

Advanced Sampled-Data Control (038798)

lecture no. 9

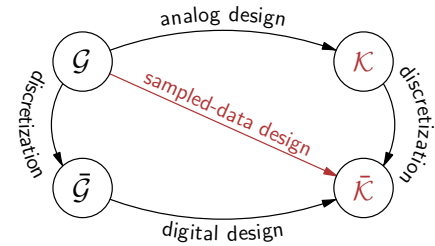
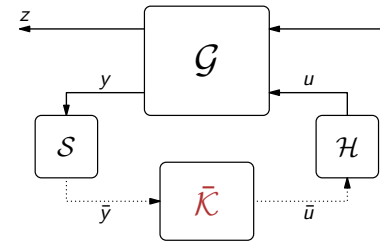
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The third approach



Remember two classical approaches:

1. digital redesign of analog controllers
2. discrete-time design

clockwise from $\mathcal{G}$ to $\bar{\mathcal{K}}$

counterclockwise from $\mathcal{G}$ to $\bar{\mathcal{K}}$

We are in the position to explore another pathway:

3. sampled-data design (design $\bar{\mathcal{K}}$ directly for analog performance)
 - ☺ solves the original, i.e. non-approximated, problem
 - ☺ controller designed is controller implemented

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Content

Continuous-time output feedback

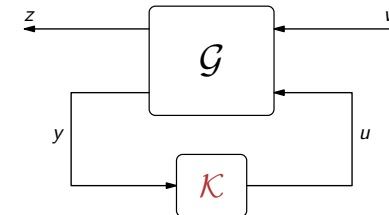
Discrete-time output feedback

State-space realizations in the lifted domain

Type I standard problem

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The standard H_2 problem



Given

$$G(s) = \begin{bmatrix} G_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & G_{yu}(s) \end{bmatrix},$$

design causal internally stabilizing K that minimizes $\|T_{zw}\|_2$, where

$$\begin{aligned} T_{zw}(s) &= \mathcal{F}_l(G(s), K(s)) \\ &= G_{zw}(s) + G_{zu}(s)K(s)(I - G_{yu}(s)K(s))^{-1}G_{yw}(s), \end{aligned}$$

which is a *nonlinear* function of K .

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Assumptions

We consider finite-dimensional generalized plants given by their state-space realizations of the form

$$G(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array} \right],$$

assuming that

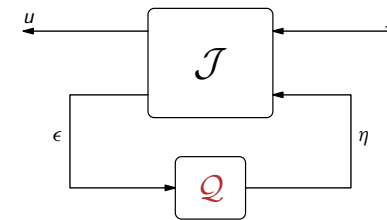
- (A, B_u) is stabilizable
- (C_y, A) is detectable
- $\begin{bmatrix} A - sI & B_u \\ C_z & D_{zu} \end{bmatrix}$ has full column rank for all $s \in j\mathbb{R}$ and $D'_{zu}D_{zu} = I$
- $\begin{bmatrix} A - sI & B_w \\ C_y & D_{yw} \end{bmatrix}$ has full row rank for all $s \in j\mathbb{R}$ and $D_{yw}D'_{yw} = I$

Remark: Normalization assumptions on the “ D ” parts are made to simplify the formulae. They can be easily relaxed to $D'_{zu}D_{zu} > 0$ and $D_{yw}D'_{yw} > 0$.

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Youla-Kučera parametrization

All internally stabilizing controllers can be parametrized as



i.e. $K(s) = \mathcal{F}_l(J(s), Q(s))$, where

$$J(s) = \left[\begin{array}{c|cc} A + B_u F_u + L_y C_y & -L_y & B_u \\ \hline F_u & 0 & I \\ -C_y & I & 0 \end{array} \right]$$

for any F_u and L_y such that $A + B_u F_u$ and $A + L_y C_y$ are Hurwitz, and Q is an arbitrary stable system.

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All stable closed-loop systems

Then all stable closed-loop mappings $w \mapsto z$ are

$$T_{zw}(s) = N_{zw}(s) - N_u(s)Q(s)N_y(s)$$

(affine function of Q), where

$$\left[\begin{array}{c|c} N_{zw}(s) & N_u(s) \\ \hline N_y(s) & 0 \end{array} \right] = \left[\begin{array}{cc|cc} A + B_u F_u & -L_y C_y & -L_y D_{yw} & B_u \\ 0 & A + L_y C_y & B_w + L_y D_{yw} & 0 \\ \hline C_z + D_{zu} F_u & C_z & 0 & D_{zu} \\ 0 & C_y & D_{yw} & 0 \end{array} \right].$$

Thus, the standard H_2 problem reduces to a two-sided H_2 model matching problem, whose solution verifies

$$\langle N_{zw} - N_u Q_{\text{opt}} N_y, N_u Q N_y \rangle_2 = \langle N_u^\sim N_{zw} N_y^\sim - N_u^\sim N_u Q_{\text{opt}} N_y N_y^\sim, Q \rangle_2 = 0,$$

for all $Q \in H_2$.

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Choice of F_u and L_y

The solution is greatly simplified if we

- choose F_u so that $N_u(s)$ is inner and L_y so that $N_y(s)$ is co-inner

Because

$$N_u(s) = \left[\begin{array}{c|c} A + B_u F_u & B_u \\ \hline C_z + D_{zu} F_u & D_{zu} \end{array} \right] \quad \text{and} \quad N_y(s) = \left[\begin{array}{c|c} A + L_y C_y & B_w + L_y D_{yw} \\ \hline C_y & D_{yw} \end{array} \right]$$

the formulae from Lecture 8 (Slides 19 and 20) apply if we pick

$$F_u = -(B'_u X + D'_{zu} C_z) \quad \text{and} \quad L_y = -(Y C'_y + B_w D'_{yw}),$$

where $X \geq 0$ and $Y \geq 0$ are the stabilizing solutions to the CARE's

$$\begin{aligned} A'X + XA + C'_z C_z - (XB_u + C'_z D_{zu})(B'_u X + D'_{zu} C_z) &= 0, \\ AY + YA' + B_w B'_w - (Y C'_y + B_w D'_{yw})(C_y Y + D_{yw} B'_w) &= 0 \end{aligned}$$

(exist under our assumptions). In this case, $Q_{\text{opt}} = \text{proj}_{H_2} N_u^\sim N_{zw} N_y^\sim$.

