

Advanced Sampled-Data Control (038798)

lecture no. 10

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Content

Type IV standard problem

Background: hSTPBC

Direct approach via hSTPBC

Extraction from analog parametrization

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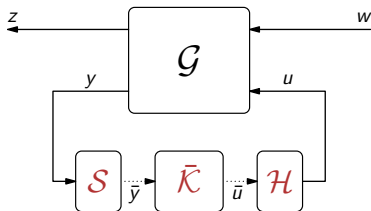
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Standard Type IV sampled-data H_2 problem



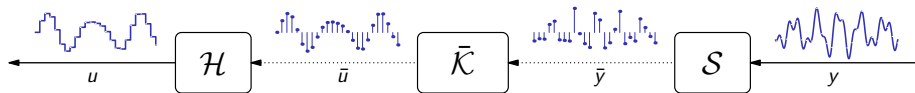
Given $\mathcal{G}$ with

$$G(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array} \right],$$

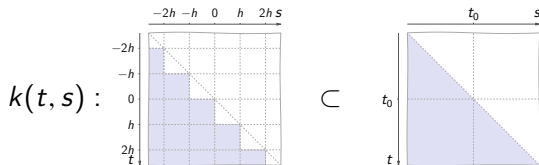
design causal internally stabilizing sampled-data controller $\mathcal{K} = \mathcal{H}\bar{\mathcal{K}}\mathcal{S}$ that minimizes $\|\mathcal{T}_{zw}\|_2$, where

$$\mathcal{T}_{zw} = \mathcal{F}_l(\mathcal{G}, \mathcal{H}\bar{\mathcal{K}}\mathcal{S}) = \mathcal{G}_{zw} + \mathcal{G}_{zu}\mathcal{H}\bar{\mathcal{K}}\mathcal{S}(I - \mathcal{G}_{yu}\mathcal{H}\bar{\mathcal{K}}\mathcal{S})^{-1}\mathcal{G}_{yw}.$$

Sampled-data causality



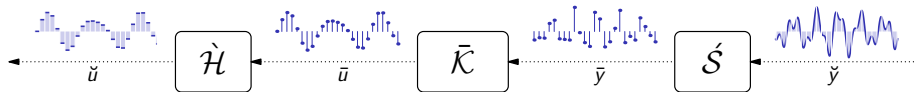
Condition “ $k(t, s) = 0$ whenever $s > \lfloor \frac{t}{h} \rfloor h$ ” constrains



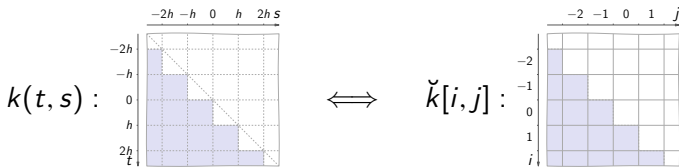
Thus,

- any sampled-data controller *must* have a “staircase” kernel although it’s not clear (yet?) whether this property is sufficient.

Sampled-data causality in the lifted domain



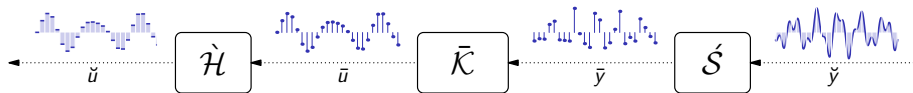
Simpler in the lifted domain:



It is clear now (also, we already saw in Lecture 4, Slide 39 that the causality of $\mathcal{S}$ implies the strict causality of $\mathcal{S}'$) that

- lifting of a sampled-data controller is always strictly causal.

Sampled-data causality in the lifted domain (contd)

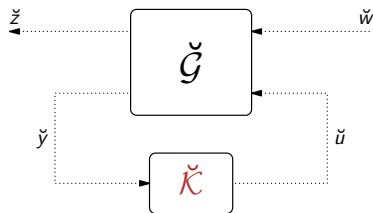


But we saw in Lecture 9, Slide 26 that any strictly causal lifted system with a finite state dimension is of the form $\mathcal{H}\bar{\mathcal{K}}\mathcal{S}$. Thus,

- $\check{\mathcal{K}}$ with a finite-dimensional state is the lifting of $\mathcal{H}\bar{\mathcal{K}}\mathcal{S}$ iff $\check{\mathcal{K}}$ is strictly causal.

This condition is much easier to handle than the finite rank condition in the non-causal case (the Rank Theorem in Lecture 6, Slide 18).

Lifted reformulation



Lifting renders it pure discrete LTI problem of minimizing $\|\check{\mathcal{T}}_{zw}\|_2$, where

$$\check{\mathcal{T}}_{zw} = \mathcal{F}_l(\check{\mathcal{G}}, \check{\mathcal{K}}) = \check{\mathcal{G}}_{zw} + \check{\mathcal{G}}_{zu}\check{\mathcal{K}}(I - \check{\mathcal{G}}_{yu}\check{\mathcal{K}})^{-1}\check{\mathcal{G}}_{yw},$$

by a *strictly causal* stabilizing $\check{\mathcal{K}}$. Thus, we again end up with a

- discrete H_2 problem with a strictly causal controller, whose solution we know.

Assumptions in the lifted domain

Having

$$\begin{bmatrix} \check{G}_{zw}(z) & \check{G}_{zu}(z) \\ \check{G}_{yw}(z) & \check{G}_{yu}(z) \end{bmatrix} = \left[\begin{array}{c|cc} \bar{A} & \acute{B}_w & \acute{B}_u \\ \hline \grave{C}_z & \check{D}_{zw} & \check{D}_{zu} \\ \grave{C}_y & \check{D}_{yw} & \check{D}_{yu} \end{array} \right],$$

we assume that

- $(\bar{A}, \acute{B}_u)$ is stabilizable
- $(\grave{C}_y, \bar{A})$ is detectable
- $\begin{bmatrix} \bar{A}-zI & \acute{B}_u \\ \grave{C}_z & \check{D}_{zu} \end{bmatrix}$ is left invertible for all $z \in \mathbb{T}$
- $\begin{bmatrix} \bar{A}-zI & \acute{B}_w \\ \grave{C}_y & \check{D}_{yw} \end{bmatrix}$ is right invertible for all $z \in \mathbb{T}$

Lifted solution

Let

$$\begin{aligned}\dot{F}_u &= -(\check{D}_{zu}^* \check{D}_{zu} + \dot{B}_u^* \bar{X} \dot{B}_u)^{-1}(\check{D}_{zu}^* \dot{C}_z + \dot{B}_u^* \bar{X} \bar{A}) \\ \dot{L}_y &= -(\dot{B}_w \check{D}_{yw}^* + \bar{A} \bar{Y} \dot{C}_y^*)(\check{D}_{yw} \check{D}_{yw}^* + \dot{C}_y \bar{Y} \dot{C}_y^*)^{-1}\end{aligned}$$

where $\bar{X} \geq 0$ and $\bar{Y} \geq 0$ are the stabilizing solutions to the DARE's

$$\begin{aligned}\bar{X} &= \bar{A}' \bar{X} \bar{A} + \dot{C}_z^* \dot{C}_z - (\dot{C}_z^* \check{D}_{zu} + \bar{A}' \bar{X} \dot{B}_u)(\check{D}_{zu}^* \check{D}_{zu} + \dot{B}_u^* \bar{X} \dot{B}_u)^{-1}(\check{D}_{zu}^* \dot{C}_z + \dot{B}_u^* \bar{X} \bar{A}), \\ \bar{Y} &= \bar{A} \bar{Y} \bar{A}' + \dot{B}_w \dot{B}_w^* - (\dot{B}_w \check{D}_{yw}^* + \bar{A} \bar{Y} \dot{C}_y^*)(\check{D}_{yw} \check{D}_{yw}^* + \dot{C}_y \bar{Y} \dot{C}_y^*)^{-1}(\check{D}_{yw} \dot{B}_w^* + \dot{C}_y \bar{Y} \bar{A}').\end{aligned}$$

