

Advanced Sampled-Data Control (038798)

lecture no. 11

Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT



Content

Digital redesign revisited

Stability-preserving redesign

H_∞ loop shaping design

H_∞ loop shaping **redesign**

Side remark: compact hSTPBC representation of transfer functions

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Digital redesign revisited

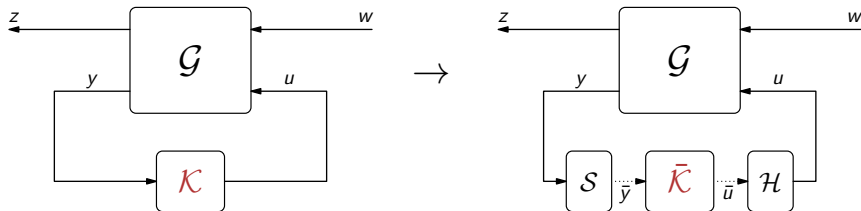
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Side remark: compact hSTPBC representation of transfer functions

Digital redesign: methodology (classical control)



1. design a “good” analog $\mathcal{K}$
2. find $\bar{\mathcal{K}}$ such that $\mathcal{K} \approx \mathcal{H}\bar{\mathcal{K}}\mathcal{S}$ for given $\mathcal{S}$ and $\mathcal{H}$

Bad news: there’s hardly any theory to support it in classical control, just plenty of rules of thumb...

Good news: they normally do the trick, if sampling rate is “fast enough”...

Modern approach: more rigor

Some possible directions:

1. optimal approximation of $\mathcal{K}$:

$$\text{minimize } \|\mathcal{K} - \mathcal{H}\bar{\mathcal{K}}\mathcal{S}\| \quad \text{or} \quad \text{minimize } \|(\mathcal{K} - \mathcal{H}\bar{\mathcal{K}}\mathcal{S})\mathcal{P}\|$$

(the latter may be viewed as approximating the loop) for some choice of norm. Both can be cast as standard Type I sampled-data problems with the generalized plants

$$\mathcal{G}_a = \begin{bmatrix} \mathcal{K} & -I \\ I & 0 \end{bmatrix} \quad \text{or} \quad \mathcal{G}_a = \begin{bmatrix} \mathcal{K}\mathcal{P} & -I \\ \mathcal{P} & 0 \end{bmatrix}.$$

If $\mathcal{P}$ and $\mathcal{K}$ are stable, the internal stability of this standard problem is equivalent to the stability of $\bar{\mathcal{K}}$. This approach does not guarantee the closed-loop stability of the original problem though...

Modern approach: more rigor (contd)

Some possible directions:

2. optimal approximation of the closed-loop behavior:

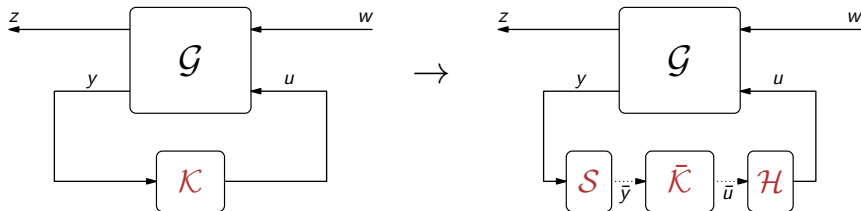
$$\text{minimize } \|\mathcal{F}_l(\mathcal{G}, \mathcal{K}) - \mathcal{F}_l(\mathcal{G}, \mathcal{H}\bar{\mathcal{K}}S)\|$$

This problem can be cast as a standard Type I sampled-data problem too, now with the generalized plant

$$\mathcal{G}_a = \begin{bmatrix} \mathcal{F}_l(\mathcal{G}, \mathcal{K}) - \mathcal{G}_{zw} & -\mathcal{G}_{zu} \\ \mathcal{G}_{yw} & \mathcal{G}_{yu} \end{bmatrix}.$$

The internal stability of this standard problem requires the internal stability of $\mathcal{F}_l(\mathcal{G}, \mathcal{H}\bar{\mathcal{K}}S)$. Therefore, the approach does guarantee the closed-loop stability of the original problem . . . But the order of the redesigned controller might be much higher than that of $\mathcal{K}$.

Type IV: a new hope



1. design a “good” analog $\mathcal{K}$
2. find $\mathcal{S}$, $\bar{\mathcal{K}}$, and $\mathcal{H}$ such that $\mathcal{K} \approx \mathcal{H}\bar{\mathcal{K}}\mathcal{S}$ (in a broad sense)

Incorporating A/D and D/A converters into the redesign process may

- result in better performance (control-oriented $\mathcal{S}$ and $\mathcal{H}$)
- substantially simplify the procedure (shall see it)
- be feasible technologically (silicon revolution)

Type IV: approaches

We may still endeavor to achieve

1. optimal approximation of $\mathcal{K}$, like

$$\text{minimize } \|(\mathcal{K} - \mathcal{H}\bar{\mathcal{K}}S)\mathcal{P}\|$$

2. optimal approximation of the closed-loop behavior, like

$$\text{minimize } \|\mathcal{F}_l(\mathcal{G}, \mathcal{K}) - \mathcal{F}_l(\mathcal{G}, \mathcal{H}\bar{\mathcal{K}}S)\|$$

Both would be similar to Type I redesign procedures... But we saw in the last part of Lecture 10 that in Type IV setting we can

3. extract sampled-data controllers from a parametrization of admissible analog controllers

by imposing the strict causality condition on the lifted controller. This idea fits perfectly into the redesign philosophy and worked well in the H_2 design. So let us try it on other redesign problems.