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Calculating $N_u^\sim N_{zw} N_y^\sim$ et cetera

Introduce matrices $A_F := A + B_u F_u$, $C_F := C_z + D_{zu} F_u$, $A_L := A + L_y C_y$, and $B_L := B_w + L_y D_{yw}$. Then

$$\begin{aligned} N_{zw}(s) &= \left[\begin{array}{cc|c} A_F & -L_y C_y & -L_y D_{yw} \\ 0 & A_L & B_L \\ \hline C_F & C_z & 0 \end{array} \right] \\ &= \left[\begin{array}{c|c} A_L & B_L \\ \hline C_z & 0 \end{array} \right] - \left[\begin{array}{c|c} A_F & L_y \\ \hline C_F & 0 \end{array} \right] \left[\begin{array}{c|c} A_L & B_L \\ \hline C_y & D_{yw} \end{array} \right] =: N_v(s) - N_F(s) N_y(s) \end{aligned}$$

We already saw the systems N_v and N_y in the estimation problem (modulo notation differences) and we know that $N_v N_y^\sim \in H_2^\perp$. Then, by the formula of Slide 10 of Lecture 8,

$$N_u^\sim(s) N_F(s) = \left[\begin{array}{c|c} A_F & L_y \\ \hline 0 & 0 \end{array} \right] + \left[\begin{array}{c|c} A_F & B_u \\ \hline L_y' X & 0 \end{array} \right]^\sim = \left[\begin{array}{c|c} A_F & B_u \\ \hline L_y' X & 0 \end{array} \right]^\sim \in H_2^\perp,$$

so that $N_u^\sim N_{zw} N_y^\sim = N_u^\sim (N_v N_y^\sim) - N_u^\sim N_F \in H_2^\perp$ too and $Q_{\text{opt}} = 0$.

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Calculating $N_u^\sim N_{zw} N_y^\sim$ et cetera (contd)

Therefore,

$$K_{\text{opt}}(s) = \mathcal{F}_1(J(s), 0) = - \left[\begin{array}{c|c} A + B_u F_u + L_y C_y & L_y \\ \hline F_u & 0 \end{array} \right],$$

which is the classical optimal LQG controller (observer-based). The optimal performance $\|T_{zw}\|_2^2 = \|N_{zw}\|_2^2 = \|N_v - N_F N_y\|_2^2$. From the solution to the estimation problem, $N_v \perp Q N_y$ for all $Q \in H_2$. Hence,

$$\begin{aligned} \|T_{zw}\|_2^2 &= \|N_v\|_2^2 + \|N_F N_y\|_2^2 = \|N_v\|_2^2 + \|N_F\|_2^2 \\ &= \text{tr}(C_z Y C_z') + \text{tr}(L_y' X L_y) = \text{tr}(C_z Y C_z') + \text{tr}(X L_y L_y') \\ &= \text{tr}(C_z Y C_z') + \text{tr}(X(A Y + Y A' + B_w B_w')) \\ &= \text{tr}(B_w' X B_w) + \text{tr}(C_z Y C_z') + \text{tr}(X A Y + Y A' X) \end{aligned}$$

which includes the LQR and Kalman filter costs $\text{tr}(B_w' X B_w)$ and $\text{tr}(C_z Y C_z')$, respectively, in it.

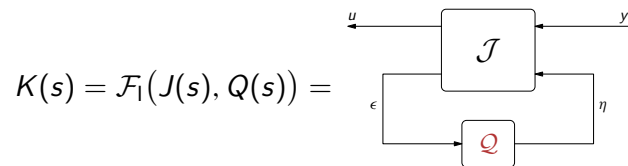
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Suboptimal solutions

It follows from $\langle N_{zw} - N_u Q_{\text{opt}} N_y, N_u Q N_y \rangle_2 = \langle N_{zw}, N_u Q N_y \rangle_2 = 0$ that

$$\|N_{zw} - N_u Q N_y\|_2^2 = \|N_{zw}\|_2^2 + \|N_u Q N_y\|_2^2 = \|N_{zw}\|_2^2 + \|Q\|_2^2$$

(by Pythagoras and because N_u is inner and N_y is co-inner). Thus, given a $\gamma > \|N_{zw}\|_2$, all γ -suboptimal controllers are



$$K(s) = \mathcal{F}_1(J(s), Q(s)) =$$

for any Q such that $\|Q\|_2^2 \leq \gamma^2 - \gamma_{\text{opt}}^2$.

Remark: This parametrization yields also all γ -suboptimal *time-varying* K 's when Q is an arbitrary time-varying system such that $\|Q\|_2^2 \leq \gamma^2 - \gamma_{\text{opt}}^2$.

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Content

Continuous-time output feedback

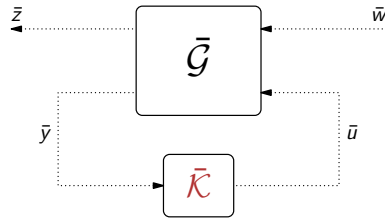
Discrete-time output feedback

State-space realizations in the lifted domain

Type I standard problem

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The standard H_2 problem with strictly causal controller



Given

$$\bar{G}(z) = \begin{bmatrix} \bar{G}_{zw}(z) & \bar{G}_{zu}(z) \\ \bar{G}_{yw}(z) & \bar{G}_{yu}(z) \end{bmatrix},$$

design **strictly causal** internally stabilizing $\bar{K}$ that minimizes $\|\bar{T}_{zw}\|_2$, where

$$\begin{aligned} \bar{T}_{zw}(z) &= \mathcal{F}_l(\bar{G}(z), \bar{K}(z)) \\ &= \bar{G}_{zw}(z) + \bar{G}_{zu}(z)\bar{K}(z)(I - \bar{G}_{yu}(z)\bar{K}(z))^{-1}\bar{G}_{yw}(z). \end{aligned}$$

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Assumptions

We consider finite-dimensional generalized plants given by their state-space realizations of the form

$$\bar{G}(z) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array} \right],$$

assuming that

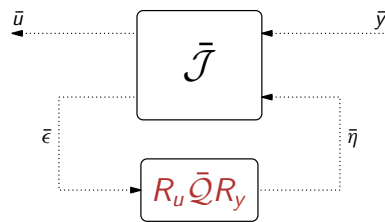
- (A, B_u) is stabilizable
- (C_y, A) is detectable
- $\begin{bmatrix} A - zI & B_u \\ C_z & D_{zu} \end{bmatrix}$ has full column rank for all $z \in \mathbb{T}$
- $\begin{bmatrix} A - zI & B_w \\ C_y & D_{yw} \end{bmatrix}$ has full row rank for all $z \in \mathbb{T}$

Remark: Neither D_{zw} nor D_{yu} is assumed to be zero. This is done to render the solution applicable to lifted systems later on.

