

Advanced Sampled-Data Control (038798)

lecture no. 8

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Content

State-space machinery

Continuous-time H_2 estimation

Discrete-time H_2 estimation

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Poles, zeros, minimality

Transfer functions in terms of state-space realization (A, B, C, D) :

$$\left. \begin{aligned} G(s) &= D + C(sI - A)^{-1}B \\ \bar{G}(z) &= D + C(zI - A)^{-1}B \end{aligned} \right\} =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

Basic definitions:

- realization *poles* are the eigenvalues of A
- *invariant zeros* are $\lambda \in \mathbb{C}$ at which $\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$ loses its normal rank
- realization is *minimal* iff (A, B) is controllable and (C, A) is observable
(and then realization poles and invariant zeros coincide with poles and transmission zeros of transfer functions)

Algebra of transfer functions

- sum:

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right] + \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array} \right]$$

- product:

$$\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right] \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right] = \left[\begin{array}{cc|c} A_2 & B_2 C_1 & B_2 D_1 \\ 0 & A_1 & B_1 \\ \hline C_2 & D_2 C_1 & D_2 D_1 \end{array} \right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2 C_1 & A_2 & B_2 D_1 \\ \hline D_2 C_1 & C_2 & D_2 D_1 \end{array} \right]$$

- inverse:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]^{-1} = \left[\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array} \right]$$

- conjugate (different for continuous- and discrete-time systems):

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]^{\sim} = \left[\begin{array}{c|c} -A' & C' \\ \hline -B' & D' \end{array} \right] \quad \text{or} \quad \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]^{\sim} = \left[\begin{array}{c|c} A^{-'} & A^{-'}C' \\ \hline -B'A^{-'} & D' - B'A^{-'}C' \end{array} \right]$$

Linear matrix equations

Sylvester: Given $A_1 \in \mathbb{F}^{p \times p}$, $A_2 \in \mathbb{F}^{m \times m}$, $Q \in \mathbb{F}^{p \times m}$, solve in $X \in \mathbb{F}^{p \times m}$

$$A_1 X + X A_2 + Q = 0.$$

X exists for every Q and is unique iff $\lambda_1 + \lambda_2 \neq 0$ for all $\lambda_1 \in \text{spec } A_1$ and $\lambda_2 \in \text{spec } A_2$.

Stein: Given $A_1 \in \mathbb{F}^{p \times p}$, $A_2 \in \mathbb{F}^{m \times m}$, $Q \in \mathbb{F}^{p \times m}$, solve in $X \in \mathbb{F}^{p \times m}$

$$X = A_1 X A_2 + Q.$$

X exists for every Q and is unique iff $\lambda_1 \cdot \lambda_2 \neq 1$ for all $\lambda_1 \in \text{spec } A_1$ and $\lambda_2 \in \text{spec } A_2$.

Matrix Lyapunov equations

Continuous-time: given $A \in \mathbb{F}^{n \times n}$ and $Q = Q' \in \mathbb{F}^{n \times n}$, solve in $X \in \mathbb{F}^{n \times n}$

$$AX + XA' + Q = 0.$$

If $Q \geq 0$, then $X \geq 0$ iff A is Hurwitz and in this case

$$X = \int_{\mathbb{R}^+} e^{At} Q e^{A't} dt.$$

Discrete-time: given $A \in \mathbb{F}^{n \times n}$ and $Q = Q' \in \mathbb{F}^{n \times n}$, solve in $X \in \mathbb{F}^{n \times n}$

$$X = AXA' + Q.$$

If $Q \geq 0$, then $X \geq 0$ iff A is Schur and in this case

$$X = \sum_{i \in \mathbb{Z}^+} A^i Q (A')^i.$$

Computing H_2 norm: CT case

Let $G(s) = C(sI - A)^{-1}B$ be stable (so in H_2). By Parseval,

$$\begin{aligned}\|G\|_2^2 &= \int_{\mathbb{R}} \|g(t)\|_F^2 dt = \int_{\mathbb{R}^+} \|Ce^{At}B\|_F^2 dt \\ &= \text{tr}\left(C \int_{\mathbb{R}^+} e^{At}BB'e^{A't}dt C'\right) = \text{tr}(CW_cC') \\ &= \text{tr}\left(B' \int_{\mathbb{R}^+} e^{A't}C'Ce^{At}dt B\right) = \text{tr}(B'W_oB),\end{aligned}$$