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Both would be similar to Type I redesign procedures... But we saw in the last part of Lecture 10 that in Type IV setting we can

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Stability-preserving redesign

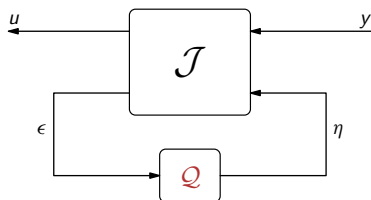
H_∞ loop shaping design

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Side remark: compact hSTPBC representation of transfer functions

Observer-based Youla-Kučera parametrization

All internally stabilizing analog controllers can be parametrized as



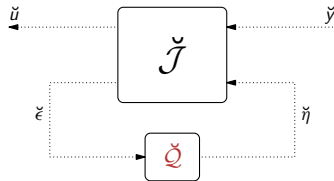
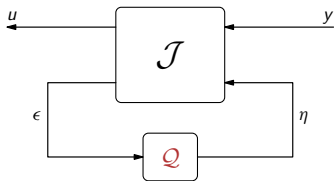
where

$$J(s) = \left[\begin{array}{c|cc} A + B_u F_u + L_y C_y & -L_y & B_u \\ \hline F_u & 0 & I \\ -C_y & I & 0 \end{array} \right]$$

for any F_u and L_y such that $A + B_u F_u$ and $A + L_y C_y$ are Hurwitz, and $\mathcal{Q}$ is an arbitrary stable system. This form can be viewed as an

- augmentation of an observer-based stabilizing controller (with innovations $\epsilon = y - C_y x_J$ and control $u = F_u x_J + \eta$).

Observer-based Youla-Kučera redesign



The redesign steps follow those of the H_2 design:

1. lift the parametrization
2. find admissible $\check{\mathcal{Q}}$'s that render $\mathcal{F}_l(\check{\mathcal{J}}, \check{\mathcal{Q}})$ strictly causal

“Admissible” here means such that $\check{\mathcal{Q}}$ is stable.

Strict causality of $\mathcal{F}_1(\check{\mathcal{J}}, \check{\mathcal{Q}})$

We know (Lecture 10, Slide 46) that

- $\mathcal{F}_1(\check{\mathcal{J}}, \check{\mathcal{Q}})$ is strictly causal iff the feedthrough part of $\check{\mathcal{Q}}$ equals that of the $(2, 2)$ part of $\check{\mathcal{J}}^{-1} = \mathcal{L}_h \mathcal{J}^{-1} \mathcal{L}_h^{-1}$.

Because

$$J^{-1} = \left[\begin{array}{c|cc} A & -B_u & L_y \\ \hline -C_y & 0 & I \\ F_u & I & 0 \end{array} \right],$$

the feedthrough part of $\check{\mathcal{Q}}$ must be the lifting of

$$\mathcal{Q}_\infty : \begin{cases} \dot{x}_Q(t) = Ax_Q(t) + L_y \epsilon(t), & x_Q(ih) = 0 \\ \eta_\infty(t) = F_u x_Q(t). \end{cases}$$

Moreover, since $\check{\mathcal{Q}}_\infty$ is static and bounded, it's always stable and then

- $\check{\mathcal{Q}}$ is stable iff (strictly causal) $\check{\mathcal{Q}}_{sc} := \check{\mathcal{Q}} - \check{\mathcal{Q}}_\infty$ is stable.

All admissible $\check{Q}$'s

Thus,

- all admissible $\check{Q}$'s that render $\mathcal{F}_l(\check{\mathcal{J}}, \check{Q})$ strictly causal are of the form $\check{Q} = \check{Q}_\infty + \check{Q}_{sc}$, where $\check{Q}_{sc}$ is an arbitrary stable *strictly causal* system.

In other words, assuming that we consider only controllers with finite state dimension,

- all admissible Q 's that render $\mathcal{F}_l(\mathcal{J}, Q)$ a sampled-data controller are of the form $Q = Q_\infty + Q_{sd}$, where

$$Q_\infty : \begin{cases} \dot{x}_Q(t) = Ax_Q(t) + L_y \epsilon(t), & x_Q(ih) = 0 \\ \eta_\infty(t) = F_u x_Q(t). \end{cases}$$

and Q_{sd} is an arbitrary stable *sampled-data* system.

All stabilizing sampled-data controllers

Plugging admissible Q 's into the Youla parametrization, with

$$x := \begin{bmatrix} x_J \\ x_Q \end{bmatrix}, \quad \eta = \eta_\infty + \eta_{sd}, \quad \eta_\infty = F_u x_Q, \quad \eta_{sd} = Q_{sd} \epsilon,$$

we end up with

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} A + B_u F_u + L_y C_y & B_u F_u \\ -L_y C_y & A \end{bmatrix} x(t) + \begin{bmatrix} -L_y \\ L_y \end{bmatrix} y(t) + \begin{bmatrix} B_u \\ 0 \end{bmatrix} \eta_{sd}(t) \\ u(t) = \begin{bmatrix} F_u & F_u \end{bmatrix} x(t) + \eta_{sd}(t) \\ \epsilon(t) = \begin{bmatrix} -C_y & 0 \end{bmatrix} x(t) + y(t) \end{cases}$$

with periodic state resets $\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} x(ih) = 0$. Similarly to what has been done in Lecture 10, Slide 48, we then introduce the transformation

$$\tilde{x}(t) := \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} x(t) = \begin{bmatrix} x_J(t) \\ x_J(t) + x_Q(t) \end{bmatrix}.$$

All stabilizing sampled-data controllers (contd)

Thus, all sampled-data stabilizing controllers are

$$\begin{cases} \dot{\tilde{x}}(t) = \begin{bmatrix} A + L_y C_y & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} \tilde{x}(t) - \begin{bmatrix} L_y \\ 0 \end{bmatrix} y(t) + \begin{bmatrix} B_u \\ B_u \end{bmatrix} \eta_{sd}(t) \\ u(t) = \begin{bmatrix} 0 & F_u \end{bmatrix} \tilde{x}(t) + \eta_{sd}(t) \\ \epsilon(t) = \begin{bmatrix} -C_y & 0 \end{bmatrix} \tilde{x}(t) + y(t) \end{cases}$$

with periodic state resets of the form $\tilde{x}_2(ih) = \tilde{x}_1(ih)$ and $\eta_{sd} = Q_{sd}\epsilon$ for an arbitrary stable sampled-data Q_{sd} .

Remark: The controller with $Q_{sd} = 0$ is dubbed the **central controller**.

Remark: The central controller can be cast as a sampling-and-hold circuit, much like the H_2 optimal controller (which is a particular case of this one with F_u and L_y chosen via solving CAREs).