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Youla-Kučera parametrization

All internally stabilizing strictly causal controllers can be parametrized as



i.e. $\bar{K}(z) = \mathcal{F}_l(\bar{J}(z), R_u \bar{Q}(z) R_y)$, where $\bar{Q}$ is an arbitrary strictly causal and stable system and

$$\bar{J}(z) = \left[\begin{array}{c|cc} A + B_u F_u + L_y C_y + L_y D_{yu} F_u & -L_y & B_u + L_y D_{yu} \\ \hline F_u & 0 & I \\ \hline -(C_y + D_{yu} F_u) & I & -D_{yu} \end{array} \right]$$

for any F_u and L_y such that $A + B_u F_u$ and $A + L_y C_y$ are Hurwitz and any square nonsingular R_u and R_y .

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All stable closed-loop systems

Then all stable closed-loop mappings $\bar{w} \mapsto \bar{z}$ are

$$\bar{T}_{zw}(z) = \bar{N}_{zw}(z) - \bar{N}_u(z)\bar{Q}(z)\bar{N}_y(z)$$

(affine function of $\bar{Q}$), where

$$\begin{bmatrix} \bar{N}_{zw}(z) & \bar{N}_u(z) \\ \bar{N}_y(z) & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A + B_u F_u & -L_y C_y & -L_y D_{yw} & B_u R_u \\ 0 & A + L_y C_y & B_w + L_y D_{yw} & 0 \\ \hline C_z + D_{zu} F_u & C_z & D_{zw} & D_{zu} R_u \\ 0 & R_y C_y & R_y D_{yw} & 0 \end{array} \right].$$

Thus, the standard H_2 problem reduces to a two-sided H_2 model matching problem, whose solution verifies

$$\langle \bar{N}_{zw} - \bar{N}_u \bar{Q}_{\text{opt}} \bar{N}_y, \bar{N}_u \bar{Q} \bar{N}_y \rangle_2 = \langle \bar{N}_u^\sim \bar{N}_{zw} \bar{N}_y^\sim - \bar{N}_u^\sim \bar{N}_u \bar{Q}_{\text{opt}} \bar{N}_y \bar{N}_y^\sim, Q \rangle_2 = 0,$$

for all $\bar{Q} \in z^{-1}H_2$.

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Choice of F_u , R_u , L_y , and R_y

The solution is greatly simplified if we

- choose F_u and R_u so that $\bar{N}_u(z)$ is inner and
- choose L_y and R_y so that $\bar{N}_y(z)$ is co-inner

Because

$$\bar{N}_u(z) = \left[\begin{array}{c|c} A + B_u F_u & B_u R_u \\ \hline C_z + D_{zu} F_u & D_{zu} R_u \end{array} \right], \quad \bar{N}_y(z) = \left[\begin{array}{c|c} A + L_y C_y & B_w + L_y D_{yw} \\ \hline R_y C_y & R_y D_{yw} \end{array} \right],$$

the formulae from Lecture 8 (Slides 21 and 22) apply if we pick

$$F_u = -R_u^2 (D'_{zu} C_z + B'_u X A) \quad \text{with } R_u = (D'_{zu} D_{zu} + B'_u X B_u)^{-1/2}$$

$$L_y = -(B_w D'_{yw} + A Y C'_y) R_y^2 \quad \text{with } R_y = (D_{yw} D'_{yw} + C_y Y C'_y)^{-1/2}$$

where $X \geq 0$ and $Y \geq 0$ are the stabilizing solutions to the DARE's

$$X = A' X A + C'_z C_z - (C'_z D_{zu} + A' X B_u) (D'_{zu} D_{zu} + B'_u X B_u)^{-1} (D'_{zu} C_z + B'_u X A),$$

$$Y = A Y A' + B_w B'_w - (B_w D'_{yw} + A Y C'_y) (D_{yw} D'_{yw} + C_y Y C'_y)^{-1} (D_{yw} B'_w + C_y Y A')$$

(exist under our assumptions). In this case, $\bar{Q}_{\text{opt}} = \text{proj}_{z^{-1}H_2} \bar{N}_u^\sim \bar{N}_{zw} \bar{N}_y^\sim$.

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Calculating $\bar{N}_u^\sim \bar{N}_{zw} \bar{N}_y^\sim$ et cetera

Introduce matrices $A_F := A + B_u F_u$, $C_F := C_z + D_{zu} F_u$, $A_L := A + L_y C_y$, and $B_L := B_w + L_y D_{yw}$. Then, repeating continuous-time arguments,

$$\begin{aligned} \bar{N}_{zw}(z) &= \left[\begin{array}{c|c} A_L & B_L \\ \hline C_F & D_{zw} \end{array} \right] - \left[\begin{array}{c|c} A_F & L_y R_y^{-1} \\ \hline C_F & 0 \end{array} \right] \left[\begin{array}{c|c} A_L & B_L \\ \hline R_y C_y & R_y D_{yw} \end{array} \right] \\ &=: \bar{N}_v(z) - \bar{N}_F(z) \bar{N}_y(z) \end{aligned}$$

We already saw these $\bar{N}_v$ and $\bar{N}_y$ in the strictly causal estimation (modulo notation differences) and we know that $\bar{N}_v \bar{N}_y^\sim \in L_2 \ominus z^{-1}H_2$. Then, by the formula of Slide 12 of Lecture 8,

$$\begin{aligned} \bar{N}_u^\sim(z) \bar{N}_F(z) &= \left[\begin{array}{c|c} A_F & L_y R_y^{-1} \\ \hline 0 & 0 \end{array} \right] + \left[\begin{array}{c|c} A_F & B_u R_u \\ \hline R_y^{-1} L'_y X & 0 \end{array} \right]^\sim \\ &= -R_u \left[\begin{array}{c|c} A_F & B_u \\ \hline (D_{yw} B'_u + C_y Y A') X & 0 \end{array} \right]^\sim R_y \in H_2^\perp, \end{aligned}$$

so that $\bar{N}_u^\sim \bar{N}_{zw} \bar{N}_y^\sim = \bar{N}_u^\sim (\bar{N}_v \bar{N}_y^\sim) - \bar{N}_u^\sim \bar{N}_F \in L_2 \ominus z^{-1}H_2$ too and $\bar{Q}_{\text{opt}} = 0$.