where W_c and W_o are the controllability and observability Gramians of G , respectively, verifying the Lyapunov equations

$$AW_c + W_cA' + BB' = 0 \quad \text{and} \quad W_oA + A'W_o + C'C = 0.$$

Computing H_2 norm: DT case

Let $\bar{G}(z) = D + C(zI - A)^{-1}B$ be stable (so in H_2). By Parseval,

$$\begin{aligned}\|\bar{G}\|_2^2 &= \sum_{i \in \mathbb{Z}} \|\bar{g}[i]\|_F^2 = \|D\|_F^2 + \sum_{i \in \mathbb{N}} \|CA^{i-1}B\|_F^2 \\ &= \|D\|_F^2 + \text{tr}\left(C \sum_{i \in \mathbb{Z}^+} [A^i B B' (A')^i] C'\right) = \text{tr}(DD' + C W_c C') \\ &= \|D\|_F^2 + \text{tr}\left(B' \sum_{i \in \mathbb{Z}^+} [(A')^i C' C A^i] B\right) = \text{tr}(D'D + B' W_o B),\end{aligned}$$

where W_c and W_o are the controllability and observability Gramians of $\bar{G}$, respectively, verifying the Lyapunov equations

$$W_c = A W_c A' + B B' \quad \text{and} \quad W_o = A' W_o A + C' C.$$

Partial fraction expansion in state space: CT case

Applying similarity transformation with $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$,

$$\begin{aligned} \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right] \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right]^{\sim} &= \left[\begin{array}{c|c} A_2 & -B_2 B'_1 \mid B_2 D'_1 \\ 0 & -A'_1 \mid C'_1 \\ \hline C_2 & -D_2 B'_1 \mid D_2 D'_1 \end{array} \right] \\ &= \left[\begin{array}{c|c} A_2 & -A_2 X - X A'_1 - B_2 B'_1 \mid B_2 D'_1 + X C'_1 \\ 0 & -A'_1 \mid C'_1 \\ \hline C_2 & -D_2 B'_1 - C_2 X \mid D_2 D'_1 \end{array} \right]. \end{aligned}$$

If $-A_1$ and A_2 have no eigenvalues in common, $A_2 X + X A'_1 + B_2 B'_1 = 0$ is solvable and then

$$\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right] \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right]^{\sim} = \left[\begin{array}{c|c} A_2 & B_2 D'_1 + X C'_1 \\ \hline C_2 & D_2 D'_1 \end{array} \right] + \left[\begin{array}{c|c} A_1 & B_1 D'_2 + X' C'_2 \\ \hline C_1 & 0 \end{array} \right]^{\sim}$$

This always works if both A_1 and A_2 are stable.

Partial fraction expansion in state space: CT case (contd)

Applying similarity transformation with $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$,

$$\begin{aligned} \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right] &\sim \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right] = \left[\begin{array}{cc|c} -A'_2 & C'_2 C_1 & C'_2 D_1 \\ 0 & A_1 & B_1 \\ \hline -B'_2 & D'_2 C_1 & D'_2 D_1 \end{array} \right] \\ &= \left[\begin{array}{cc|c} -A'_2 & A'_2 X + X A_1 + C'_2 C_1 & C'_2 D_1 + X B_1 \\ 0 & A_1 & B_1 \\ \hline -B'_2 & D'_2 C_1 + B'_2 X & D'_2 D_1 \end{array} \right]. \end{aligned}$$

If A_1 and $-A_2$ have no eigenvalues in common, $A'_2 X + X A_1 + C'_2 C_1 = 0$ is solvable and then

$$\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right] \sim \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right] = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline D'_2 C_1 + B'_2 X & D'_2 D_1 \end{array} \right] + \left[\begin{array}{c|c} A_2 & B_2 \\ \hline D'_1 C_2 + B'_1 X' & 0 \end{array} \right] \sim$$

This always works if both A_1 and A_2 are stable.