Then the optimal controller is

$$\check{K}_{\text{opt}}(z) = - \left[\frac{\bar{A} + \dot{B}_u \dot{F}_u + \dot{L}_y \dot{C}_y + \dot{L}_y \check{D}_{yu} \dot{F}_u}{\dot{F}_u} \middle| \frac{\dot{L}_y}{0} \right],$$

and the optimal performance is

$$\|\check{T}_{zw}\|_2^2 = \frac{1}{h} \left(\|\check{D}_{zw}\|_{\text{HS}}^2 + \text{tr}(\bar{X} \dot{B}_w \dot{B}_w^* + \dot{C}_z^* \dot{C}_z \bar{Y} + \bar{X} \bar{A} \bar{Y} \bar{A}' - \bar{X} \bar{Y}) \right).$$

Hmm...

Now the parts that include infinite-dimensional operators do not collapse to finite-dimensional operators (matrices). For instance, to compute the gain

$$\dot{F}_u = -(\check{D}_{zu}^* \check{D}_{zu} + \acute{B}_u^* \bar{X} \acute{B}_u)^{-1} (\check{D}_{zu}^* \dot{C}_z + \acute{B}_u^* \bar{X} \bar{A})$$

and the DARE

$$\begin{aligned} \bar{X} = & \bar{A}' \bar{X} \bar{A} + \dot{C}_z^* \dot{C}_z \\ & - (\dot{C}_z^* \check{D}_{zu} + \bar{A}' \bar{X} \acute{B}_u) (\check{D}_{zu}^* \check{D}_{zu} + \acute{B}_u^* \bar{X} \acute{B}_u)^{-1} (\check{D}_{zu}^* \dot{C}_z + \acute{B}_u^* \bar{X} \bar{A}) \end{aligned}$$

we need to compute and manipulate the infinite-dimensional operator

$$\check{O}_u := \begin{bmatrix} \check{D}_{zu}^* \\ \dot{C}_z^* \end{bmatrix} \begin{bmatrix} \check{D}_{zu} & \dot{C}_z \end{bmatrix} + \begin{bmatrix} \acute{B}_u^* \\ \bar{A}' \end{bmatrix} \bar{X} \begin{bmatrix} \acute{B}_u & \bar{A} \end{bmatrix} : \mathbb{L}^m \times \mathbb{R}^n \rightarrow \mathbb{L}^m \times \mathbb{R}^n$$

and then interpret the hold $\dot{F}_u : \mathbb{R}^n \rightarrow \mathbb{L}^m$. We shall study two ways to do that...

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The idea

Integral operators on finite time intervals (like $[0, h]$) are easier manipulated

— via their state-space equations,

much like what we do with dynamic systems over $\mathbb{R}^+$ (the idea traces back to Bart, Gohberg & Kaashoek, 1982). E.g. the operator $\check{D} : \mathbb{L} \rightarrow \mathbb{L}$

$$\check{D} : u \mapsto u(\tau) + C \int_0^\tau e^{A(\tau-\sigma)} B u(\sigma) d\sigma$$

(here $D = I$ to simplify the exposition) can be equivalently described as

$$\check{D} : \begin{cases} \dot{x}(\tau) = Ax(\tau) + Bu(\tau), & x(0) = 0 \\ y(\tau) = Cx(\tau) + u(\tau) \end{cases}$$

so

$$\check{D}^{-1} : \begin{cases} \dot{x}(\tau) = (A - BC)x(\tau) + By(\tau), & x(0) = 0 \\ u(\tau) = -Cx(\tau) + y(\tau) \end{cases}$$

which is only a matter of swapping u and y and is *way easier* than inverting $\check{D}$ as an integral operator. This logic works in more general settings...

hSTPBC

Our main tool is **hybrid systems with two-point boundary conditions**, which are operators $\Sigma : \mathbb{L}^m \times \mathbb{R}^{\bar{m}} \rightarrow \mathbb{L}^p \times \mathbb{R}^{\bar{p}}$ mapping $\begin{bmatrix} u \\ \bar{u} \end{bmatrix} \mapsto \begin{bmatrix} y \\ \bar{y} \end{bmatrix}$ as follows:

$$\Sigma : \begin{cases} \dot{x}(\tau) = Ax(\tau) + Bu(\tau), & \Omega x(0) + \Upsilon x(h) = \bar{B}\bar{u} \\ y(\tau) = Cx(\tau) + Du(\tau) \\ \bar{y} = \bar{C}_0 x(0) + \bar{C}_h x(h) + \bar{D}\bar{u} \end{cases}$$

with *square* matrices Ω and Υ shaping boundary conditions of $x(\tau)$.

Similarity transformations do not change hSTPBC's:

$$\Sigma : \begin{cases} \dot{\tilde{x}}(\tau) = T^{-1}AT\tilde{x}(\tau) + T^{-1}Bu(\tau), & S\Omega T\tilde{x}(0) + S\Upsilon T\tilde{x}(h) = S\bar{B}\bar{u} \\ y(\tau) = CT\tilde{x}(\tau) + Du(\tau) \\ \bar{y} = \bar{C}_0 T\tilde{x}(0) + \bar{C}_h T\tilde{x}(h) + \bar{D}\bar{u} \end{cases}$$

for any nonsingular T and S .

hSTPBC: solution

We should know that

$$x(\tau) = e^{A\tau}x(0) + \int_0^\tau e^{A(\tau-\sigma)}Bu(\sigma)d\sigma,$$

from which

$$\Omega x(0) + \Upsilon x(h) = (\Omega + \Upsilon e^{Ah})x(0) + \Upsilon \int_0^h e^{A(h-\sigma)}Bu(\sigma)d\sigma = \bar{B}\bar{u}.$$

This equation is solvable in $x(0)$ for all possible $u \in \mathbb{L}^m$ and $\bar{u} \in \mathbb{R}^{\bar{m}}$ if

– $\Xi := \Omega + \Upsilon e^{Ah}$ is invertible,

which is the **well-posedness** condition of this hSTPBC (always well posed if either “ $\Omega = I$ and $\Upsilon = 0$ ” or “ $\Omega = 0$ and $\Upsilon = I$ ”).

hSTPBC: solution (contd)