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Calculating $N_u^\sim N_{zw} N_y^\sim$ et cetera (contd)

Therefore,

$$\bar{K}_{\text{opt}}(z) = \mathcal{F}_l(\bar{J}(z), 0) = - \left[\begin{array}{c|c} A + B_u F_u + L_y C_y + L_y D_{yu} F_u & L_y \\ \hline F_u & 0 \end{array} \right],$$

which is the classical optimal LQG controller (observer-based). The optimal performance $\|\bar{T}_{zw}\|_2^2 = \|\bar{N}_{zw}\|_2^2 = \|\bar{N}_v - \bar{N}_F \bar{N}_y\|_2^2$. From the solution to the strictly causal estimation problem, $\bar{N}_v \perp \bar{Q} \bar{N}_y$ for all $\bar{Q} \in z^{-1}H_2$. Hence,

$$\begin{aligned} \|\bar{T}_{zw}\|_2^2 &= \|\bar{N}_v\|_2^2 + \|\bar{N}_F \bar{N}_y\|_2^2 = \|\bar{N}_v\|_2^2 + \|\bar{N}_F\|_2^2 \\ &= \text{tr}(D_{zw} D'_{zw} + C_z Y C'_z) + \text{tr}(R_y^{-1} L'_y X L_y R_y^{-1}) \\ &= \text{tr}(D_{zw} D'_{zw}) + \text{tr}(C_z Y C'_z) + \text{tr}(X L_y R_y^{-2} L'_y) \\ &= \text{tr}(D_{zw} D'_{zw}) + \text{tr}(C_z Y C'_z) + \text{tr}(X (A Y A' - Y + B_w B'_w)) \\ &= \text{tr}(D_{zw} D'_{zw}) + \text{tr}(B'_w X B_w) + \text{tr}(C_z Y C'_z) + \text{tr}(X A Y A' - X Y) \\ &= \|D_{zw}\|_F^2 + \text{tr}(X B_w B'_w + C'_z C_z Y + X A Y A' - X Y). \end{aligned}$$

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Content

Continuous-time output feedback

Discrete-time output feedback

State-space realizations in the lifted domain

Type I standard problem

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Transfer function of finite-dimensional $\mathcal{L}_h \mathcal{G} \mathcal{L}_h^{-1}$

Let $\mathcal{G}$ be a finite-dimensional LTI system with $g(t) = D\delta(t) + Ce^{At}B\mathbb{1}(t)$. The transfer function of its lifting has then the kernel (with $\tau \in [-h, h]$)

$$\begin{aligned}\check{G}(z; \tau) &= (\mathfrak{Z}\{\check{g}\})(\tau) = \sum_{i \in \mathbb{Z}} \check{g}[i](\tau) z^{-i} = \sum_{i \in \mathbb{Z}} g(ih + \tau) z^{-i} \\ &= D\delta(\tau) + Ce^{A\tau}B\mathbb{1}(\tau) + \sum_{i \in \mathbb{N}} Ce^{A(ih+\tau)} z^{-i} B \\ &= D\delta(\tau) + Ce^{A\tau}B\mathbb{1}(\tau) + Ce^{A(h+\tau)}(zI - e^{Ah})^{-1}B\end{aligned}$$

so that

$$\begin{aligned}\check{Y}(z; \tau) &= D\check{U}(z; \tau) + C \int_0^\tau e^{A(\tau-\sigma)} B \check{U}(z; \sigma) d\sigma \\ &\quad + Ce^{A\tau} (zI - e^{Ah})^{-1} \int_0^h e^{A(h-\sigma)} B \check{U}(z; \sigma) d\sigma.\end{aligned}$$

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Lifted state-space realization

Introduce operators

$$\begin{aligned}\bar{A} : \mathbb{R}^n &\rightarrow \mathbb{R}^n & \bar{\chi} &\mapsto e^{Ah} \bar{\chi} \\ \bar{B} : \mathbb{L}^m &\rightarrow \mathbb{R}^n & \check{\zeta} &\mapsto \int_0^h e^{A(h-\sigma)} B \check{\zeta}(\sigma) d\sigma \\ \bar{C} : \mathbb{R}^n &\rightarrow \mathbb{L}^p & \bar{\chi} &\mapsto Ce^{A\tau} \bar{\chi} \\ \bar{D} : \mathbb{L}^m &\rightarrow \mathbb{L}^p & \check{\zeta} &\mapsto D\check{\zeta}(\tau) + C \int_0^\tau e^{A(\tau-\sigma)} B \check{\zeta}(\sigma) d\sigma\end{aligned}$$

Then the transfer function of $\check{\mathcal{G}}$ is

$$\check{G}(z) = \bar{D} + \bar{C}(zI - \bar{A})^{-1}\bar{B} =: \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}$$

and the quadruple $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is called its state-space realization. It is a
 — standard discrete state-space realization except that its parameters are operators over $\mathbb{L} := L_2[0, h]$ rather than matrices

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Impulse response of finite-dimensional $\mathcal{L}_h \mathcal{G} \mathcal{L}_h^{-1}$

The impulse response of $\check{\mathcal{G}}$ in terms of state-space realization is

$$\check{g}[i] = \begin{cases} 0 & \text{if } i < 0 \\ \bar{D} & \text{if } i = 0 \quad (\text{infinite rank}) \\ \bar{C}\bar{A}^{i-1}\bar{B} & \text{if } i > 0 \quad (\text{finite rank}) \end{cases}$$

The “system matrix” of $\check{\mathcal{G}}$ is then

$$[\check{\mathcal{G}}] = \begin{array}{c|cccc} & -2 & -1 & 0 & 1 \\ \hline -2 & \bar{D} & & & \\ -1 & \bar{C}\bar{B} & \bar{D} & & \\ 0 & \bar{C}\bar{A}\bar{B} & \bar{C}\bar{B} & \bar{D} & \\ 1 & \bar{C}\bar{A}^2\bar{B} & \bar{C}\bar{A}\bar{B} & \bar{C}\bar{B} & \bar{D} \\ i & & & & \end{array}$$

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Extracting the feedthrough part from $\mathcal{L}_h \mathcal{G} \mathcal{L}_h^{-1}$

