) = -\frac{1}{\sqrt{\sigma}} e^{-t/\sqrt{\sigma}} \mathbb{1}_{[0,h]}(t) \xrightarrow{\sigma \rightarrow 0} -\delta(t).$$

Optimal behavior with $\rho \rightarrow 0$ and $\sigma \rightarrow 0$

Thus, if control effort is unlimited and measurement noise is negligible, the optimal controller is



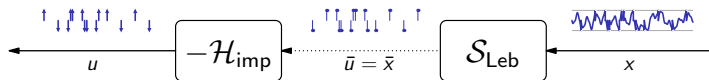
The closed-loop system, with $u(ih + \sigma) = -\delta(\sigma)x(ih)$, behaves as

$$\begin{aligned} x(ih + \tau) &= x(ih) + \int_0^\tau (u(ih + \sigma) + w(ih + \sigma)) d\sigma \\ &= x(ih) - x(ih) + \int_0^\tau w(ih + \sigma) d\sigma = \int_0^\tau w(ih + \sigma) d\sigma \end{aligned}$$

Thus, the system is

- reset to zero at every sampling instance ih .

Lebesgue sampling



Fix some level $\delta_x > 0$ and then

- sample whenever $|x(t)| = \delta_x$ and reset $x(t)$ to zero after every sample.

Bernhardsson & Åström (1999) proved¹ that the mean sampling time in this case is

$$h_{av} = \delta_x^2$$

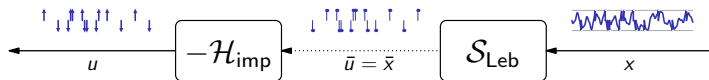
and that the variance

$$\text{var}(x) = \frac{\delta_x^2}{6} = \frac{h_{av}}{6},$$

which outperforms the conventional (Riemann) sampling by a **factor of 3**!

¹Using the theory of Markovian diffusion processes...

Lebesgue sampling: open questions



Performance gain is impressive, but

- can we do better by other strategies?
- can this be generalized to more general processes?
(it can for general first-order systems, for example)
- can this be generalized to better posed setups?
(like in the presence of measurement noise, limited control effort, et cetera)
- can this be generalized to different control goals?

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A (very brief) introduction to Lyapunov functions

Consider the system

$$\dot{x}(t) = f(x(t)) \quad \text{with } f(0) = 0.$$

If we can find a function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $V(0) = 0$ and $V(x) > 0$ whenever $x \neq 0$, $\lim_{\|x\| \rightarrow \infty} V(x) = \infty$, and

$$\dot{V}(x) := \frac{dV(x(t))}{dt} = \frac{\partial V(x)}{\partial x} \frac{dx}{dt} < 0, \quad \forall x \neq 0$$

then the equilibrium $x \equiv 0$ is globally asymptotically stable. If the system is linear, $f(x) = Ax$, a common choice is $V(x) = x'Px$ for $P = P' > 0$. Then

$$\dot{V}(x) = 2x'PAx = x'(PA + A'P)x = -x'Qx,$$

where Q verifies the Lyapunov equation $PA + A'P + Q = 0$. It is a known fact that if $Q > 0$, then $P > 0$ iff A is Hurwitz. This $V(x) = x'Px$ may be thought of as an energy function.

A (very brief) introduction to Lyapunov functions

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Sampling generation for state feedback

Let $\dot{x}(t) = Ax(t) + Bu(t)$ and assume that the feedback law $u(t) = Fx(t)$ is satisfactory (and $A_F := A + BF$ is Hurwitz). Consider the sampled-data control law

$$u(t) = Fx(t_i), \quad \forall t \in [t_i, t_{i+1})$$

(i.e. with $\mathcal{S}_{\text{idl}}$ and $\mathcal{H}_{\text{ZOH}}$). Our goal is to

- choose the sampling instances $\{t_i\}$ guaranteeing stability.