If Σ is well-posed,

$$\begin{aligned} x(\tau) &= e^{A\tau} \Xi^{-1} \left(\bar{B} \bar{u} - \Upsilon \int_0^h e^{A(h-\sigma)} B u(\sigma) d\sigma \right) + \int_0^\tau e^{A(\tau-\sigma)} B u(\sigma) d\sigma \\ &= e^{A\tau} \Xi^{-1} \left(\Omega \int_0^\tau e^{-A\sigma} B u(\sigma) d\sigma - \Upsilon e^{Ah} \int_\tau^h e^{-A\sigma} B u(\sigma) d\sigma + \bar{B} \bar{u} \right) \end{aligned}$$

and then

$$\begin{aligned} y(\tau) &= Du(\tau) + Ce^{A\tau} \Xi^{-1} \left(\Omega \int_0^\tau e^{-A\sigma} B u(\sigma) d\sigma - \Upsilon e^{Ah} \int_\tau^h e^{-A\sigma} B u(\sigma) d\sigma \right) \\ &\quad + Ce^{A\tau} \Xi^{-1} \bar{B} \bar{u} \\ \bar{y} &= (-\bar{C}_0 \Xi^{-1} \Upsilon e^{Ah} + \bar{C}_h e^{Ah} \Xi^{-1} \Omega) \int_0^h e^{-A\sigma} B u(\sigma) d\sigma \\ &\quad + (\bar{D} + (\bar{C}_0 + \bar{C}_h e^{Ah}) \Xi^{-1} \bar{B}) \bar{u} \end{aligned}$$

with the subsystem $u \mapsto y$ is causal / anti-causal whenever $\Upsilon = 0$ / $\Omega = 0$.

hSTPBC: adjoint

It can be shown that

$$\Sigma^* : \begin{cases} \dot{x}(\tau) = -A'x(\tau) - C'u(\tau), & \tilde{\Omega}'x(0) + \tilde{\Upsilon}'x(h) = \bar{C}'\bar{u} \\ y(\tau) = B'x(\tau) + D'u(\tau) \\ \bar{y} = \bar{B}_0'x(0) - \bar{B}_h'x(h) + (\bar{D}' + \bar{B}_0'\bar{C}_0' + \bar{B}_h'\bar{C}_h')\bar{u} \end{cases}$$

where $\tilde{\Omega}$ and $\tilde{\Upsilon}$ are any matrices such that

$$\Omega\tilde{\Omega} = \Upsilon\tilde{\Upsilon} \quad \text{and} \quad \tilde{\Omega}'\tilde{\Omega} + \tilde{\Upsilon}'\tilde{\Upsilon} > 0,$$

$\bar{B}_0$ and $\bar{B}_h$ are any matrices such that

$$\Omega\bar{B}_0 + \Upsilon\bar{B}_h = \bar{B},$$

and

$$\bar{C} := -\bar{C}_0'\tilde{\Omega} + \bar{C}_h'\tilde{\Upsilon}.$$

Moreover, if Σ is well posed, so is Σ^* .

$\check{D}$ as hSTPBC

The hSTPBC¹

$$\Sigma_D : \begin{cases} \dot{x}(\tau) = Ax(\tau) + Bu(\tau), & x(0) = 0 \\ y(\tau) = Cx(\tau) + Du(\tau) \end{cases}$$

corresponds to the (causal) operator $\mathbb{L}^m \rightarrow \mathbb{L}^p$

$$u \mapsto Du(\tau) + C \int_0^\tau e^{A(\tau-\sigma)} Bu(\sigma) d\sigma,$$

which is of the form of the “ $\check{D}$ ” part of lifted state-space realizations, i.e.

$$\check{D} = \Sigma_D.$$

¹Corresponds to $\Omega = I$, $\Upsilon = 0$, $\bar{B} = 0$, $\bar{C}_0 = \bar{C}_h = 0$, and $\bar{D} = 0$.

$\dot{C}$ as hSTPBC

The hSTPBC²

$$\Sigma_C : \begin{cases} \dot{x}(\tau) = Ax(\tau), & x(0) = \bar{u} \\ y(\tau) = Cx(\tau) \end{cases}$$

corresponds to the operator $\mathbb{R}^n \rightarrow \mathbb{L}^p$

$$\bar{u} \mapsto Ce^{A\tau} \bar{u},$$

which is of the form of the “ $\dot{C}$ ” part of lifted state-space realizations, i.e.

$$\dot{C} = \Sigma_C.$$

²Corresponds to $\Omega = I$, $\Upsilon = 0$, $B = 0$, $D = 0$, $\bar{B} = I$, $\bar{C}_0 = \bar{C}_h = 0$, and $\bar{D} = 0$.

$\acute{B}$ as hSTPBC

The hSTPBC³

$$\Sigma_B : \begin{cases} \dot{x}(\tau) = Ax(\tau) + Bu(\tau), & x(0) = 0 \\ \bar{y} = x(h) \end{cases}$$

corresponds to the operator $\mathbb{L}^m \rightarrow \mathbb{R}^n$

$$u \mapsto \int_0^h e^{A(h-\sigma)} Bu(\sigma) d\sigma,$$

which is of the form of the “ $\acute{B}$ ” part of lifted state-space realizations, i.e.

$$\acute{B} = \Sigma_B.$$

³Corresponds to $\Omega = I$, $\Upsilon = 0$, $C = 0$, $D = 0$, $\bar{B} = 0$, $\bar{C}_0 = 0$, $\bar{C}_h = I$, and $\bar{D} = 0$.

$\bar{A}$ as hSTPBC

The hSTPBC⁴

$$\Sigma_A : \begin{cases} \dot{x}(\tau) = Ax(\tau), & x(0) = \bar{u} \\ \bar{y} = x(h) \end{cases}$$

corresponds to the operator $\mathbb{R}^n \rightarrow \mathbb{R}^n$ (which is a matrix $\mathbb{R}^{n \times n}$)

$$\bar{u} \mapsto e^{Ah} \bar{u},$$

which is of the form of the “ $\bar{A}$ ” part of lifted state-space realizations, i.e.

$$\bar{A} = \Sigma_A.$$

⁴Corresponds to $\Omega = I$, $\Upsilon = 0$, $B = 0$, $C = 0$, $D = 0$, $\bar{B} = I$, $\bar{C}_0 = 0$, $\bar{C}_h = I$, $\bar{D} = 0$.

State-space realizations as hSTPBC

The hSTPBC's above have common “state” dynamics, so that

$$\begin{bmatrix} \check{D} & \check{C} \\ \check{B} & \bar{A} \end{bmatrix} : \begin{cases} \dot{x}_1(\tau) = Ax_1(\tau) + Bu_1(\tau), & x_1(0) = \bar{u}_1 \\ y_1(\tau) = Cx_1(\tau) + Du_1(\tau) \\ \bar{y}_1 = x_1(h) \end{cases}$$

and then

$$\begin{bmatrix} \check{D} & \check{C} \\ \check{B} & \bar{A} \end{bmatrix}^* : \begin{cases} \dot{x}_2(\tau) = -A'x_2(\tau) - C'u_2(\tau), & x_2(h) = \bar{u}_2 \\ y_2(\tau) = B'x_2(\tau) + D'u_2(\tau) \\ \bar{y}_2 = x_2(0) \end{cases}$$

Manipulations with these operators are a matter of plugging right signals in now...