A finite-dimensional shift-invariant $\check{\mathcal{G}}$ can always be decomposed as

$$\check{\mathcal{G}} = \check{\mathcal{G}}_\infty + \check{\mathcal{G}}_{\text{sc}} : \begin{array}{c|cccc} & -2 & -1 & 0 & 1 \\ \hline -2 & \bar{D} & & & \\ -1 & \bar{C}\bar{B} & \bar{D} & & \\ 0 & \bar{C}\bar{A}\bar{B} & \bar{C}\bar{B} & \bar{D} & \\ 1 & \bar{C}\bar{A}^2\bar{B} & \bar{C}\bar{A}\bar{B} & \bar{C}\bar{B} & \bar{D} \\ i & & & & \end{array} = \begin{array}{c|cccc} & -2 & -1 & 0 & 1 \\ \hline -2 & \bar{D} & & & \\ -1 & & \bar{D} & & \\ 0 & & & \bar{D} & \\ 1 & & & & \bar{D} \\ i & & & & \end{array} + \begin{array}{c|cccc} & -2 & -1 & 0 & 1 \\ \hline -2 & & & & \\ -1 & \bar{C}\bar{B} & & & \\ 0 & \bar{C}\bar{A}\bar{B} & \bar{C}\bar{B} & & \\ 1 & \bar{C}\bar{A}^2\bar{B} & \bar{C}\bar{A}\bar{B} & \bar{C}\bar{B} & \\ i & & & & \end{array},$$

where $\check{\mathcal{G}}_\infty$ is the static lifted system with the transfer function

$$\check{G}_\infty(z) = \bar{D}$$

(**feedthrough part** of $\check{\mathcal{G}}$) and $\check{\mathcal{G}}_{\text{sc}}$ is a strictly causal system with the transfer function

$$\check{G}_{\text{sc}}(z) = \bar{C}(zI - \bar{A})^{-1}\bar{B}$$

(**strictly causal part** of $\check{\mathcal{G}}$).

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Feedthrough part of $\mathcal{L}_h \mathcal{G} \mathcal{L}_h^{-1}$

In the lifted domain it acts as the following mapping $\check{u} \mapsto \check{y}$:

$$\check{y}[i] = \check{D}\check{u}[i] \iff \check{y}[i](\tau) = D\check{u}[i](\tau) + C \int_0^\tau e^{A(\tau-\sigma)} B \check{u}[i](\sigma) d\sigma$$

In the continuous-time domain this reads

$$y(ih + \tau) = Du(ih + \tau) + C \int_0^\tau e^{A(\tau-\sigma)} Bu(ih + \sigma) d\sigma$$

Thus, the feedthrough part of $\check{\mathcal{G}}$ acts in the continuous-time domain as

$$\mathcal{G}_\infty : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(ih) = 0 \\ y(t) = Cx(t) + Du(t), \end{cases}$$

i.e. it is an analog system, **reset** at every time instance $t_i = ih$, for all $i \in \mathbb{Z}$.

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Strictly causal part of $\mathcal{L}_h \mathcal{G} \mathcal{L}_h^{-1}$ (contd)

It can be decomposed as

$$\check{\mathcal{G}}_{sc}(z) = \check{C}(z(I - \bar{A})^{-1})z^{-1}\check{B} =: \check{\Phi}_c(z) \bar{G}_A(z) \check{\Psi}_B(z)$$

i.e. $\check{\mathcal{G}}_{sc} = \check{\mathcal{H}}_c \bar{G}_A \check{\mathcal{S}}_B$, where

$\check{\mathcal{S}}_B$ is the static sampler with the sampling function $\psi_B(t) = e^{At} B \mathbb{1}_{[0,h]}(t)$,

$\bar{G}_A$ is the causal pure discrete system with the transfer function

$$\bar{G}_A(z) = z \left[\begin{array}{c|c} e^{Ah} & I \\ \hline I & 0 \end{array} \right] = \left[\begin{array}{c|c} e^{Ah} & I \\ \hline e^{Ah} & e^{Ah} \end{array} \right],$$

$\check{\mathcal{H}}_c$ is the static hold with the hold function $\phi_c(t) = Ce^{At} \mathbb{1}_{[0,h]}(t)$.

(note, $\psi_B(t) \in \mathbb{R}^{n \times m}$ and $\phi_c(t) \in \mathbb{R}^{p \times n}$, i.e. they are not square in general).

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Transfer function of $\check{\mathcal{S}}_{idl}$ et cetera

The transfer function $\check{\Psi}_{idl}(z)$ of the lifting of $\mathcal{S}_{idl}$ has the kernel

$$\check{\Psi}_{idl}(z; \sigma) = \delta(\sigma),$$

so that

$$\bar{Y}(z) = z^{-1} \int_0^h \check{\Psi}_{idl}(z; h - \sigma) \check{Y}(z; \sigma) d\sigma = z^{-1} \check{Y}(z; h) = \check{Y}(z; 0).$$

Hence, given a strictly proper $G(s) = C(sI - A)^{-1}B$,

$$\check{\Psi}_{idl}(z) \check{G}(z) = \check{\Psi}_{idl}(z) \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array} \right] = z^{-1} \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array} \right] = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array} \right] =: \check{G}(z),$$

where $\bar{C} = C\bar{A}$ (so $z^{-1}\bar{C} = \check{\Psi}_{idl}(z)\check{C}$) and $\bar{D} = C\bar{B}$ (so $z^{-1}\bar{D} = \check{\Psi}_{idl}(z)\check{D}$).

An important consequence of this is that

- $\check{\mathcal{G}} = \check{\mathcal{S}}_{idl} \check{\mathcal{G}}$ is always strictly causal.

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Transfer function of $\check{\mathcal{H}}_{zoh}$ et cetera

We know (Lecture 7, Slide 28) that $\check{\Phi}_{zoh}(z)$ has the kernel

$$\check{\Phi}_{zoh}(z; \tau) = 1,$$

so that $\check{U}(z; \tau) = \bar{U}(z)$. Hence, given a $G(s) = D + C(sI - A)^{-1}B$,

$$\check{G}(z) \check{\Phi}_{zoh}(z) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array} \right] \check{\Phi}_{zoh}(z) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array} \right] =: \check{G}(z),$$

where

$$\begin{aligned} \bar{B} &= \check{B} \check{\Phi}_{zoh}(z) : \mathbb{R}^m \rightarrow \mathbb{R}^n & \bar{\xi} &\mapsto \int_0^h e^{A\sigma} d\sigma B \bar{\xi} \\ \bar{D} &= \check{D} \check{\Phi}_{zoh}(z) : \mathbb{R}^m \rightarrow \mathbb{L}^p & \bar{\xi} &\mapsto \left(D + C \int_0^\tau e^{A\sigma} d\sigma B \right) \bar{\xi} \\ & & &\mapsto [C \ D] \exp \left(\left[\begin{array}{cc} A & B \\ 0 & 0 \end{array} \right] \tau \right) \left[\begin{array}{c} 0 \\ I \end{array} \right] \bar{\xi} \end{aligned}$$

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Example: lifted transfer function of an integrator

Let $\mathcal{G}_i$ be the LTI system with the transfer function

$$G_i(s) = \frac{1}{s} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array} \right].$$