The closed-loop system in this case is

$$\dot{x}(t) = Ax(t) + BFx(t_i) = A_F x(t) + BFe(t),$$

where $e(t) := x(t_i) - x(t)$ for every $t \in [t_i, t_{i+1})$ (note that $e(t_i) = 0$, $\forall i$). Pick the Lyapunov function candidate $V = x'Px$, where $PA_F + A_F'P = -Q$ for some $Q > 0$ ($P > 0$ exists for every $Q > 0$). We have:

$$\dot{V} = 2x'P(A_F x + BFe) = \begin{bmatrix} x' & e' \end{bmatrix} \begin{bmatrix} -Q & PBF \\ F'B'P & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}.$$

Sampling generation for state feedback (contd)

The matrix

$$\begin{bmatrix} -Q & PBF \\ F'B'P & 0 \end{bmatrix} \not\leq 0$$

(why?), hence we cannot guarantee that $\dot{V} < 0$ for all t . But

$$\dot{V}|_{t=t_i} = -x'(t_i)Qx(t_i) < 0,$$

so we may consider generating $\{t_i\}$ to prevent $\dot{V}$ to become nonnegative.

A possible algorithm (Tabuada, 2007) would be to sample whenever

$$\begin{bmatrix} x'(t) & e'(t) \end{bmatrix} \begin{bmatrix} -Q & PBF \\ F'B'P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = -\alpha x'(t)Qx(t)$$

for some given $\alpha \in (0, 1)$.

Sampling generation for state feedback (contd)

Equivalently, the i th sampling instance is chosen to satisfy

$$\begin{aligned}
 0 &= \begin{bmatrix} x'(t_i) & x'(t_{i-1}) - x'(t_i) \end{bmatrix} \begin{bmatrix} (\alpha - 1)Q & PBF \\ F'B'P & 0 \end{bmatrix} \begin{bmatrix} x(t_i) \\ x(t_{i-1}) - x(t_i) \end{bmatrix} \\
 &= \begin{bmatrix} x'(t_i) & x'(t_{i-1}) \end{bmatrix} \begin{bmatrix} (\alpha - 1)Q - PBF - F'B'P & PBF \\ F'B'P & 0 \end{bmatrix} \begin{bmatrix} x(t_i) \\ x(t_{i-1}) \end{bmatrix} \\
 &= \begin{bmatrix} x'(t_i) & x'(t_{i-1}) \end{bmatrix} \begin{bmatrix} \alpha Q + PA + A'P & PBF \\ F'B'P & 0 \end{bmatrix} \begin{bmatrix} x(t_i) \\ x(t_{i-1}) \end{bmatrix}.
 \end{aligned}$$

This choice guarantees that

$$\dot{V} \leq -\alpha x' Q x < 0,$$

which, in turn, guarantees stability and certain rate of convergence. It can be shown that this choice also guarantees the existence of $\underline{h} > 0$ such that

$$t_{i+1} - t_i \geq \underline{h}, \quad \forall i \in \mathbb{Z}^+.$$

General observations

Lyapunov-based results are quite rich, they apply to

- 😊 nonlinear systems
- 😊 handle external disturbances
- 😊 systems with time-varying delays

At the same time,

- 😊 output-feedback extensions are nontrivial
- 😊 choices of Lyapunov functions are not intuitive
- 😊 performance guarantees are frequently plagued by conservatism
- 😊 not design friendly (mainly an analysis tool)

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Networked control

A flavor of robustification

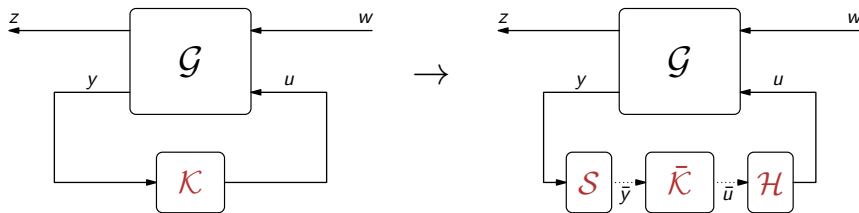
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Setup

Almost like that studied in Lecture 11:



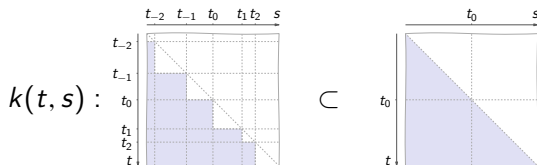
but now assume that the sampling is not periodic, but rather is done under an arbitrary, unknown a priori, sequence of sampling instances

$$\{t_i\}_{i \in \mathbb{Z}^+}, \quad \text{with } 0 = t_0 < t_1 < \dots \text{ and } \lim_{i \rightarrow \infty} t_i = \infty.$$

We assume also that $\exists \bar{h} < \infty$ such that $\sup_i h_i \leq \bar{h}$, where $h_i := t_{i+1} - t_i$ (this $\bar{h}$ need not be known).

The idea (not much different)

Because (Lecture 2, Slide 39)



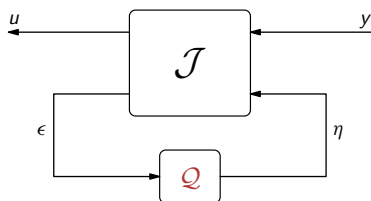
we may endeavor to

- extract the sampled-data controller of interest from a parametrization of admissible analog controllers.

In the lifted domain (with a non-uniform split) this still reads *strictly causal* and all reasonings used in Lectures 10 and 11 apply mutatis mutandis.

Observer-based Youla-Kučera parametrization

All internally stabilizing analog controllers can be parametrized as



where

$$J(s) = \left[\begin{array}{c|cc} A + B_u F_u + L_y C_y & -L_y & B_u \\ \hline F_u & 0 & I \\ -C_y & I & 0 \end{array} \right]$$

for any F_u and L_y such that $A + B_u F_u$ and $A + L_y C_y$ are Hurwitz, and Q is an arbitrary stable system.

All admissible $\mathcal{Q}$'s

Repeating the steps of Lecture 11,

- all admissible $\mathcal{Q}$'s that render $\mathcal{F}_l(\mathcal{J}, \mathcal{Q})$ a sampled-data controller are of the form $\mathcal{Q} = \mathcal{Q}_\infty + \mathcal{Q}_{sd}$, where

$$\mathcal{Q}_\infty : \begin{cases} \dot{x}_Q(t) = Ax_Q(t) + L_y \epsilon(t), & x_Q(t_i) = 0 \\ \eta_\infty(t) = F_u x_Q(t). \end{cases}$$

and $\mathcal{Q}_{sd}$ is an arbitrary stable *sampled-data* system for the same $\{t_i\}$.

- this $\mathcal{Q}_\infty$ is stable whenever $\sup_i h_i \leq \bar{h} < \infty$, which is assumed by Lemma 1 from Lecture 11, the $L_2(\mathbb{R}^+)$ -induced norm of $\mathcal{Q}_\infty$ is upperbounded by the $L_2[0, \bar{h}]$ -induced norm of the system with the transfer function $F_u(sI - A)^{-1}L_y$, which is bounded

All stabilizing sampled-data controllers

Then all sampled-data stabilizing controllers are

$$\begin{cases} \dot{\tilde{x}}(t) = \begin{bmatrix} A + L_y C_y & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} \tilde{x}(t) - \begin{bmatrix} L_y \\ 0 \end{bmatrix} y(t) + \begin{bmatrix} B_u \\ B_u \end{bmatrix} \eta_{sd}(t) \\ u(t) = \begin{bmatrix} 0 & F_u \end{bmatrix} \tilde{x}(t) + \eta_{sd}(t) \\ \epsilon(t) = \begin{bmatrix} -C_y & 0 \end{bmatrix} \tilde{x}(t) + y(t) \end{cases}$$

with aperiodic state resets of the form $\tilde{x}_2(\mathbf{t}_i) = \tilde{x}_1(\mathbf{t}_i)$ at every time instance t_i and $\eta_{sd} = Q_{sd}\epsilon$ for an arbitrary stable sampled-data Q_{sd} .