More options

All stabilizing controllers can be parametrized around any given stabilizing controller $\mathcal{K}_0$. Let

$$P(s) = \left[\begin{array}{c|c} A & B_u \\ \hline C_y & 0 \end{array} \right] \quad \text{and} \quad K_0(s) = \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array} \right].$$

Then all stabilizing controllers are $\mathcal{F}_I(\mathcal{J}_0, \mathcal{Q})$, where

$$J_0(s) = \left[\begin{array}{ccc|cc} A_0 & 0 & 0 & B_0 & -L_0 \\ 0 & A_0 & B_0 C_y & 0 & -L_0 \\ 0 & B_u C_0 & A + B_u D_0 C_y & 0 & B_u \\ \hline C_0 & 0 & 0 & D_0 & I \\ -F_0 & F_0 & -C_y & I & 0 \end{array} \right]$$

for any F_0 and L_0 such that $A_0 + B_0 F_0$ and $A_0 + L_0 C_0$ are Hurwitz and $\mathcal{Q}$ is an arbitrary stable system. The extraction of sampled-data controllers from this parametrization is straightforward then...

Content

Digital redesign revisited

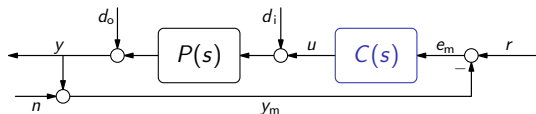
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Side remark: compact hSTPBC representation of transfer functions

SISO closed-loop transfer functions



$$e = r - y = \underbrace{\frac{S}{1+PC}}_1 r - \underbrace{\frac{T_d}{1+PC}}_P d_i - \underbrace{\frac{S}{1+PC}}_1 d_o + \underbrace{\frac{T=1-S}{1+PC}}_1 n$$

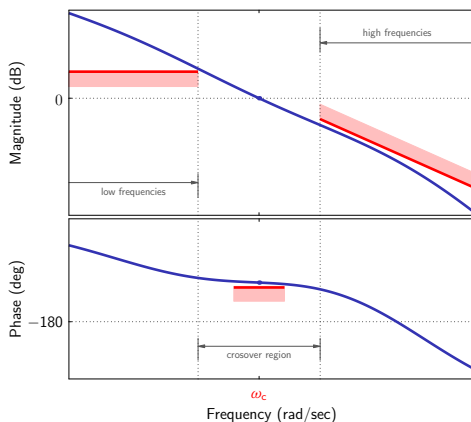
$$u = \underbrace{\frac{C}{1+PC}}_{T_c} r - \underbrace{\frac{PC}{1+PC}}_T d_i - \underbrace{\frac{C}{1+PC}}_{T_c} d_o - \underbrace{\frac{C}{1+PC}}_{T_c} n$$

We wanna have closed-loop stability and keep

- $|S(j\omega)| \ll 1$ within spectra of r , Pd_i , and d_o
- $|T(j\omega)| \ll 1$ within spectrum of n
- $|T_c(j\omega)| \not\gg 1 \iff |T(j\omega)| \not\gg |P(j\omega)|, \forall \omega$

SISO loop shaping

The loop transfer function is $L(s) := P(s)C(s)$. Then

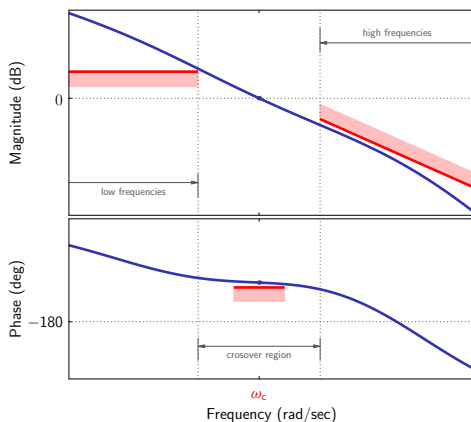


- $|S(j\omega)| \ll 1 \iff |L(j\omega)| \gg 1$
- $|T(j\omega)| \ll 1 \iff |L(j\omega)| \ll 1$
- ratio $|T(j\omega)|/|P(j\omega)|$ is related to the ratio of their bandwidths
- closed-loop system is stable iff $L(j\omega)$ verifies the Nyquist criterion
- closed-loop system is robust if $L(j\omega)$ is “far” from the critical point (measured by stability margins, like the phase margin)

Magnitude shaping is relatively easy, phase shaping is

SISO loop shaping

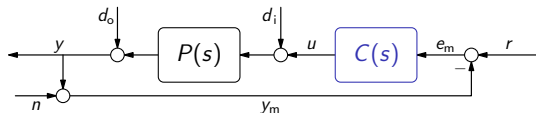
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Magnitude shaping is relatively easy, phase shaping is knotty...

MIMO closed-loop transfer functions



$$e = \underbrace{(I + PC)^{-1}}_{S_o} r - \underbrace{(I + PC)^{-1}P}_{T_d} d_i - \underbrace{(I + PC)^{-1}}_{S_o} d_o + \underbrace{(I + PC)^{-1}PC}_{T_o=1-S_o} n$$

$$u = \underbrace{C(I + PC)^{-1}}_{T_c} r - \underbrace{C(I + PC)^{-1}P}_{T_i} d_i - \underbrace{C(I + PC)^{-1}}_{T_c} (d_o + n)$$

We wanna have closed-loop stability and keep

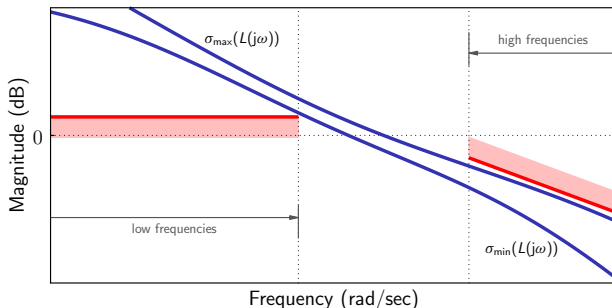
- $\sigma_{\max}(S_o(j\omega)) \ll 1$ within spectra of r , Pd_i , and d_o
- $\sigma_{\max}(T_o(j\omega)) \ll 1$ within spectrum of n
- $\sigma_{\max}(T_c(j\omega)) \not\gg 1 \iff \sigma_{\max}(T(j\omega)) \not\gg \sigma_{\min}(P(j\omega)), \forall \omega$

MIMO magnitude shaping

Let $L_o(s) := P(s)C(s)$, then

- $\sigma_{\max}(S_o(j\omega)) \ll 1 \iff \sigma_{\min}(L_o(j\omega)) \gg 1$
- $\sigma_{\max}(T_o(j\omega)) \ll 1 \iff \sigma_{\max}(L_o(j\omega)) \ll 1$

so we have:



MIMO phase shaping by McFarlane & Glover

The idea in essence:

- cast **phase shaping** as an H_∞ **robust stabilization** problem (as “far from the critical point” may be interpreted as a robustness requirement)

The latter

- can be solved analytically, via the H_∞ optimization
- applies to MIMO systems

Design steps

1. Shape loop magnitude via weights as $P_{\text{msh}}(s) := W_o(s)P(s)W_i(s)$
 - typically P, PI, or PID, may include LPF
 - can cancel (stable) poles or zeros of $P(s)$ if required
 - in MIMO case can decouple
2. Design $C_{\text{msh}}(s)$ for $P_{\text{msh}}(s)$ via robust stabilization
 - robustness level, say α_{max} , serves as a success indicator
3. If successful, pick $C(s) = W_i(s)C_{\text{msh}}(s)W_o(s)$, which yields

$$\begin{aligned}
 L_o(s) &= P(s)C(s) = W_o^{-1}(s)P_{\text{msh}}(s)C_{\text{msh}}(s)W_o(s) \\
 &= W_o^{-1}(s)L_{\text{msh},o}(s)W_o(s) \\
 L_i(s) &= C(s)P(s) = W_i(s)C_{\text{msh}}(s)P_{\text{msh}}(s)W_i^{-1}(s) \\
 &= W_i(s)L_{\text{msh},i}(s)W_i^{-1}(s)
 \end{aligned}$$

(in the SISO case, the actual loop is exactly the designed loop).

Properties of the resulted system

- closed-loop stability is guaranteed
- controller order = plant order + $2 \times$ (weights order)
- integral actions can be easily enforced in $C(s)$ (since $W_i(s)$ and $W_o(s)$ are parts of the resulting controller)

Choice of robust stability problem

Rationale:

- not necessarily reflects physics of the problem
- rather, should possess favorable properties from the design viewpoint

Robust stability with respect to normalized coprime factor uncertainties:

- ⚡ impedes cancelations of lightly damped poles/zeros
 - underlying H_∞ problem equivalent to minimizing $\left\| \begin{bmatrix} T_1 & T_2 \\ T_d & S_0 \end{bmatrix} \right\|_\infty$, which penalizes all four closed-loop transfer functions
- ⚡ has noniterative solution
 - nontrivial for H_∞ problems
- ⚡ may preserve magnitude shaping

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Robust stability with respect to **normalized coprime factor uncertainties**:

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underlying H_∞ problem equivalent to minimizing $\left\| \begin{bmatrix} T_i & T_c \\ T_d & S_o \end{bmatrix} \right\|_\infty$, which penalizes all four closed-loop transfer functions
- ☺ has noniterative solution
nontrivial for H_∞ problems
- ☺ may preserve magnitude shaping

Solution to the robust stabilization problem

Let $P_{\text{msh}}(s) = C(sI - A)^{-1}B$. The maximal attainable **robustness radius**

$$\alpha_{\max} = \frac{1}{\sqrt{1 + \rho(YX)}} < 1,$$

where $X \geq 0$ and $Y \geq 0$ are the stabilizing solutions to the CAREs

$$A'X + XA - XBB'X + C'C = 0 \quad \text{and} \quad AY + YA' - YC'CY + BB' = 0.$$

For a given $\gamma > 1/\alpha_{\max} > 1$, the matrix $Z_\gamma := ((1 - \gamma^{-2})I - \gamma^{-2}YX)^{-1}$ is well defined and then all γ -suboptimal stabilizing controllers are given by

$$C_{\text{msh}} = \mathcal{F}_l(\mathcal{J}_\gamma, Q)$$

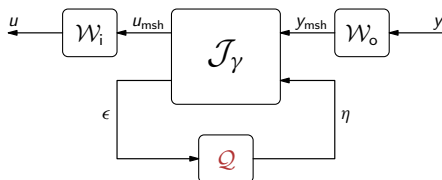
where

$$J_\gamma(s) = \left[\begin{array}{c|cc} A - BB'X - Z_\gamma YC'C & Z_\gamma YC' & Z_\gamma B \\ \hline -B'X & 0 & I \\ -C & I & 0 \end{array} \right]$$

and Q is an arbitrary stable system such that $\|Q\|_{L_2(\mathbb{R}^+) \rightarrow L_2(\mathbb{R}^+)} < \sqrt{\gamma^2 - 1}$.

All admissible H_∞ loop shaping solutions

Thus, the set of all admissible solutions is



Note that if $\|Q\|_{L_2(\mathbb{R}^+) \rightarrow L_2(\mathbb{R}^+)} \not\leq \sqrt{\gamma^2 - 1}$, we cannot even guarantee the stability of the closed-loop system. This structure suggests that

- sampled-data approximation of this controller can be obtained via imposing an appropriate constraint on Q ,

like what we have done in the case of the H_2 optimization and stabilization.

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Digital redesign revisited

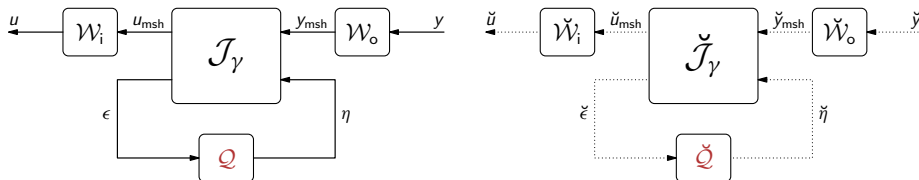
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H_∞ loop shaping design

H_∞ loop shaping **redesign**

Side remark: compact hSTPBC representation of transfer functions

Extraction in the lifted domain



The redesign steps again follow those of the H_2 design:

1. lift the parametrization, $\check{\mathcal{C}} = \check{\mathcal{W}}_i \mathcal{F}_l(\check{\mathcal{J}}_\gamma, \check{\mathcal{Q}}) \check{\mathcal{W}}_o$
2. find admissible $\check{\mathcal{Q}}$'s that render $\check{\mathcal{C}}$ strictly causal

(because the cascade of a causal and a strictly causal systems is strictly causal)

“Admissible” here means such that $\|\check{\mathcal{Q}}\|_{\ell_2^{\mathbb{N}}(\mathbb{Z}^+) \rightarrow \ell_2^{\mathbb{N}}(\mathbb{Z}^+)} < \sqrt{\gamma^2 - 1}$, which includes stability and causality.