Then

$$\check{\mathcal{G}}_i(z) = \check{D}_i + \check{C}_i \frac{1}{z-1} \check{B}_i,$$

where

$$\check{B}_i \check{\zeta} = \int_0^h \zeta(\sigma) d\sigma, \quad (\check{C}_i \bar{\chi})(\tau) = \bar{\chi}, \quad \text{and} \quad (\check{D}_i \check{\zeta})(\tau) = \int_0^\tau \zeta(\sigma) d\sigma.$$

Note that this $\check{C}_i = \check{\Phi}_{\text{ZOH}}(z)$ and $\frac{1}{zh} \check{B}_i = \check{\Psi}_{\text{avg}}(z)$. Thus,

$$\check{\mathcal{G}}_i(z) = \check{D}_i + \check{\Phi}_{\text{ZOH}}(z) \frac{zh}{z-1} \check{\Psi}_{\text{avg}}(z).$$

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Content

Continuous-time output feedback

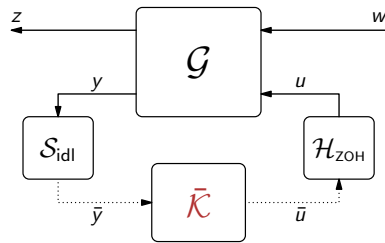
Discrete-time output feedback

State-space realizations in the lifted domain

Type I standard problem

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Standard Type I sampled-data H_2 problem



Given $\mathcal{G}$ with

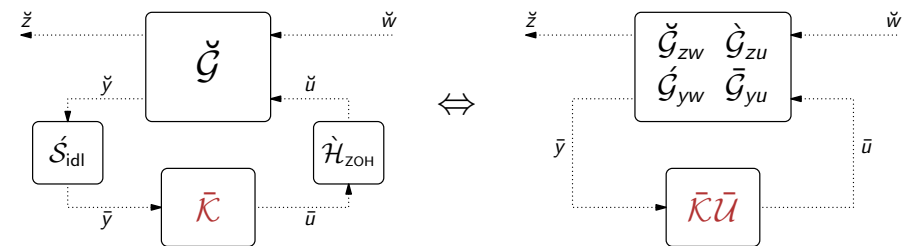
$$G(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & 0 & 0 \end{array} \right],$$

the ideal sampler $\mathcal{S}_{\text{idl}}$ and the zero-order hold $\mathcal{H}_{\text{ZOH}}$, design causal internally stabilizing discrete $\bar{\mathcal{K}}$ that minimizes $\|\mathcal{T}_{zw}\|_2$, where

$$\begin{aligned} \mathcal{T}_{zw} &= \mathcal{F}_1(\mathcal{G}, \mathcal{H}_{\text{ZOH}} \bar{\mathcal{K}} \mathcal{S}_{\text{idl}}) = \mathcal{G}_{zw} + \mathcal{G}_{zu} \mathcal{H}_{\text{ZOH}} \bar{\mathcal{K}} \mathcal{S}_{\text{idl}} (I - \mathcal{G}_{yu} \mathcal{H}_{\text{ZOH}} \bar{\mathcal{K}} \mathcal{S}_{\text{idl}})^{-1} \mathcal{G}_{yw} \\ &= \mathcal{G}_{zw} + \mathcal{G}_{zu} \mathcal{H}_{\text{ZOH}} (I - \bar{\mathcal{K}} \mathcal{S}_{\text{idl}} \mathcal{G}_{yu} \mathcal{H}_{\text{ZOH}})^{-1} \bar{\mathcal{K}} \mathcal{S}_{\text{idl}} \mathcal{G}_{yw}. \end{aligned}$$

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Lifted reformulation



Lifting renders it pure discrete LTI problem of minimizing $\|\check{\mathcal{T}}_{zw}\|_2$, where

$$\check{\mathcal{T}}_{zw} = \check{\mathcal{G}}_{zw} + \check{\mathcal{G}}_{zu} \check{\mathcal{H}}_{\text{ZOH}} (I - \bar{\mathcal{K}} \check{\mathcal{S}}_{\text{idl}} \check{\mathcal{G}}_{yu} \check{\mathcal{H}}_{\text{ZOH}})^{-1} \bar{\mathcal{K}} \check{\mathcal{S}}_{\text{idl}} \check{\mathcal{G}}_{yw}.$$

As we know, $\check{\mathcal{S}}_{\text{idl}} \check{\mathcal{G}}_{yw}$ and $\check{\mathcal{S}}_{\text{idl}} \check{\mathcal{G}}_{yu}$ are strictly causal, hence

$$\check{\mathcal{T}}_{zw} = \underbrace{\check{\mathcal{G}}_{zw} + \check{\mathcal{G}}_{zu} \check{\mathcal{H}}_{\text{ZOH}}}_{\check{\mathcal{G}}_{zu}} (I - \underbrace{\bar{\mathcal{K}} \bar{\mathcal{U}} \bar{\mathcal{U}}^{-1} \check{\mathcal{S}}_{\text{idl}} \check{\mathcal{G}}_{yu} \check{\mathcal{H}}_{\text{ZOH}}}_{\check{\mathcal{G}}_{yu}})^{-1} \underbrace{\bar{\mathcal{K}} \bar{\mathcal{U}} \bar{\mathcal{U}}^{-1} \check{\mathcal{S}}_{\text{idl}} \check{\mathcal{G}}_{yw}}_{\check{\mathcal{G}}_{yw}}$$

and we end up with a discrete H_2 problem with a strictly causal controller.

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Lifted generalized plant

We have

$$\begin{bmatrix} \check{G}_{zw}(z) & \check{G}_{zu}(z) \\ \check{G}_{yw}(z) & \check{G}_{yu}(z) \end{bmatrix} = \left[\begin{array}{c|cc} \bar{A} & \dot{B}_w & \bar{B}_u \\ \hline \check{C}_z & \check{D}_{zw} & \check{D}_{zu} \\ \check{C}_y & \dot{D}_{yw} & \bar{D}_{yu} \end{array} \right]$$

where $\bar{A} = e^{Ah}$, $\bar{B}_u = \int_0^h e^{A\sigma} d\sigma B_u$,

$$\dot{B}_w : \mathbb{L}^{n_w} \rightarrow \mathbb{R}^n \quad \check{\zeta} \mapsto \int_0^h e^{A(h-\sigma)} B_w \check{\zeta}(\sigma) d\sigma$$

$$\check{C}_z : \mathbb{R}^n \rightarrow \mathbb{L}^{n_z} \quad \bar{\chi} \mapsto C_z e^{A\tau} \bar{\chi}$$

$$\check{D}_{zw} : \mathbb{L}^{n_w} \rightarrow \mathbb{L}^{n_z} \quad \check{\zeta} \mapsto C_z \int_0^\tau e^{A(\tau-\sigma)} B_w \check{\zeta}(\sigma) d\sigma$$

$$\check{D}_{zu} : \mathbb{R}^{n_u} \rightarrow \mathbb{L}^{n_z} \quad \bar{\xi} \mapsto \begin{bmatrix} C_z & D_{zu} \end{bmatrix} \exp\left(\begin{bmatrix} A & B_u \\ 0 & 0 \end{bmatrix} \tau\right) \begin{bmatrix} 0 \\ I \end{bmatrix} \bar{\xi}$$

and $\begin{bmatrix} \bar{C}_y & \dot{D}_{yw} & \bar{D}_{yu} \end{bmatrix} = C_y \begin{bmatrix} \bar{A} & \dot{B}_w & \bar{B}_u \end{bmatrix}$.