Connection 1

Let

$$\check{O}_1 := \begin{bmatrix} \check{D} & \check{C} \\ \check{B} & \bar{A} \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} \check{D} & \check{C} \\ \check{B} & \bar{A} \end{bmatrix} = \begin{bmatrix} \check{D}^* \\ \check{C}^* \end{bmatrix} \begin{bmatrix} \check{D} & \check{C} \end{bmatrix} + \begin{bmatrix} \check{B}'^* \\ \bar{A}' \end{bmatrix} X \begin{bmatrix} \check{B} & \bar{A} \end{bmatrix}$$

Then, by aggregating x_2 and x_1 and considering $u_2 = y_1$ and $\bar{u}_2 = X\bar{y}_1$, we end up with the following hSTPBC representation of $\check{O}_1$:

$$\begin{cases} \dot{x}(\tau) = \begin{bmatrix} -A' & -C'A \end{bmatrix} x(\tau) + \begin{bmatrix} -C'D \\ B \end{bmatrix} u(\tau), & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} x(0) + \begin{bmatrix} I & -X \\ 0 & 0 \end{bmatrix} x(h) = \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{u} \\ y(\tau) = \begin{bmatrix} B' & D'C \end{bmatrix} x(\tau) + D'Du(\tau) \\ \bar{y} = \begin{bmatrix} I & 0 \end{bmatrix} x(0) \end{cases}$$

(here $u := u_1$, $\bar{u} := \bar{u}_1$, $y := y_2$, and $\bar{y} := \bar{y}_2$).

Remark: The addition of the finite-rank part shows up only in the shape of the boundary conditions.

Connection 2

Let

$$\check{O}_2 := \begin{bmatrix} \check{D} & \check{C} \\ \check{B} & \bar{A} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} \check{D} & \check{C} \\ \check{B} & \bar{A} \end{bmatrix}^* = \begin{bmatrix} \check{D} \\ \check{B} \end{bmatrix} [\check{D}^* \quad \check{B}^*] + \begin{bmatrix} \check{C} \\ \bar{A} \end{bmatrix} Y [\check{C}^* \quad \bar{A}']$$

Then, by aggregating x_1 and x_2 and considering $u_1 = y_2$ and $\bar{u}_1 = Y\bar{y}_2$, we end up with the following hSTPBC representation of $\check{O}_2$:

$$\begin{cases} \dot{x}(\tau) = \begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix} x(\tau) + \begin{bmatrix} BD' \\ -C' \end{bmatrix} u(\tau), & \begin{bmatrix} I & -Y \\ 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} x(h) = \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{u} \\ y(\tau) = \begin{bmatrix} C & DB' \end{bmatrix} x(\tau) + DD' u(\tau) \\ \bar{y} = \begin{bmatrix} I & 0 \end{bmatrix} x(0) \end{cases}$$

(here $u := u_2$, $\bar{u} := \bar{u}_2$, $y := y_1$, and $\bar{y} := \bar{y}_1$).

Remark: The addition of the finite-rank part shows up only in the shape of the boundary conditions, again.

Schur transformation

The transformation

$$\mathcal{S}_u \left(\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \right) = \begin{bmatrix} M_{11}^{-1} & -M_{11}^{-1}M_{12} \\ M_{21}M_{11}^{-1} & M_{22} - M_{21}M_{11}^{-1}M_{12} \end{bmatrix}$$

is called the **upper Schur transformation** of M (its $(2,2)$ entry is the Schur complement of M_{11}). A useful observation is that $\mathcal{S}_u(M)$ can be performed via manipulating inputs and outputs of M . Namely, let

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} u_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} M_{11}^{-1} & -M_{11}^{-1}M_{12} \\ M_{21}M_{11}^{-1} & M_{22} - M_{21}M_{11}^{-1}M_{12} \end{bmatrix} \begin{bmatrix} y_1 \\ u_2 \end{bmatrix},$$

i.e. $\mathcal{S}_u(M)$ is the result of mere swapping the first input and output of M .

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Hmm... (contd)

Now the parts that include infinite-dimensional operators do not collapse to finite-dimensional operators (matrices). For instance, to compute the gain

$$\dot{F}_u = -(\check{D}_{zu}^* \check{D}_{zu} + \acute{B}_u^* \bar{X} \acute{B}_u)^{-1}(\check{D}_{zu}^* \dot{C}_z + \acute{B}_u^* \bar{X} \bar{A})$$

and the DARE

$$\begin{aligned} \bar{X} = & \bar{A}' \bar{X} \bar{A} + \dot{C}_z^* \dot{C}_z \\ & - (\dot{C}_z^* \check{D}_{zu} + \bar{A}' \bar{X} \acute{B}_u)(\check{D}_{zu}^* \check{D}_{zu} + \acute{B}_u^* \bar{X} \acute{B}_u)^{-1}(\check{D}_{zu}^* \dot{C}_z + \acute{B}_u^* \bar{X} \bar{A}) \end{aligned}$$

we need to compute and manipulate the infinite-dimensional operator

$$\check{O}_u := \begin{bmatrix} \check{D}_{zu}^* \\ \dot{C}_z^* \end{bmatrix} \begin{bmatrix} \check{D}_{zu} & \dot{C}_z \end{bmatrix} + \begin{bmatrix} \acute{B}_u^* \\ \bar{A}' \end{bmatrix} \bar{X} \begin{bmatrix} \acute{B}_u & \bar{A} \end{bmatrix} : \mathbb{L}^m \times \mathbb{R}^n \rightarrow \mathbb{L}^m \times \mathbb{R}^n$$

and then interpret the hold $\dot{F}_u : \mathbb{R}^n \rightarrow \mathbb{L}^m$.

The $\bar{X}$ part

A key observation is that

$$\mathcal{S}_u(\check{O}_u) = \begin{bmatrix} \times & \dot{\bar{F}}_u \\ \times & \bar{X} \end{bmatrix}$$

We already saw that $\check{O}_u$ has the following hSTPBC realization:

$$\check{O}_u : \begin{cases} \dot{x}(\tau) = \begin{bmatrix} -A' & -C_z' C_z \\ 0 & A \end{bmatrix} x(\tau) + \begin{bmatrix} -C_z' D_{zu} \\ B \end{bmatrix} u(\tau) \\ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} x(0) + \begin{bmatrix} I & -\bar{X} \\ 0 & 0 \end{bmatrix} x(h) = \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{u} \\ y(\tau) = \begin{bmatrix} B_u' & D_{zu}' C_z \end{bmatrix} x(\tau) + D_{zu}' D_{zu} u(\tau) \\ \bar{y} = \begin{bmatrix} I & 0 \end{bmatrix} x(0) \end{cases}$$

To derive its Schur transform we only need to swap u and y (and after that we only need to account for the effect of the discrete input).