Strict causality of $\mathcal{F}_l(\check{\mathcal{J}}_\gamma, \check{\mathcal{Q}})$

We know (Lecture 10, Slide 46) that

- $\mathcal{F}_l(\check{\mathcal{J}}_\gamma, \check{\mathcal{Q}})$ is strictly causal iff the feedthrough part of $\check{\mathcal{Q}}$ equals that of the $(2, 2)$ part of $\check{\mathcal{J}}_\gamma^{-1} = \mathcal{L}_h \mathcal{J}_\gamma^{-1} \mathcal{L}_h^{-1}$.

Because

$$J_\gamma^{-1} = \left[\begin{array}{c|cc} A + (Z_\gamma - I)BB'X & Z_\gamma B & Z_\gamma YC' \\ \hline C & 0 & I \\ B'X & I & 0 \end{array} \right],$$

the feedthrough part of $\check{\mathcal{Q}}$ must be the lifting of

$$\mathcal{Q}_\infty : \begin{cases} \dot{x}_Q(t) = (A + (Z_\gamma - I)BB'X)x_Q(t) + Z_\gamma YC'\epsilon(t), & x_Q(ih) = 0 \\ \eta(t) = B'X x_Q(t). \end{cases}$$

But

- how does this constraint affect admissibility?

Technical preliminaries

Let $\check{\mathcal{G}}$ be a stable and causal system in the lifted domain with the impulse response $\check{g}[i, j]$ and $\check{\mathcal{G}}_\infty$, whose impulse response

$$\check{g}_\infty[i, j] = \begin{cases} \check{g}[i, j] & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

be its feedthrough part.

Remember the definition of induced norms. Given an operator $O : \mathbb{U} \rightarrow \mathbb{Y}$,

$$\|O\|_{\mathbb{U} \rightarrow \mathbb{Y}} := \sup_{u \in \mathbb{U}, \|u\|_{\mathbb{U}}=1} \|Ou\|_{\mathbb{Y}}.$$

Technical preliminaries (contd)

Lemma 1 Let $i_0 \in \mathbb{Z}$ be any index such that $\|\check{g}[i_0, i_0]\|_{\mathbb{L} \rightarrow \mathbb{L}} \geq \|\check{g}[i, i]\|_{\mathbb{L} \rightarrow \mathbb{L}}$ for all $i \in \mathbb{Z}$. Then

$$\|\check{G}_\infty\|_{\ell_2^{\mathbb{L}}(\mathbb{Z}) \rightarrow \ell_2^{\mathbb{L}}(\mathbb{Z})} = \|\check{g}[i_0, i_0]\|_{\mathbb{L} \rightarrow \mathbb{L}}$$

and it is attainable by $\check{u}[i] = u_0 \bar{\delta}[i_0]$ for some $u_0 \in \mathbb{L}$.

Proof Because $\check{G}_\infty$ is static, for any $\check{u} \in \ell_2^{\mathbb{L}}(\mathbb{Z})$ such that $\|\check{u}\|_2 = 1$

$$\begin{aligned} \|\check{G}_\infty \check{u}\|_2^2 &= \sum_{i \in \mathbb{Z}} \|\check{g}[i, i] \check{u}[i]\|_2^2 \leq \sum_{i \in \mathbb{Z}} \|\check{g}[i, i]\|_{\mathbb{L} \rightarrow \mathbb{L}}^2 \|\check{u}[i]\|_2^2 \\ &\leq \|\check{g}[i_0, i_0]\|_{\mathbb{L} \rightarrow \mathbb{L}}^2 \sum_{i \in \mathbb{Z}} \|\check{u}[i]\|_2^2 = \|\check{g}[i_0, i_0]\|_{\mathbb{L} \rightarrow \mathbb{L}}^2, \end{aligned}$$

which implies that $\|\check{G}_\infty\|_{\ell_2^{\mathbb{L}}(\mathbb{Z}) \rightarrow \ell_2^{\mathbb{L}}(\mathbb{Z})} \leq \|\check{g}[i_0, i_0]\|_{\mathbb{L} \rightarrow \mathbb{L}}$. This upper bound is attainable whenever $\check{u}[i] = u_0 \bar{\delta}[i_0]$ for any $u_0 \in \mathbb{L}$ such that $\|u_0\|_{\mathbb{L}} = 1$ and $\|\check{g}[i_0, i_0] \check{u}_0\|_2 = \|\check{g}[i_0, i_0]\|_{\mathbb{L} \rightarrow \mathbb{L}}$. □

Technical preliminaries (contd)

Lemma 2 $\|\check{\mathcal{G}}\|_{\ell_2^{\mathbb{L}}(\mathbb{Z}) \rightarrow \ell_2^{\mathbb{L}}(\mathbb{Z})} \geq \|\check{\mathcal{G}}_\infty\|_{\ell_2^{\mathbb{L}}(\mathbb{Z}) \rightarrow \ell_2^{\mathbb{L}}(\mathbb{Z})}.$

Proof For any $\check{u}$ such that $\check{u}[i] = u_0 \bar{\delta}[i_0]$ with u_0 such that $\|u_0\|_{\mathbb{L}} = 1$ and $\|\check{g}[i_0, i_0] \check{u}_0\|_2 = \|\check{g}[i_0, i_0]\|_{\mathbb{L} \rightarrow \mathbb{L}},$

$$\begin{aligned} \|\check{\mathcal{G}}\check{u}\|_2^2 &= \sum_{i \geq i_0} \|\check{g}[i, i_0] u_0\|_2^2 = \|\check{g}[i_0, i_0]\|_{\mathbb{L} \rightarrow \mathbb{L}}^2 + \sum_{i > i_0} \|\check{g}[i, i_0] u_0\|_2^2 \\ &\geq \|\check{g}[i_0, i_0]\|_{\mathbb{L} \rightarrow \mathbb{L}}^2 = \|\check{\mathcal{G}}_\infty\|_{\ell_2^{\mathbb{L}}(\mathbb{Z}) \rightarrow \ell_2^{\mathbb{L}}(\mathbb{Z})}^2. \end{aligned}$$

Hence, we have that

$$\|\check{\mathcal{G}}\|_{\ell_2^{\mathbb{L}}(\mathbb{Z}) \rightarrow \ell_2^{\mathbb{L}}(\mathbb{Z})} \geq \|\check{\mathcal{G}}\check{u}\|_2 \geq \|\check{\mathcal{G}}_\infty\|_{\ell_2^{\mathbb{L}}(\mathbb{Z}) \rightarrow \ell_2^{\mathbb{L}}(\mathbb{Z})},$$

which proves the statement. □

Admissibility of $\check{Q}$

By Lemma 2, we know that

- $\check{Q}$ is admissible *only if* $\|\check{Q}_\infty\|_{\ell_2^{\mathbb{L}}(\mathbb{Z}^+) \rightarrow \ell_2^{\mathbb{L}}(\mathbb{Z}^+)} < \sqrt{\gamma^2 - 1}$.