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Assumptions in the lifted domain

Having

$$\begin{bmatrix} \check{G}_{zw}(z) & \check{G}_{zu}(z) \\ \check{G}_{yw}(z) & \check{G}_{yu}(z) \end{bmatrix} = \left[\begin{array}{c|cc} \bar{A} & \dot{B}_w & \bar{B}_u \\ \hline \check{C}_z & \check{D}_{zw} & \check{D}_{zu} \\ \check{C}_y & \dot{D}_{yw} & \bar{D}_{yu} \end{array} \right],$$

we assume that

- $(\bar{A}, \bar{B}_u)$ is stabilizable
- $(\bar{C}_y, \bar{A})$ is detectable
- $\begin{bmatrix} \bar{A}-zI & \bar{B}_u \\ \check{C}_z & \check{D}_{zu} \end{bmatrix}$ has full column rank for all $z \in \mathbb{T}$
- $\begin{bmatrix} \bar{A}-zI & \dot{B}_w \\ \check{C}_y & \dot{D}_{yw} \end{bmatrix}$ has full row rank for all $z \in \mathbb{T}$

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Lifted solution

Let

$$\begin{aligned} \bar{F}_u &= -(\dot{D}_{zu}^* \dot{D}_{zu} + \bar{B}_u' \bar{X} \bar{B}_u)^{-1} (\dot{D}_{zu}^* \check{C}_z + \bar{B}_u' \bar{X} \bar{A}) \\ \bar{L}_y &= -(\dot{B}_w \dot{D}_{yw}^* + \bar{A} \bar{Y} \bar{C}_y') (\dot{D}_{yw} \dot{D}_{yw}^* + \bar{C}_y \bar{Y} \bar{C}_y')^{-1} (\dot{D}_{yw} \dot{B}_w^* + \bar{C}_y \bar{Y} \bar{A}'). \end{aligned}$$

where $\bar{X} \geq 0$ and $\bar{Y} \geq 0$ are the stabilizing solutions to the DARE's

$$\begin{aligned} \bar{X} &= \bar{A}' \bar{X} \bar{A} + \check{C}_z^* \check{C}_z - (\check{C}_z^* \dot{D}_{zu} + \bar{A}' \bar{X} \bar{B}_u) (\dot{D}_{zu}^* \dot{D}_{zu} + \bar{B}_u' \bar{X} \bar{B}_u)^{-1} (\dot{D}_{zu}^* \check{C}_z + \bar{B}_u' \bar{X} \bar{A}), \\ \bar{Y} &= \bar{A} \bar{Y} \bar{A}' + \dot{B}_w \dot{B}_w^* - (\dot{B}_w \dot{D}_{yw}^* + \bar{A} \bar{Y} \bar{C}_y') (\dot{D}_{yw} \dot{D}_{yw}^* + \bar{C}_y \bar{Y} \bar{C}_y')^{-1} (\dot{D}_{yw} \dot{B}_w^* + \bar{C}_y \bar{Y} \bar{A}'). \end{aligned}$$

Then the optimal controller is

$$\bar{K}_{\text{opt}}(z) = -z \left[\begin{array}{c|c} \bar{A} + \bar{B}_u \bar{F}_u + \bar{L}_y \bar{C}_y + \bar{L}_y \bar{D}_{yu} \bar{F}_u & \bar{L}_y \\ \hline \bar{F}_u & 0 \end{array} \right],$$

and the optimal performance is

$$\|\check{T}_{zw}\|_2^2 = \frac{1}{h} \left(\|\check{D}_{zw}\|_{\text{HS}}^2 + \text{tr}(\bar{X} \dot{B}_w \dot{B}_w^* + \check{C}_z^* \check{C}_z \bar{Y} + \bar{X} \bar{A} \bar{Y} \bar{A}' - \bar{X} \bar{Y}) \right).$$

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Peeling off: a key observation

All parts of the solution that involve infinite-dimensional operators are

$$\begin{bmatrix} \bar{Q}_z & \bar{S}_z \\ \bar{S}_z' & \bar{R}_z \end{bmatrix} := \begin{bmatrix} \check{C}_z^* \\ \dot{D}_{zu}^* \end{bmatrix} \begin{bmatrix} \check{C}_z & \dot{D}_{zu} \end{bmatrix}, \quad \begin{bmatrix} \bar{Q}_w & \bar{S}_w' \\ \bar{S}_w & \bar{R}_w \end{bmatrix} := \begin{bmatrix} \dot{B}_w \\ \dot{D}_{yw} \end{bmatrix} \begin{bmatrix} \dot{B}_w^* & \dot{D}_{yw}^* \end{bmatrix}.$$

Indeed, the DAREs then read

$$\begin{aligned} \bar{X} &= \bar{A}' \bar{X} \bar{A} + \bar{Q}_z - (\bar{S}_z + \bar{A}' \bar{X} \bar{B}_u) (\bar{R}_z + \bar{B}_u' \bar{X} \bar{B}_u)^{-1} (\bar{S}_z' + \bar{B}_u' \bar{X} \bar{A}), \\ \bar{Y} &= \bar{A} \bar{Y} \bar{A}' + \bar{Q}_w - (\bar{S}_w' + \bar{A} \bar{Y} \bar{C}_y') (\bar{R}_w + \bar{C}_y \bar{Y} \bar{C}_y')^{-1} (\bar{S}_w + \bar{C}_y \bar{Y} \bar{A}'). \end{aligned}$$

with the corresponding gains

$$\begin{aligned} \bar{F}_u &= -(\bar{R}_z + \bar{B}_u' \bar{X} \bar{B}_u)^{-1} (\bar{S}_z' + \bar{B}_u' \bar{X} \bar{A}), \\ \bar{L}_y &= -(\bar{S}_w' + \bar{A} \bar{Y} \bar{C}_y') (\bar{R}_w + \bar{C}_y \bar{Y} \bar{C}_y')^{-1}. \end{aligned}$$

Thus, if we know how to compute those “Q-R-S” triples, we are done.