Partial fraction expansion in state space: DT case

If $X - A_2XA'_1 = B_2B'_1$, then $\Theta_1(z) := (zI - A_2)^{-1}B_2B'_1(\frac{1}{z}I - A'_1)^{-1}$ reads

$$\begin{aligned}\Theta_1(z) &= (zI - A_2)^{-1}(X - A_2XA'_1)(\frac{1}{z}I - A'_1)^{-1} \\ &= (zI - A_2)^{-1}((I - \frac{1}{z}A_2)X + A_2X(\frac{1}{z}I - A'_1))(\frac{1}{z}I - A'_1)^{-1} \\ &= \frac{1}{z}X(\frac{1}{z}I - A'_1)^{-1} + (zI - A_2)^{-1}A_2X \\ &= X + (zI - A_2)^{-1}A_2X + XA'_1(\frac{1}{z}I - A'_1)^{-1}\end{aligned}$$

(since $\zeta(\zeta I - A)^{-1} = I + A(\zeta I - A)^{-1}$) and then

$$\begin{aligned}\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]^\sim &= D_2D'_1 + C_2XC'_1 \\ &+ \left[\begin{array}{c|c} A_2 & B_2D'_1 + A_2XC'_1 \\ \hline C_2 & 0 \end{array}\right] + \left[\begin{array}{c|c} A_1 & B_1D'_2 + A_1X'C'_2 \\ \hline C_1 & 0 \end{array}\right]^\sim\end{aligned}$$

This always works if both A_1 and A_2 are stable.

Partial fraction expansion in state space: DT case (contd)

If $X - A_2'XA_1 = C_2'C_1$, then $\Theta_2(z) := (\frac{1}{z}I - A_2')^{-1}C_2'C_1(zI - A_1)^{-1}$ reads

$$\begin{aligned}\Theta_2(z) &= (\tfrac{1}{z}I - A_2')^{-1}(X - A_2'XA_1)(zI - A_1)^{-1} \\ &= (\tfrac{1}{z}I - A_2')^{-1}((I - zA_2')X + A_2'X(zI - A_1))(zI - A_1)^{-1} \\ &= zX(zI - A_1)^{-1} + (\tfrac{1}{z}I - A_2')^{-1}A_2'X \\ &= X + XA_1(zI - A_1)^{-1} + (\tfrac{1}{z}I - A_2')^{-1}A_2'X\end{aligned}$$

and then

$$\begin{aligned}\left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] &\sim \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = D_2'D_1 + B_2'XB_1 \\ &\quad + \left[\begin{array}{c|c} A_1 & B_1 \\ \hline D_2'C_1 + B_2'XA_1 & 0 \end{array}\right] + \left[\begin{array}{c|c} A_2 & B_2 \\ \hline D_1'C_2 + B_1'X'A_2 & 0 \end{array}\right] \sim\end{aligned}$$

This always works if both A_1 and A_2 are stable.

H_2 CARE

Continuous-time algebraic Riccati equation, aka CARE, is a quadratic matrix equation of the form

$$A'X + XA + C'C - (XB + C'D)(D'D)^{-1}(B'X + D'C) = 0.$$

Its solution is said to be *stabilizing* if the matrix $A + BF$ is Hurwitz, where $F := -(D'D)^{-1}(B'X + D'C)$. The stabilizing solution exists iff

- (A, B) is stabilizable
- $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ has full column rank for all $s \in j\mathbb{R} \cup \infty$

and then $X = X' \geq 0$ (in fact, $X > 0$ iff (A, B, C, D) has no invariant zeros in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$).

H_2 CARE as Sylvester and as Lyapunov

The stabilizing solution X verifies the Sylvester equation

$$(A + BF)'X + XA + (C + DF)'C = 0 = A'X + X(A + BF) + C'(C + DF)$$

Alternatively, X verifies the Lyapunov equation

$$(A + BF)'X + X(A + BF) + (C + DF)'(C + DF) = 0,$$

i.e. X is the observability Gramian of $(A + BF, B, C + DF, D)$, which is the numerator of a right coprime factorization

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{c|c} A + BF & BR \\ \hline C + DF & DR \end{array} \right] \left[\begin{array}{c|c} A + BF & BR \\ \hline F & R \end{array} \right]^{-1}$$

for any square and nonsingular R .

H_2 DARE

Discrete-time algebraic Riccati equation, aka DARE, is a quadratic matrix equation of the form

$$X = A'XA + C'C - (C'D + A'XB)(D'D + B'XB)^{-1}(D'C + B'XA).$$

Its solution is said to be *stabilizing* if the matrix $D'D + B'XB$ is nonsingular and $A + BF$ is Schur, where $F := -(D'D + B'XB)^{-1}(D'C + B'XA)$. The stabilizing solution exists iff

- (A, B) is stabilizable
- $\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}$ has full column rank for all $z \in \mathbb{T}$

and then $X = X' \geq 0$ (in fact, $X > 0$ iff (A, B, C, D) has no invariant zeros in $\mathbb{D}_1$).