This condition is also sufficient as we may always pick $\check{Q} = \check{Q}_\infty$. Moreover, because $\check{Q}_\infty$ is shift invariant in our case,

- $\check{Q}$ is admissible iff $\|\check{q}[0]\|_{\mathbb{L} \rightarrow \mathbb{L}} < \sqrt{\gamma^2 - 1}$,

which, in turn, is equivalent to the condition that the $\mathbb{L}$ -induced norm of the $\mathbb{L} \rightarrow \mathbb{L}$ operator described by

$$\begin{cases} \dot{x}(\tau) = (A + (Z_\gamma - I)BB'X)x(\tau) + Z_\gamma YC'u(\tau), & x(0) = 0 \\ y(\tau) = B'Xx(\tau) \end{cases}$$

is smaller than $\sqrt{\gamma^2 - 1}$.

Admissibility of $\check{Q}$ (contd)

In other words, there is an admissible $\check{Q}$ that renders $\check{C}$ strictly causal iff the $L_2[0, h]$ -induced norm of the LTI system with the transfer function

$$Q_J(s) = \left[\begin{array}{c|c} \frac{A + (Z_\gamma - I)BB'X}{B'X} & \frac{Z_\gamma YC'}{0} \end{array} \right]$$

is smaller than γ . So the next question is

- how to compute the $L_2[0, h]$ -induced norm of an LTI system?

More technical preliminaries

Lemma 3 Let $O : \mathbb{H}_i \rightarrow \mathbb{H}_o$ be a bounded Hilbert space operator. Then

$$\|O\|_{\mathbb{H}_i \rightarrow \mathbb{H}_o} < \gamma \iff \gamma^2 I - O^* O > 0 \iff \gamma^2 I - O O^* > 0.$$

Proof It is readily seen that

$$\|O\|_{\mathbb{H}_i \rightarrow \mathbb{H}_o}^2 < \gamma^2 \iff \gamma^2 \|u\|_{\mathbb{H}_i}^2 - \|Ou\|_{\mathbb{H}_o}^2 > 0, \quad \forall u \neq 0.$$

But

$$\gamma^2 \|u\|_{\mathbb{H}_i}^2 - \|Ou\|_{\mathbb{H}_o}^2 = \gamma^2 \langle u, u \rangle_{\mathbb{H}_i} - \langle Ou, Ou \rangle_{\mathbb{H}_o} = \langle (\gamma^2 I - O^* O)u, u \rangle_{\mathbb{H}_i},$$

from which the first “iff” follows immediately. The second “iff” follows by the fact that $\|O\|_{\mathbb{H}_i \rightarrow \mathbb{H}_o} = \|O^*\|_{\mathbb{H}_o \rightarrow \mathbb{H}_i}$. □

More technical preliminaries (contd)

Lemma 4 Consider a family of integral operators $O_\theta : L_2[0, \theta] \rightarrow L_2[0, \theta]$ having a common *bounded* kernel $o(t, s)$, i.e.

$$O_\theta : \zeta \mapsto \xi, \quad \text{where } \xi(t) = \int_0^\theta o(t, s)\zeta(s)ds \text{ for all } t \in [0, \theta].$$

Then $\|O_\theta\|_{L_2[0, \theta] \rightarrow L_2[0, \theta]}$ is a monotonically increasing continuous function of θ and its limit at $\theta \downarrow 0$ is zero.

Proof Let P_θ be the projection operator truncating any signal to $[0, \theta]$, i.e.

$$(P_\theta u)(t) = \begin{cases} u(t) & \text{if } t \in [0, \theta] \\ 0 & \text{otherwise} \end{cases}$$

Then

$$O_{\theta_2} = P_{\theta_2} O_{\theta_1} P_{\theta_2}$$

whenever $\theta_1 > \theta_2 \geq 0$.

More technical preliminaries (contd)

Proof (contd) Now, let $\mathbb{U}_i := L_2[0, \theta_i]$ and

$$\mathbb{U}_{12} := \{u \in L_2[0, \theta_1] \mid u(t) = 0 \text{ whenever } t > \theta_2\} \subset \mathbb{U}_1$$

(i.e. $\mathbb{U}_{12}$ comprises all signals from $\mathbb{U}_1$ such that $u = P_{\theta_2} u$). We then have:

$$\begin{aligned} \sup_{u \in \mathbb{U}_1, \|u\|_2=1} \|O_{\theta_1} u\|_2^2 &\geq \sup_{u \in \mathbb{U}_{12}, \|u\|_2=1} \|O_{\theta_1} u\|_2^2 = \sup_{u \in \mathbb{U}_{12}, \|u\|_2=1} \|O_{\theta_1} P_{\theta_2} u\|_2^2 \\ &\geq \sup_{u \in \mathbb{U}_{12}, \|u\|_2=1} \|P_{\theta_2} O_{\theta_1} P_{\theta_2} u\|_2^2 = \sup_{u \in \mathbb{U}_2, \|u\|_2=1} \|O_{\theta_2} u\|_2^2 \end{aligned}$$

In other words,

$$\|O_{\theta_1}\|_{\mathbb{U}_1 \rightarrow \mathbb{U}_1} \geq \|O_{\theta_2}\|_{\mathbb{U}_2 \rightarrow \mathbb{U}_2}$$

whenever $\theta_1 > \theta_2 \geq 0$, which proves monotonicity. The proof of continuity is more tedious. The result for the limit follows from the known fact that $\lim_{\theta \downarrow 0} Q_\theta = 0$. □