The $\bar{X}$ part (contd)

Thus, assuming $D'_{zu}D_{zu} = I$,

$$\begin{bmatrix} \dot{\bar{F}}_u \\ \bar{X} \end{bmatrix} : \begin{cases} \dot{x}(\tau) = H_u x(\tau), & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} x(0) + \begin{bmatrix} I & -\bar{X} \\ 0 & 0 \end{bmatrix} x(h) = \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{u} \\ u(\tau) = \begin{bmatrix} -B'_u & -D'_{zu}C_z \end{bmatrix} x(\tau) \\ \bar{y} = \begin{bmatrix} I & 0 \end{bmatrix} x(0) \end{cases}$$

where

$$H_u := \begin{bmatrix} -A' & -C'_z C_z \\ 0 & A \end{bmatrix} - \begin{bmatrix} -C'_z D_{zu} \\ B_u \end{bmatrix} \begin{bmatrix} B'_u & D'_{zu} C_z \end{bmatrix}.$$

Applying similarity transform with $T = \begin{bmatrix} I & \bar{X} \\ 0 & I \end{bmatrix}$ and $S = I$, we get:

$$\begin{bmatrix} \dot{\bar{F}}_u \\ \bar{X} \end{bmatrix} : \begin{cases} \dot{\tilde{x}}(\tau) = \tilde{H}_u \tilde{x}(\tau), & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \tilde{x}(0) + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tilde{x}(h) = \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{u} \\ u(\tau) = \begin{bmatrix} -B'_u & -B'_u \bar{X} - D'_{zu} C_z \end{bmatrix} \tilde{x}(\tau) \\ \bar{y} = \begin{bmatrix} I & \bar{X} \end{bmatrix} \tilde{x}(0) \end{cases}$$

where $\tilde{H}_u := \begin{bmatrix} I & -\bar{X} \\ 0 & I \end{bmatrix} H_u \begin{bmatrix} I & \bar{X} \\ 0 & I \end{bmatrix}$.

The $\bar{X}$ part (contd)

With some algebra,

$$\begin{aligned}\tilde{H}_u &= \begin{bmatrix} -A' & -\bar{X}A - A'\bar{X} - C_z'C_z \\ 0 & A \end{bmatrix} \\ &\quad - \begin{bmatrix} -\bar{X}B_u - C_z'D_{zu} \\ B_u \end{bmatrix} \begin{bmatrix} B_u' & B_u'\bar{X} + D_{zu}'C_z \end{bmatrix} \\ &= \begin{bmatrix} -(\cdot)' & -\bar{X}A - A'\bar{X} - C_z'C_z + (\bar{X}B_u + C_z'D_{zu})(B_u'\bar{X} + D_{zu}'C_z) \\ -B_uB_u' & A - B_u(B_u'\bar{X} + D_{zu}'C_z) \end{bmatrix}\end{aligned}$$

An educated guess is to take $\bar{X} = X$, where X is any solution to the CARE $XA + A'X + C_z'C_z - (XB_u + C_z'D_{zu})(B_u'X + D_{zu}'C_z) = 0$ (the admissibility of this choice is yet to be proved though). In this case

$$\tilde{H}_u = \begin{bmatrix} -A_F' & 0 \\ -B_uB_u' & A_F \end{bmatrix},$$

where $A_F := A + B_uF$ for the gain $F_u = -B_u'X - D_{zu}'C_z$.

The $\bar{X}$ part (contd)

Thus,

$$\begin{bmatrix} \dot{\bar{F}}_u \\ \bar{X} \end{bmatrix} : \begin{cases} \dot{\tilde{x}}(\tau) = \begin{bmatrix} -A'_F & 0 \\ -B'_u B'_u & A_F \end{bmatrix} \tilde{x}(\tau), & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \tilde{x}(0) + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tilde{x}(h) = \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{u} \\ u(\tau) = \begin{bmatrix} -B'_u & F_u \end{bmatrix} \tilde{x}(\tau) \\ \bar{y} = \begin{bmatrix} I & X \end{bmatrix} \tilde{x}(0) \end{cases}$$

The first component of the state in this case $\tilde{x}_1(\tau) \equiv 0$, which can be seen from its isolation from both $\bar{u}$ and $\tilde{x}_2(\tau)$ and from $\tilde{x}_1(h) = 0$. Hence,

$$\begin{bmatrix} \dot{\bar{F}}_u \\ \bar{X} \end{bmatrix} : \begin{cases} \dot{\tilde{x}}_2(\tau) = A_F \tilde{x}_2(\tau), & \tilde{x}_2(0) = \bar{u} \\ u(\tau) = F_u \tilde{x}_2(\tau) \\ \bar{y} = X \tilde{x}_2(0) \end{cases}$$

for which

$$\bar{y} = X \bar{u}.$$

Thus, $\bar{X} = X$, which agrees with the guess of the previous slide and proves that a solution of the CARE is a solution of the lifted DARE as well.

The $\bar{X}$ part (contd)

The analog output of the hSTPBC (again by Slide 15) is

$$u(\tau) = F_u e^{A_F \tau} \bar{u},$$

i.e. $\dot{F}_u : \bar{u} \mapsto F_u e^{A_F \tau} \bar{u}$. Now, we need conditions for $\bar{A} + \dot{B}_u \dot{F}_u$ being Schur. To this end,

$$\begin{aligned} \bar{A} + \dot{B}_u \dot{F}_u &= e^{Ah} + \int_0^h e^{A(h-\sigma)} B_u F_u e^{A_F \sigma} d\sigma \\ &= e^{Ah} + \begin{bmatrix} I & 0 \end{bmatrix} \exp \left(\begin{bmatrix} A & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} h \right) \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= e^{Ah} + \begin{bmatrix} I & I \end{bmatrix} \exp \left(\begin{bmatrix} A & 0 \\ 0 & A + B_u F_u \end{bmatrix} h \right) \begin{bmatrix} -I \\ I \end{bmatrix} \\ &= e^{Ah} - e^{Ah} + e^{(A+B_u F_u)h} = e^{(A+B_u F_u)h}, \end{aligned}$$

so $\bar{A} + \dot{B}_u \dot{F}_u$ is Schur iff $A + B_u F_u$ is Hurwitz. Hence,

- $\bar{X}$ is stabilizing iff X is stabilizing.

The $\bar{X}$ part (contd)

Thus, if the CARE

$$XA + A'X + C_z' C_z - (XB_u + C_z' D_{zu})(B_u' X + D_{zu}' C_z) = 0$$

has the stabilizing solution X , we have that

- $\bar{X} = X$
- $\dot{\bar{F}}_u$ is the lifting of the static hold with $\phi_u(t) = F_u e^{(A+B_u F_u)t} \mathbb{1}_{[0,h]}(t)$

where $F_u := -XB_u' - D_{zu}' C_z$ is the optimal LQR gain for the corresponding continuous-time problem. It can be shown that

- $(\bar{A}, \dot{\bar{B}}_u)$ is stabilizable (wrt $\mathbb{D}_1$) iff (A, B_u) is stabilizable (wrt $\mathbb{C} \setminus \bar{\mathbb{C}}_0$)
- $\begin{bmatrix} \bar{A}-zI & \dot{\bar{B}}_u \\ \dot{\bar{C}}_z & \dot{\bar{D}}_{zu} \end{bmatrix}$ left invertible $\forall z \in \mathbb{T}$ iff $\begin{bmatrix} A-sI & B_u \\ C_z & D_{zu} \end{bmatrix}$ left invertible $\forall s \in j\mathbb{R}$

which implies that the solvability conditions in the lifted domain reduce to those for the continuous-time problem.