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Peeling off: the w -part

The operator

$$\psi_w(z) := z^{-1} \begin{bmatrix} \dot{B}_w \\ \dot{D}_{yw} \end{bmatrix} : \check{\zeta} \mapsto z^{-1} \begin{bmatrix} I \\ C_y \end{bmatrix} \int_0^h e^{A(h-\sigma)} B_w \check{\zeta}(\sigma) d\sigma$$

is the lifted transfer function of a static sampler. We then know that

$$\psi_w(z) \psi_w^\sim(z) = \begin{bmatrix} I \\ C_y \end{bmatrix} \int_0^h e^{A\sigma} B_w B_w' e^{A'\sigma} d\sigma \begin{bmatrix} I & C_y' \end{bmatrix} = \begin{bmatrix} \bar{Q}_w & \bar{S}_w' \\ \bar{S}_w & \bar{R}_w \end{bmatrix}.$$

Moreover, if

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{bmatrix} := \exp \left(\begin{bmatrix} A & B_w B_w' \\ 0 & -A' \end{bmatrix} h \right),$$

the arguments of the proof of Van Loan's formulae (see Lecture 3, Slide 40) yield

$$\int_0^h e^{A\sigma} B_w B_w' e^{A'\sigma} d\sigma = \Gamma_{12} \Gamma_{11}'.$$

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Peeling off: the z -part

The operator

$$\phi_z(z) := \begin{bmatrix} \dot{C}_z & \dot{D}_{zu} \end{bmatrix} : \begin{bmatrix} \bar{\chi} \\ \check{\xi} \end{bmatrix} \mapsto \begin{bmatrix} C_z & D_{zu} \end{bmatrix} \exp \left(\begin{bmatrix} A & B_u \\ 0 & 0 \end{bmatrix} \tau \right) \begin{bmatrix} \bar{\chi} \\ \check{\xi} \end{bmatrix}$$

is the lifted transfer function of a static hold. We then know that

$$\phi_z^\sim(z) \phi_z(z) = \int_0^h e^{\begin{bmatrix} A' & 0 \\ B_u' & 0 \end{bmatrix} \tau} \begin{bmatrix} C_z' C_z & C_z' D_{zu} \\ D_{zu}' C_z & D_{zu}' D_{zu} \end{bmatrix} e^{\begin{bmatrix} A & B_u \\ 0 & 0 \end{bmatrix} \tau} d\tau = \begin{bmatrix} \bar{Q}_z & \bar{S}_z \\ \bar{S}_z' & \bar{R}_z \end{bmatrix}$$

and we can use Van Loan's formulae again to end up with the single matrix exponential

$$\exp \left(\begin{bmatrix} -A' & 0 & C_z' C_z & C_z' D_{zu} \\ -B_u' & 0 & D_{zu}' C_z & D_{zu}' D_{zu} \\ 0 & 0 & A & B_u \\ 0 & 0 & 0 & 0 \end{bmatrix} h \right).$$

whose (3, 4) subblock equals $\bar{B}_u$, as a matter of fact.

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Peeling off: assumptions

First, note that

$$\text{rank} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \text{rank} \begin{bmatrix} M_1 \\ M_2' M_2 \end{bmatrix}, \quad \text{rank} \begin{bmatrix} M_1 & M_2 \end{bmatrix} = \text{rank} \begin{bmatrix} M_1 & M_2 M_2' \end{bmatrix}.$$

Hence,

$$\begin{aligned} \text{rank} \begin{bmatrix} \bar{A} - zI & \bar{B}_u \\ \dot{C}_z & \dot{D}_{zu} \end{bmatrix} &= \text{rank} \begin{bmatrix} \bar{A} - zI & \bar{B}_u \\ \dot{C}_z^* \dot{C}_z & \dot{C}_z^* \dot{D}_{zu} \\ \dot{D}_{zu}^* \dot{C}_z & \dot{D}_{zu}^* \dot{D}_{zu} \end{bmatrix} = \text{rank} \begin{bmatrix} \bar{A} - zI & \bar{B}_u \\ \bar{Q}_z & \bar{S}_z \\ \bar{S}_z' & \bar{R}_z \end{bmatrix} \\ \text{rank} \begin{bmatrix} \bar{A} - zI & \dot{B}_w \\ \bar{C}_y & \dot{D}_{yw} \end{bmatrix} &= \text{rank} \begin{bmatrix} \bar{A} - zI & \dot{B}_w \dot{B}_w^* & \dot{B}_w \dot{D}_{yw}^* \\ \bar{C}_y & \dot{D}_{yw} \dot{B}_w^* & \dot{D}_{yw} \dot{D}_{yw}^* \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} \bar{A} - zI & \bar{Q}_w & \bar{S}_w' \\ \bar{C}_y & \bar{S}_w & \bar{R}_w \end{bmatrix} \end{aligned}$$

meaning that the rank conditions can be checked via matrix operations too.

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Peeling off: performance

Since $\|\check{T}_{zw}\|_2^2 = \frac{1}{h} (\|\check{D}_{zw}\|_{\text{HS}}^2 + \text{tr}(\bar{X} \bar{Q}_w + \bar{Q}_z \bar{Y} + \bar{X} \bar{A} \bar{Y} \bar{A}' - \bar{X} \bar{Y}))$, it is left to compute the Hilbert-Schmidt norm of the integral operator $\check{D}_{zw}$, having the kernel $d_{zw}(t, s) = C_z e^{A(t-s)} B_w \mathbb{1}(t-s)$. We have:

$$\begin{aligned} \|\check{D}_{zw}\|_{\text{HS}}^2 &= \int_0^h \int_0^h \text{tr}(d_{zw}(t, s) d_{zw}'(t, s)) ds dt \\ &= \int_0^h \int_0^t \text{tr}(C_z e^{A(t-s)} B_w B_w' e^{A'(t-s)} C_z') ds dt \\ &= \text{tr} \left(e^{A'h} \int_0^h e^{-A'(h-t)} C_z' C_z \int_0^t e^{A(t-s)} B_w B_w' e^{-A's} ds dt \right) \\ &= \text{tr} \left(\begin{bmatrix} e^{A'h} & 0 & 0 \end{bmatrix} \exp \left(\begin{bmatrix} -A' & C_z' C_z & 0 \\ 0 & A & B_w B_w' \\ 0 & 0 & -A' \end{bmatrix} h \right) \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \right), \end{aligned}$$

so we again ended up with a matrix exponential.

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