$L_2[0, h]$ -induced norms of LTI systems

Let $G(s) = C(sI - A)^{-1}B$. Denote by $\check{G}_\theta : L_2[0, \theta] \rightarrow L_2[0, \theta]$ the integral operator with the kernel $g_\theta(t, s) = Ce^{A(t-s)}B\mathbb{1}(t-s)$ and then by $\|\check{G}_\theta\|$ its $L_2[0, \theta]$ -induced norm, so that $\|\check{G}_h\|$ equals the $L_2[0, h]$ -induced norms of $\mathcal{G}$. Our goal is to find conditions for $\|\check{G}_h\| < \gamma$. To this end, note that

- $\lim_{\theta \downarrow 0} \|\check{G}_\theta\| = 0$, so the bound always holds for sufficiently small θ .
- $\lim_{\theta \rightarrow \infty} \|\check{G}_\theta\| = \|G\|_\infty$ (bounded iff $\mathcal{G}$ is stable).
- as $\|\check{G}_\theta\|$ monotonically increases and is continuous as a function of θ , every $0 < \gamma < \|G\|_\infty$ is attainable under some θ and

$$\begin{aligned} \|\check{G}_h\| \not< \gamma &\iff \gamma^2 I - \check{G}_h^* \check{G}_h \not> 0 \\ &\iff \gamma^2 I - \check{G}_\theta^* \check{G}_\theta \text{ is singular for some } \theta \in [0, h] \end{aligned}$$

Thus, we can find the minimal θ for which $\|\check{G}_\theta\| \not< \gamma$ via

- finding the minimal θ for which $(\gamma^2 I - \check{G}_\theta^* \check{G}_\theta)^{-1}$ is not well posed.

$L_2[0, h]$ -induced norms of LTI systems (contd)

Represent $\check{G}_\theta$ as the following hSTPBC:

$$\begin{cases} \dot{x}_1(\tau) = Ax_1(\tau) + Bu(\tau), & x_1(0) = 0 \\ y(\tau) = Cx_1(\tau) \end{cases}$$

Then $\check{G}_\theta^*$ corresponds to

$$\begin{cases} \dot{x}_2(\tau) = -A'x_2(\tau) - C'y(\tau), & x_2(\theta) = 0 \\ z(\tau) = B'x_2(\tau) \end{cases}$$

and $\gamma^2 I - \check{G}_\theta^* \check{G}_\theta$ —to

$$\begin{cases} \dot{x}(\tau) = \begin{bmatrix} -A' & -C'C \end{bmatrix} x(\tau) + \begin{bmatrix} 0 \\ B \end{bmatrix} u(\tau), & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} x(0) + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x(\theta) = 0 \\ z(\tau) = -\begin{bmatrix} B' & 0 \end{bmatrix} x(\tau) + \gamma^2 u(\tau) \end{cases}$$

$L_2[0, h]$ -induced norms of LTI systems (contd)

Finally, $(\gamma^2 I - \check{G}_\theta^* \check{G}_\theta)^{-1}$ is obtained by swapping u and z :

$$\begin{cases} \dot{x}(\tau) = [\gamma^{-2} A'_{BB'}, -C'_A C] x(\tau) + \gamma^{-2} \begin{bmatrix} 0 \\ B \end{bmatrix} u(\tau), & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} x(0) + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x(\theta) = 0 \\ u(\tau) = \gamma^{-2} \begin{bmatrix} B' & 0 \end{bmatrix} x(\tau) + \gamma^{-2} z(\tau) \end{cases}$$

Define the function

$$\begin{bmatrix} \Gamma_{\gamma,11}(\theta) & \Gamma_{\gamma,12}(\theta) \\ \Gamma_{\gamma,21}(\theta) & \Gamma_{\gamma,22}(\theta) \end{bmatrix} := \exp \left(\begin{bmatrix} -A' & -C'_A C \\ \gamma^{-2} B B' & A \end{bmatrix} \theta \right).$$

This system is well posed iff

$$\Xi = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{\gamma,11}(\theta) & \Gamma_{\gamma,12}(\theta) \\ \Gamma_{\gamma,21}(\theta) & \Gamma_{\gamma,22}(\theta) \end{bmatrix} = \begin{bmatrix} \Gamma_{\gamma,11}(\theta) & \Gamma_{\gamma,12}(\theta) \\ 0 & I \end{bmatrix}$$

is nonsingular. Thus,

$$- \quad \|\check{G}_h\| < \gamma \iff \det \Gamma_{\gamma,11}(\theta) \neq 0 \text{ for all } \theta \in [0, h].$$

$L_2[0, h]$ -induced norms of LTI systems: example

Let

$$G(s) = \frac{1}{s}$$

(so that $A = 0$ and $B = C = 1$). Then

$$\begin{bmatrix} \Gamma_{\gamma,11}(\theta) & \Gamma_{\gamma,12}(\theta) \\ \Gamma_{\gamma,21}(\theta) & \Gamma_{\gamma,22}(\theta) \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & -1 \\ \gamma^{-2} & 0 \end{bmatrix} \theta\right) = \begin{bmatrix} \cos \frac{\theta}{\gamma} & -\gamma \sin \frac{\theta}{\gamma} \\ \gamma^{-1} \sin \frac{\theta}{\gamma} & \cos \frac{\theta}{\gamma} \end{bmatrix}$$

and $\det \Gamma_{\gamma,11}(\theta) = 0$ iff $\frac{\theta}{\gamma} = \frac{\pi}{2}$. Thus,

$$\|\check{G}_h\| < \gamma \iff h < \frac{\pi}{2}\gamma.$$

Remark: In general, the condition reduces to transcendental equations and cannot be solved analytically. The MATLAB command "compnorm" can be used in numerical simulations (combined with a binary search over γ or h).

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Remark: In general, the condition reduces to transcendental equations and cannot be solved analytically. The MATLAB command “compnorm” can be used in numerical simulations (combined with a binary search over γ or h).