The central controller

Let

$$\begin{bmatrix} \Gamma_{11}(\tau) & \Gamma_{12}(\tau) \\ 0 & \Gamma_{22}(\tau) \end{bmatrix} := \exp \left(\begin{bmatrix} A + L_y C_y & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} \tau \right)$$

then

$$\begin{cases} \tilde{x}(t_i + \tau) = \begin{bmatrix} \Gamma_{11}(\tau) & \Gamma_{12}(\tau) \\ 0 & \Gamma_{22}(\tau) \end{bmatrix} \tilde{x}(t_i) - \int_0^\tau \begin{bmatrix} \Gamma_{11}(\tau - \sigma) L_y \\ 0 \end{bmatrix} y(t_i + \sigma) d\sigma \\ u(t_i + \tau) = F_u \Gamma_{22}(\tau) \tilde{x}_2(t_i) \end{cases}$$

Thus, $u(t)$ depends only on $\tilde{x}_2(t_i)$. If communication is reliable,

- the sensor and actuator sides can be separated

and only part passing through the communication channel is $\tilde{x}_2(t_i)$.

H_2 norm of time-varying systems

We know that if $\mathcal{G} \in H_2$ is LTI, its H_2 -norm is

$$\|\mathcal{G}\|_2^2 = \int_0^\infty \|g(t)\|_F^2 dt.$$

If $\mathcal{G}$ is h -periodic, we have to account for variations of the impulse response $g(t, s)$ for different $s \in [0, h]$. The norm is then

$$\|\mathcal{G}\|_2^2 = \frac{1}{h} \int_0^h \int_s^\infty \|g(t, s)\|_F^2 dt ds.$$

What if $g(t, s)$ is not periodic? It appears natural to generalize it as

$$\|\mathcal{G}\|_2^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_s^\infty \|g(t, s)\|_F^2 dt ds.$$

But this is only a

→ seminorm, i.e. it might happen that $\|\mathcal{G}\|_2 = 0$ for $\mathcal{G} \neq 0$.

H_2 norm of time-varying systems

We know that if $\mathcal{G} \in H_2$ is LTI, its H_2 -norm is

$$\|\mathcal{G}\|_2^2 = \int_0^\infty \|g(t)\|_F^2 dt.$$

If $\mathcal{G}$ is h -periodic, we have to account for variations of the impulse response $g(t, s)$ for different $s \in [0, h]$. The norm is then

$$\|\mathcal{G}\|_2^2 = \frac{1}{h} \int_0^h \int_s^\infty \|g(t, s)\|_F^2 dt ds.$$

What if $g(t, s)$ is not periodic? It appears natural to generalize it as

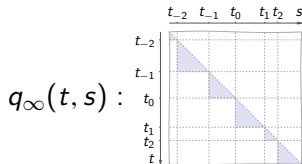
$$\|\mathcal{G}\|_2^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_s^\infty \|g(t, s)\|_F^2 dt ds.$$

But this is only a

- **seminorm**, i.e. it might happen that $\|\mathcal{G}\|_2 = 0$ for $\mathcal{G} \neq 0$.

H_2 norm of $\mathcal{Q}_\infty$

Consider $\mathcal{Q}_\infty$ from Slide 36, whose impulse response



In this case,

$$\begin{aligned}
 \|\mathcal{Q}_\infty\|_2^2 &= \lim_{i \rightarrow \infty} \frac{1}{t_i} \int_0^{t_i} \int_s^\infty \|q_\infty(t, s)\|_F^2 dt ds \\
 &= \lim_{i \rightarrow \infty} \frac{1}{t_i} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \int_s^\infty \|q_\infty(t, s)\|_F^2 dt ds \\
 &= \lim_{i \rightarrow \infty} \frac{1}{t_i} \sum_{j=0}^{i-1} \int_0^{h_j} \int_0^{h_j-s} \|F_u e^{At} L_y\|_F^2 dt ds.
 \end{aligned}$$