The $\bar{Y}$ part

Similarly, it can be proved that

- $(\dot{C}_y, \bar{A})$ is detectable (wrt $\mathbb{D}_1$) iff (C_y, A) is detectable (wrt $\mathbb{C} \setminus \bar{\mathbb{C}}_0$)
- $\begin{bmatrix} \bar{A}-zI & \dot{B}_w \\ \dot{C}_y & \dot{D}_{yw} \end{bmatrix}$ right invertible $\forall z \in \mathbb{T}$ iff $\begin{bmatrix} A-sI & B_w \\ C_y & D_{yw} \end{bmatrix}$ right invertible $\forall s \in \mathbb{j}\mathbb{R}$

Then the CARE

$$AY + YA' + B_w B_w' - (YC_y' + B_w D_{yw}') (C_y Y + D_{yw} B_w') = 0$$

has the stabilizing solution Y and

- $\bar{Y} = Y$
- $z^{-1} \dot{L}_y$ is the lifting of the sampler with $\psi_y(t) = e^{(A+L_y C_y)t} L_y \mathbb{1}_{[0,h]}(t)$

where $L_y := -YC_y' - B_w D_{yw}'$ is the corresponding continuous-time Kalman filter gain.

Optimal controller

The optimal controller

$$\check{K}_{\text{opt}}(z) = -\check{F}_u \left[\frac{\bar{A}_K}{I} \middle| \frac{I}{0} \right] \check{L}_y,$$

then corresponds to the following dynamics:

$$\mathcal{K}_{\text{opt}} : \begin{cases} \bar{x}_K[i+1] = \bar{A}_K \bar{x}_K[i] - \int_0^h e^{(A+L_y C_y)(h-\sigma)} L_y y(ih + \sigma) d\sigma \\ u(ih + \tau) = F_u e^{(A+B_u F_u)\tau} \bar{x}_K[i] \end{cases}$$

and it's only left to peel-off the “A” matrix of its discrete part,

$$\bar{A}_K := \bar{A} + \check{B}_u \check{F}_u + \check{L}_y \check{C}_y + \check{L}_y \check{D}_{yu} \check{F}_u = \begin{bmatrix} \check{L}_y & I \end{bmatrix} \begin{bmatrix} \check{D}_{yu} & \check{C}_y \\ \check{B}_u & \bar{A} \end{bmatrix} \begin{bmatrix} \check{F}_u \\ I \end{bmatrix}.$$

The $\bar{A}_K$ part

The hSTPBC's for all three components are

$$\begin{bmatrix} \dot{\bar{F}}_u \\ I \end{bmatrix} : \begin{cases} \dot{x}_1(\tau) = (A + B_u F_u)x_1(\tau), & x_1(0) = \bar{u} \\ y_1(\tau) = F_u \tilde{x}_2(\tau) \\ \bar{y}_1 = \bar{u} \end{cases}$$

$$\begin{bmatrix} \check{D}_{yu} & \check{C}_y \\ \bar{B}_u & \bar{A} \end{bmatrix} : \begin{cases} \dot{x}_2(\tau) = A x_2(\tau) + B_u u_2(\tau), & x_2(0) = \bar{u}_2 \\ y_2(\tau) = C_y x_2(\tau) \\ \bar{y}_2 = x_2(h) \end{cases}$$

$$\begin{bmatrix} \check{L}_y & I \end{bmatrix} : \begin{cases} \dot{x}_3(\tau) = (A + L_y C_y)x_3(\tau) + L_y u_3(\tau), & x_3(0) = 0 \\ \bar{y} = x_3(h) + \bar{u}_3 \end{cases}$$

and we need to plug $u_2 = y_1$, $\bar{u}_2 = \bar{y}_1$, $u_3 = y_2$, $\bar{u}_3 = \bar{y}_2$, and aggregate the “states” x_1 , x_2 , x_3 .

The $\bar{A}_K$ part (contd)

Hence, applying similarity with $T = \begin{bmatrix} I & -I & -I \\ 0 & I & I \end{bmatrix}$ and $S = T^{-1} = \begin{bmatrix} I & I & 0 \\ 0 & I & I \end{bmatrix}$,

$$\bar{A}_K : \begin{cases} \dot{x}(\tau) = \begin{bmatrix} A + L_y C_y & L_y C_y & 0 \\ 0 & A & B_u F_u \\ 0 & 0 & A + B_u F_u \end{bmatrix} x(\tau), & x(0) = \begin{bmatrix} 0 \\ I \\ I \end{bmatrix} \bar{u} \\ \bar{y} = \begin{bmatrix} I & I & 0 \end{bmatrix} x(h) \end{cases}$$

$$: \begin{cases} \dot{\tilde{x}}(\tau) = \begin{bmatrix} A + L_y C_y & 0 & B_u F_u \\ 0 & A & 0 \\ 0 & 0 & A + B_u F_u \end{bmatrix} \tilde{x}(\tau), & \tilde{x}(0) = \begin{bmatrix} I \\ 0 \\ I \end{bmatrix} \bar{u} \\ \bar{y} = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \tilde{x}(h) \end{cases}$$

$$: \begin{cases} \dot{\hat{x}}(\tau) = \begin{bmatrix} A + L_y C_y & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} \hat{x}(\tau), & \hat{x}(0) = \begin{bmatrix} I \\ I \end{bmatrix} \bar{u} \\ \bar{y} = \begin{bmatrix} I & 0 \end{bmatrix} \hat{x}(h) \end{cases}$$

$$= \begin{bmatrix} I & 0 \end{bmatrix} \exp \left(\begin{bmatrix} A + L_y C_y & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} h \right) \begin{bmatrix} I \\ I \end{bmatrix}.$$

Optimal sampler and hold

The optimal hold is static, with the hold function

$$\phi_u(t) = F_u e^{(A+B_u F_u)t} \mathbb{1}_{[0,h]}(t),$$

which effectively imitates the optimal analog LQR control signal during the intersample. It is optimal for the Type III problem as well.

The optimal sampler is also static, with the sampling function

$$\psi_y(t) = -e^{(A+L_y C_y)t} L_y \mathbb{1}_{[0,h]}(t),$$

which imitates the processing of the measurement y by the analog Kalman filter. It is optimal for the Type II problem as well.

Content

Type IV standard problem

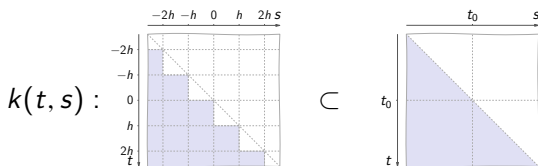
Background: hSTPBC

Direct approach via hSTPBC

Extraction from analog parametrization

The idea

Because



we may endeavor to

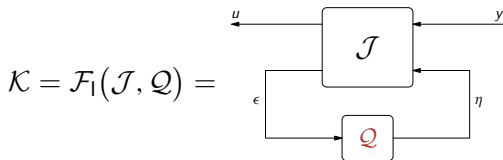
- extract the sampled-data controller of interest from a parametrization of admissible analog controllers.

Imposing the “staircase” constraint on $k(t, s)$ might be hard as most design procedures do not explicitly operate with controller kernels in time domain. The constraint is more tangible in the lifted domain, where the task is to

- find the least harming *strictly causal* controller among lifted parametrizations.