An admissible $\mathcal{Q}$

Summarizing, there is an admissible $\mathcal{Q}$ that renders the γ -suboptimal H_∞ loop shaping controller sampled-data iff

$$\begin{bmatrix} I & 0 \end{bmatrix} \exp \left(\begin{bmatrix} -A' - XBB'(Z'_\gamma - I) & -XBB'X \\ \frac{1}{\gamma^2 - 1} Z_\gamma Y C' C Y Z'_\gamma & A + (Z_\gamma - I)BB'X \end{bmatrix} \theta \right) \begin{bmatrix} I \\ 0 \end{bmatrix}$$

is nonsingular for all $\theta \in [0, h]$. One such $\mathcal{Q}$ may be described by

$$\mathcal{Q} : \begin{cases} \dot{x}_Q(t) = (A + (Z_\gamma - I)BB'X)x_Q(t) + Z_\gamma Y C' \epsilon(t), & x_Q(ih) = 0 \\ \eta(t) = B'X x_Q(t) \end{cases}$$

where $X \geq 0$ and $Y \geq 0$ are the stabilizing solutions to the CAREs

$$A'X + XA - XBB'X + C'C = 0 \quad \text{and} \quad AY + YA' - YC'CY + BB' = 0$$

and $Z_\gamma = ((1 - \gamma^{-2})I - \gamma^{-2}YX)^{-1}$.

An admissible $\mathcal{C}_{\text{msh}}$

With some tedious algebra, the controller is

$$\begin{cases} \dot{\tilde{x}}(t) = \begin{bmatrix} A - YC'C & -BB'X \\ 0 & A - BB'X \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} YC' \\ 0 \end{bmatrix} y_{\text{msh}}(t), \\ u_{\text{msh}}(t) = \begin{bmatrix} 0 & -B'X \end{bmatrix} \tilde{x}(t) \end{cases}$$

with periodic state resets of the form $\tilde{x}_2(ih) = Z_\gamma \tilde{x}_1(ih)$, which is of almost the same form as before. Differences are

- the state reset mechanism is a bit different (depends on Z_γ)
- the input is a filtered version of y , namely, $y_{\text{msh}} = \mathcal{W}_o y$
- the actual control signal u is a filtered version of u_{msh} , $u = \mathcal{W}_i u_{\text{msh}}$
(it makes sense to use a low-pass $\mathcal{W}_i$, to render u continuous or even differentiable)

Remark: It is worth emphasizing that the controller is stabilizing if and *only* if the $L_2(\mathbb{R}^+)$ -induced norm of $\mathcal{G}$ is smaller than $\sqrt{\gamma^2 - 1}$.

Content

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H_∞ loop shaping redesign

Side remark: compact hSTPBC representation of transfer functions

Compact representation

Consider the hSTPBC

$$\check{O}(z) : \begin{cases} \dot{x}(\tau) = Ax(\tau) + Bu(\tau), & zx(0) - x(h) = 0 \\ y(\tau) = Cx(\tau) + Du(\tau) \end{cases}$$

Its solution is (because $(zI - e^{Ah})^{-1}z = I + (zI - e^{Ah})^{-1}e^{Ah}$)

$$\begin{aligned} y(\tau) &= Du(\tau) + Ce^{A\tau}(zI - e^{Ah})^{-1} \\ &\quad \times \left(z \int_0^\tau e^{-A\sigma} Bu(\sigma) d\sigma + e^{Ah} \int_\tau^h e^{-A\sigma} Bu(\sigma) d\sigma \right) \\ &= Du(\tau) + C \int_0^\tau e^{A(\tau-\sigma)} Bu(\sigma) d\sigma \\ &\quad + Ce^{A\tau} (zI - e^{Ah})^{-1} \int_0^h e^{A(h-\sigma)} Bu(\sigma) d\sigma \end{aligned}$$

i.e. $y = (\check{D} + \check{C}(zI - \bar{A})^{-1}\check{B})u$. Thus $\check{O}(\cdot) = \check{G}(z)$ for all $z \in \mathbb{C}$ for which it exists. This may be an alternative to the integral operator representation.

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i.e. $y = (\check{D} + \check{C}(zI - \bar{A})^{-1}\check{B})u$. Thus $\check{O}(z) = \check{\check{G}}(z)$ for all $z \in \mathbb{C}$ for which it exist. This may be an alternative to the integral operator representation.

Compact representation (contd)

With some more effort, it can be shown that the transfer function

$$\begin{bmatrix} \check{G}_{zw}(z) & \check{G}_{zu}(z) \\ \check{G}_{yw}(z) & \check{G}_{yu}(z) \end{bmatrix} = \begin{bmatrix} \check{G}_{zw}(z) & \check{G}_{zu}(z)\check{\Phi}_{\text{ZOH}}(z) \\ \check{\Psi}_{\text{idl}}(z)\check{G}_{yw}(z) & \check{\Psi}_{\text{idl}}(z)\check{G}_{yu}(z)\check{\Phi}_{\text{ZOH}}(z) \end{bmatrix}$$

can be described by the hSTPBC

$$\left\{ \begin{array}{l} \frac{d}{d\tau} \check{X}(z; \tau) = \begin{bmatrix} A & B_u \\ 0 & 0 \end{bmatrix} \check{X}(z; \tau) + \begin{bmatrix} B_w \\ 0 \end{bmatrix} \check{W}(z; \tau) \\ \begin{bmatrix} zI & 0 \\ 0 & I \end{bmatrix} \check{X}(z; 0) - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \check{X}(z; h) = \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{U}(z) \\ \check{Z}(z; \tau) = \begin{bmatrix} C_z & D_{zu} \end{bmatrix} \check{X}(z; \tau) + D_{zw} \check{W}(z; \tau) \\ \bar{Y}(z) = \begin{bmatrix} C_y & 0 \end{bmatrix} \check{X}(z; 0) \end{array} \right.$$

Alternative form of the H_2 optimal controller

The lifted transfer function $\check{K}_{\text{opt}}(z)$ can be expressed as the hSTPBC

$$\left\{ \begin{array}{l} \frac{d}{d\tau} \check{X}_K(z; \tau) = \begin{bmatrix} A + L_y C_y & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} \check{X}_K(z; \tau) - \begin{bmatrix} L_y \\ 0 \end{bmatrix} \check{Y}(z; \tau) \\ \begin{bmatrix} zI & 0 \\ -I & I \end{bmatrix} \check{X}_K(z; 0) - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \check{X}_K(z; h) = 0 \\ \check{U}(z; \tau) = \begin{bmatrix} 0 & F_u \end{bmatrix} \check{X}_K(z; h) \end{array} \right.$$