H_2 DARE as Stein and as Lyapunov

The stabilizing solution X verifies the Stein equation

$$X = (A + BF)'XA + (C + DF)'C = A'X(A + BF) + C'(C + DF)$$

Alternatively, X verifies the Lyapunov equation

$$X = (A + BF)'X(A + BF) + (C + DF)'(C + DF),$$

i.e. X is the observability Gramian of $(A + BF, B, C + DF, D)$, which is the numerator of a right coprime factorization

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] = \left[\begin{array}{c|c} A + BF & BR \\ \hline C + DF & DR \end{array} \right] \left[\begin{array}{c|c} A + BF & BR \\ \hline F & R \end{array} \right]^{-1},$$

for any square and nonsingular R , again.

Spectral factorization via H_2 CARE

Let

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

and assume that A is Hurwitz and $\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$ is left invertible $\forall s \in j\mathbb{R} \cup \infty$. It is required to construct a spectral factor W , $W^{-1} \in H_\infty$ such that

$$G^\sim(s)G(s) = W^\sim(s)W(s).$$

It is more convenient to cast it as finding a right coprime factorization

$$G(s) = N(s)M^{-1}(s), \quad N, M \in H_\infty$$

with an *inner* $N(s)$, i.e. such that $N^\sim(s)N(s) = I$ (as $G \in H_\infty$, $M^{-1} \in H_\infty$ too and then $W = M^{-1}$ is the spectral factor). The factorization $G = NW$ is called *inner-outer factorization* of G .

Remark: W.l.o.g. we may assume that $D'D = I$. Otherwise, just substitute $\begin{bmatrix} B \\ D \end{bmatrix} \rightarrow \begin{bmatrix} B \\ D \end{bmatrix} (D'D)^{-1/2}$ and then scale resulting $W(s) \rightarrow (D'D)^{1/2} W(s)$.

Spectral factorization via H_2 CARE

Let

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

and assume that A is Hurwitz and $\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$ is left invertible $\forall s \in j\mathbb{R} \cup \infty$. It is required to construct a spectral factor W , $W^{-1} \in H_\infty$ such that

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It is more convenient to cast it as finding a right coprime factorization

$$G(s) = N(s)M^{-1}(s), \quad N, M \in H_\infty$$

with an *inner* $N(s)$, i.e. such that $N^\sim(s)N(s) = I$ (as $G \in H_\infty$, $M^{-1} \in H_\infty$ too and then $W = M^{-1}$ is the spectral factor). The factorization $G = NW$ is called **inner-outer factorization** of G .

Remark: W.l.o.g. we may assume that $D'D = I$. Otherwise, just substitute $\begin{bmatrix} B \\ D \end{bmatrix} \rightarrow \begin{bmatrix} B \\ D \end{bmatrix} (D'D)^{-1/2}$ and then scale resulting $W(s) \rightarrow (D'D)^{1/2} W(s)$.

Spectral factorization via H_2 CARE (contd)

It is known that with any F such that $A + BF$ is Hurwitz

$$\begin{bmatrix} N(s) \\ M(s) \end{bmatrix} = \left[\begin{array}{c|c} \frac{A + BF}{C + DF} & \frac{B}{D} \\ \hline F & I \end{array} \right]$$

By the formulae of Slide 10,

$$N^\sim(s)N(s) = I + \left[\begin{array}{c|c} \frac{A + BF}{B'X + D'C + F} & \frac{B}{0} \\ \hline & \end{array} \right] + \left[\begin{array}{c|c} \frac{A + BF}{B'X + D'C + F} & \frac{B}{0} \\ \hline & \end{array} \right]^\sim,$$

where $X \geq 0$ solves $(A + BF)'X + X(A + BF) = (C + DF)'(C + DF)$. The choice $F = -(B'X + D'C)$ results in $N^\sim(s)N(s) = I$, as required. We only need to ensure that this F is stabilizing. This is indeed true, as long as X is the *stabilizing* solution to the CARE

$$A'X + XA + C'C - (XB + C'D)(B'X + D'C) = 0,$$

which is always possible under our assumptions about $G(s)$.