All suboptimal analog controllers

We already saw (Lecture 9, Slide 11) that all γ -suboptimal solutions to the standard H_2 problem are



where $\mathcal{J}$ is an LTI system with the transfer function

$$J(s) = \left[\begin{array}{c|cc} A + B_u F_u + L_y C_y & -L_y & B_u \\ \hline F_u & 0 & I \\ -C_y & I & 0 \end{array} \right]$$

and $\mathcal{Q}$ is any H_2 system such that

$$\|\mathcal{Q}\|_2^2 \leq \gamma^2 - \gamma_{\text{opt}}^2 = \gamma^2 - \text{tr}(B_w B_w' X + Y C_z' C_z + X A Y + Y A' X).$$

All suboptimal lifted controllers

Equivalently, all γ -suboptimal lifted solutions are

$$\check{\mathcal{K}} = \mathcal{F}_l(\check{\mathcal{J}}, \check{\mathcal{Q}}) =$$

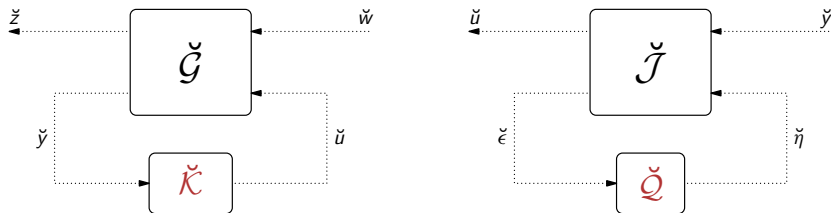
The diagram shows a feedback control system. The input $\check{u}$ enters a block labeled $\check{\mathcal{J}}$. The output of $\check{\mathcal{J}}$ is $\check{y}$. A feedback path from $\check{y}$ goes through a block labeled $\check{\mathcal{Q}}$ (in red) to produce $\check{\eta}$. This signal $\check{\eta}$ is then summed with $\check{u}$ to produce $\check{\epsilon}$, which is fed back into the input of $\check{\mathcal{J}}$.

where $\check{\mathcal{J}} = \mathcal{L}_h \mathcal{J} \mathcal{L}_h^{-1}$ and $\check{\mathcal{Q}}$ is any H_2 system such that

$$\|\check{\mathcal{Q}}\|_2^2 \leq \gamma^2 - \gamma_{\text{opt}}^2$$

and $\check{\mathcal{K}}$ corresponds to a causal $\mathcal{K}$.

Reformulation as extraction problem



Thus,

minimizing $\|\mathcal{F}_l(\check{\mathcal{G}}, \check{\mathcal{K}})\|_2$ by stabilizing strictly causal $\check{\mathcal{K}}$

$\Updownarrow$

minimizing $\|\check{\mathcal{Q}}\|_2$ subject to strict causality of $\check{\mathcal{K}} = \mathcal{F}_l(\check{\mathcal{J}}, \check{\mathcal{Q}})$

Feedthrough part of $\check{\mathcal{K}}$

In terms of system matrices,

$$\llbracket \check{\mathcal{K}} \rrbracket = \mathcal{F}_I(\llbracket \check{\mathcal{J}} \rrbracket, \llbracket \check{\mathcal{Q}} \rrbracket) = \llbracket \check{\mathcal{J}}_{11} \rrbracket + \llbracket \check{\mathcal{J}}_{12} \rrbracket \llbracket \check{\mathcal{Q}} \rrbracket (I - \llbracket \check{\mathcal{J}}_{22} \rrbracket \llbracket \check{\mathcal{Q}} \rrbracket)^{-1} \llbracket \check{\mathcal{J}}_{21} \rrbracket.$$

All components there, including the well-posed $(I - \llbracket \check{\mathcal{J}}_{22} \rrbracket \llbracket \check{\mathcal{Q}} \rrbracket)^{-1}$, are lower triangular. This implies that

- the diagonal elements of $\llbracket \check{\mathcal{K}} \rrbracket$ depend only on the diagonal elements of each $\llbracket \check{\mathcal{J}}_{ij} \rrbracket$ and of $\llbracket \check{\mathcal{Q}} \rrbracket$.

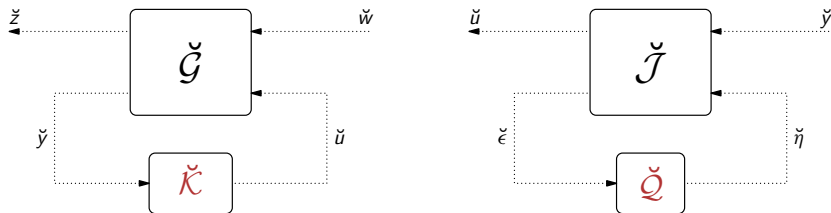
In other words,

$$\check{\mathcal{K}}_\infty = \mathcal{F}_I(\check{\mathcal{J}}_\infty, \check{\mathcal{Q}}_\infty),$$

where $\check{\mathcal{G}}_\infty$ stands for the feedthrough part of $\check{\mathcal{G}}$, and

- $\check{\mathcal{K}}$ is strictly causal $\iff \mathcal{F}_I(\check{\mathcal{J}}_\infty, \check{\mathcal{Q}}_\infty) = 0$.

Reformulation as extraction problem (contd)



Thus,

minimizing $\|\mathcal{F}_1(\check{\mathcal{G}}, \check{\mathcal{K}})\|_2$ by stabilizing strictly causal $\check{\mathcal{K}}$

$\Leftrightarrow$

minimizing $\|\check{\mathcal{Q}}\|_2$ subject to strict causality of $\check{\mathcal{K}} = \mathcal{F}_1(\check{\mathcal{J}}, \check{\mathcal{Q}})$

$\Leftrightarrow$

minimizing $\|\check{\mathcal{Q}}\|_2$ subject to $\mathcal{F}_1(\check{\mathcal{J}}_\infty, \check{\mathcal{Q}}_\infty) = 0$

Two observations

1. By Parseval (and the causality of $\check{Q}$),

$$\|\check{Q}\|_2^2 = \sum_{i \in \mathbb{Z}^+} \|\check{Q}[i]\|_{\text{HS}}^2 = \|\check{Q}_\infty\|_2^2 + \sum_{i \in \mathbb{N}} \|\check{Q}[i]\|_{\text{HS}}^2.$$

Hence, minimizing $\|\check{Q}\|_2$ subject to constraints only on its feedthrough part $\check{Q}_\infty$ always yields static $\check{Q}_{\text{opt}} = \check{Q}_\infty$.

2. If $\mathcal{F}_1(\check{J}_\infty, \check{Q}_\infty)$ is well posed, the mapping $\mathcal{F}_1(\check{J}_\infty, \check{Q}_\infty) : \check{Q}_\infty \mapsto \check{K}_\infty$ is invertible, with

$$\check{K}_\infty = \mathcal{F}_1(\check{J}_\infty, \check{Q}_\infty) \iff \check{Q}_\infty = \mathcal{F}_1(\check{J}_\infty^{-1}, \check{K}_\infty)$$

(as the (1,2) and (2,1) sub-blocks of $\check{J}_\infty$ are square and invertible).

Two observations

1. By Parseval (and the causality of $\check{Q}$),

$$\|\check{Q}\|_2^2 = \sum_{i \in \mathbb{Z}^+} \|\check{Q}[i]\|_{\text{HS}}^2 = \|\check{Q}_\infty\|_2^2 + \sum_{i \in \mathbb{N}} \|\check{Q}[i]\|_{\text{HS}}^2.$$

Hence, minimizing $\|\check{Q}\|_2$ subject to constraints only on its feedthrough part $\check{Q}_\infty$ always yields static $\check{Q}_{\text{opt}} = \check{Q}_\infty$.

2. If $\mathcal{F}_l(\check{J}_\infty, \check{Q}_\infty)$ is well posed, the mapping $\mathcal{F}_l(\check{J}_\infty, \check{Q}_\infty) : \check{Q}_\infty \mapsto \check{K}_\infty$ is invertible, with

$$\check{K}_\infty = \mathcal{F}_l(\check{J}_\infty, \check{Q}_\infty) \iff \check{Q}_\infty = \mathcal{F}_u(\check{J}_\infty^{-1}, \check{K}_\infty)$$

(as the $(1,2)$ and $(2,1)$ sub-blocks of $\check{J}_\infty$ are square and invertible).

The optimal $\mathcal{Q}$

The unique $\check{\mathcal{Q}}$ such that $\mathcal{F}_l(\check{\mathcal{J}}_\infty, \check{\mathcal{Q}}_\infty) = 0$ is the static

$$\check{\mathcal{Q}} = \mathcal{F}_u(\check{\mathcal{J}}_\infty^{-1}, 0) = \begin{bmatrix} 0 & I \end{bmatrix} \check{\mathcal{J}}_\infty^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Since $\check{\mathcal{J}}^{-1} = (\mathcal{L}_h \mathcal{J} \mathcal{L}_h^{-1})^{-1} = \mathcal{L}_h \mathcal{J}^{-1} \mathcal{L}_h^{-1}$ and

$$[J^{-1}(s)]_{22} = \begin{bmatrix} 0 & I \end{bmatrix} \left[\begin{array}{c|cc} A & -B_u & L_y \\ \hline -C_y & 0 & I \\ F_u & I & 0 \end{array} \right] \begin{bmatrix} 0 \\ I \end{bmatrix} = \left[\begin{array}{c|c} A & L_y \\ \hline F_u & 0 \end{array} \right],$$

the optimal $\check{\mathcal{Q}}_{\text{opt}}$ in the continuous-time domain acts as (Lecture 9, Slide 25)

$$\mathcal{Q}_{\text{opt}} : \begin{cases} \dot{x}_Q(t) = Ax_Q(t) + L_y \epsilon(t), & x_Q(ih) = 0 \\ \eta(t) = F_u x_Q(t). \end{cases}$$

The optimal performance

$\check{Q}_{\text{opt}}$ is static and its transfer function $\check{Q}_{\text{opt}}(z) = \check{D}_Q$ is an integral operator $\mathbb{L} \rightarrow \mathbb{L}$ with the kernel $d_Q(t, s) = F_u e^{A(t-s)} L_y \mathbb{1}(t-s)$. Then

$$\|Q_{\text{opt}}\|_2^2 = \frac{1}{h} \|\check{D}_Q\|_{\text{HS}}^2 = \frac{1}{h} \int_0^h \int_0^t \text{tr}(F_u e^{A(t-s)} L_y L_y' e^{A'(t-s)} F_u') ds dt$$

and we know how to compute it (Lecture 9, Slide 40).

The optimal $\mathcal{K}$

The mapping $\mathcal{K}_{\text{opt}} = \mathcal{F}_l(\mathcal{J}, \mathcal{Q}_{\text{opt}})$, with

$$\mathcal{J} : \begin{cases} \dot{x}_J(t) = (A + B_u F_u + L_y C_y) x_J(t) - L_y y(t) + B_u \eta(t) \\ u(t) = F_u x_J(t) + \eta(t) \\ \epsilon(t) = -C_y x_J(t) + y(t) \end{cases}$$

reads then as

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} A + B_u F_u + L_y C_y & B_u F_u \\ -L_y C_y & A \end{bmatrix} x(t) + \begin{bmatrix} -L_y \\ L_y \end{bmatrix} y(t), & \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} x(ih) = 0 \\ u(t) = \begin{bmatrix} F_u & F_u \end{bmatrix} x(t) \end{cases}$$

or, with $\tilde{x} := \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} x$, as

$$\begin{cases} \dot{\tilde{x}}(t) = \begin{bmatrix} A + L_y C_y & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} \tilde{x}(t) - \begin{bmatrix} L_y \\ 0 \end{bmatrix} y(t), & \begin{bmatrix} 0 & 0 \\ -I & I \end{bmatrix} \tilde{x}(ih) = 0 \\ u(t) = \begin{bmatrix} 0 & F_u \end{bmatrix} \tilde{x}(t) \end{cases}$$

The optimal $\mathcal{K}$ (contd)

Thus, $\mathcal{K}_{\text{opt}}$ acts as

$$\begin{cases} \dot{\tilde{x}}(t) = \begin{bmatrix} A + L_y C_y & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} \tilde{x}(t) - \begin{bmatrix} L_y \\ 0 \end{bmatrix} y(t), & \tilde{x}_2(ih) = \tilde{x}_1(ih) \\ u(t) = \begin{bmatrix} 0 & F_u \end{bmatrix} \tilde{x}(t) \end{cases}$$

Let

$$\begin{bmatrix} \Gamma_{11}(\tau) & \Gamma_{12}(\tau) \\ 0 & \Gamma_{22}(\tau) \end{bmatrix} := \exp \left(\begin{bmatrix} A + L_y C_y & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} \tau \right)$$

then

$$\begin{aligned} \tilde{x}(ih + \tau) &= \begin{bmatrix} \Gamma_{11}(\tau) & \Gamma_{12}(\tau) \\ 0 & \Gamma_{22}(\tau) \end{bmatrix} \tilde{x}(ih) - \begin{bmatrix} I \\ 0 \end{bmatrix} \int_0^\tau \Gamma_{11}(\tau - \sigma) L_y y(ih + \sigma) d\sigma \\ &= \begin{bmatrix} \Gamma_{11}(\tau) + \Gamma_{12}(\tau) \\ \Gamma_{22}(\tau) \end{bmatrix} \tilde{x}_1(ih) - \begin{bmatrix} I \\ 0 \end{bmatrix} \int_0^\tau \Gamma_{11}(\tau - \sigma) L_y y(ih + \sigma) d\sigma \end{aligned}$$

and $u(ih + \tau) = F_u \Gamma_{22}(\tau) \tilde{x}_1(ih)$ depends only on $\tilde{x}_1(ih)$.

The optimal $\mathcal{K}$ (contd)

Hence, denoting $\bar{x}_K[i] := \tilde{x}_1(ih)$, we end up with

$$\mathcal{K}_{\text{opt}} : \begin{cases} \bar{x}_K[i+1] = \bar{A}_K \bar{x}_K[i] - \int_0^h e^{(A+L_y C_y)(h-\sigma)} L_y y(ih + \sigma) d\sigma \\ u(ih + \tau) = F_u e^{(A+B_u F_u)\tau} \bar{x}_K[i] \end{cases}$$

where

$$\bar{A}_K = \begin{bmatrix} I & 0 \end{bmatrix} \exp \left(\begin{bmatrix} A + L_y C_y & B_u F_u \\ 0 & A + B_u F_u \end{bmatrix} h \right) \begin{bmatrix} I \\ I \end{bmatrix}.$$

This is exactly what we got with the direct approach.