Continuous-time spectral factorization

Let

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

and assume that A is Hurwitz and $\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$ is left invertible $\forall s \in j\mathbb{R} \cup \infty$. It is required to construct a spectral factor W , $W^{-1} \in H_\infty$ such that

$$G^\sim(s)G(s) = W^\sim(s)W(s).$$

The solution is

$$\begin{bmatrix} G(s) \\ I \end{bmatrix} W^{-1}(s) = \begin{bmatrix} A+BF & B \\ \hline C+DF & D \\ F & I \end{bmatrix} (D'D)^{-1/2}$$

where $X \geq 0$ is the stabilizing solution to the DARE

$$A'X + XA + C'C - (XB + C'D)(D'D)^{-1}(B'X + D'C) = 0,$$

and $F = -(D'D)^{-1}(B'X + D'C)$.

Continuous-time co-spectral factorization

Let

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

and assume that A is Hurwitz and $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ is right invertible $\forall s \in j\mathbb{R} \cup \infty$. It is required to construct a co-spectral factor W , $W^{-1} \in H_\infty$ such that

$$G(s)G^\sim(s) = W(s)W^\sim(s).$$

The solution is

$$W^{-1}(s) \begin{bmatrix} G(s) & I \end{bmatrix} = (DD')^{-1/2} \left[\begin{array}{c|c} A + LC & B + LD & L \\ \hline C & D & I \end{array} \right]$$

where $Y \geq 0$ is the stabilizing solution to the DARE

$$AY + YA' + BB' - (YC' + BD')(DD')^{-1}(CY + DB') = 0$$

and $L = -(YC' + BD')(DD')^{-1}$.

Discrete-time spectral factorization

Let

$$\bar{G}(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

and assume that A is Schur and $\begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}$ is left invertible for all $z \in \mathbb{T}$. It is required to construct a spectral factor $\bar{W}, \bar{W}^{-1} \in H_\infty$ such that

$$\bar{G}^\sim(z) \bar{G}(z) = \bar{W}^\sim(z) \bar{W}(z).$$

The solution is

$$\begin{bmatrix} \bar{G}(z) \\ I \end{bmatrix} \bar{W}^{-1}(z) = \begin{bmatrix} A + BF & B \\ \hline C + DF & D \\ F & I \end{bmatrix} (D'D + B'XB)^{-1/2}$$

where $X \geq 0$ is the stabilizing solution to the DARE

$$X = A'XA + C'C - (C'D + A'XB)(D'D + B'XB)^{-1}(D'C + B'XA)$$

and $F = -(D'D + B'XB)^{-1}(D'C + B'XA)$.

Discrete-time co-spectral factorization

Let

$$\bar{G}(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

and assume that A is Schur and $\begin{bmatrix} A^{-z'} & B \\ C & D \end{bmatrix}$ is right invertible for all $z \in \mathbb{T}$. It is required to construct a co-spectral factor $\bar{W}$, $\bar{W}^{-1} \in H_\infty$ such that

$$\bar{G}(z)\bar{G}^\sim(z) = \bar{W}(z)\bar{W}^\sim(z).$$

The solution is

$$\bar{W}^{-1}(z) \left[\begin{array}{c|c} \bar{G}(z) & I \end{array} \right] = (DD' + CYC')^{-1/2} \left[\begin{array}{c|c} A + LC & B + LD & L \\ \hline C & D & I \end{array} \right]$$

where $Y \geq 0$ is the stabilizing solution to the DARE

$$Y = AYA' + BB' - (BD' + AYC')(DD' + CYC')^{-1}(DB_1' + CYA')$$

and $L = -(BD' + AYC')(DD' + CYC')^{-1}$.

Content

State-space machinery

Continuous-time H_2 estimation

Discrete-time H_2 estimation

Continuous-time H_2 estimation

Let

$$\begin{bmatrix} G_v(s) \\ G_y(s) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_v & 0 \\ C_y & D_y \end{array} \right]$$

where (C_y, A) is detectable and $\begin{bmatrix} A-sI & B \\ C_y & D_y \end{bmatrix}$ has full row rank $\forall s \in j\mathbb{R} \cup \infty$. We aim at designing $K \in H_\infty \cap H_2$ that stabilizes

$$G_e(s) = G_v(s) - K(s)G_y(s)$$

and minimizes

$$\|G_e\|_2 = \|G_v - KG_y\|_2.$$

This would be a one-sided H_2 model matching problem if G_v and G_y were stable. So the first step is to regain stability.

Stabilization

Let L_y be any matrix such that $A + L_y C_y$ is Hurwitz and $\det R_y \neq 0$. Define

$$\begin{bmatrix} N_v(s) & M_v(s) \\ N_y(s) & M_y(s) \end{bmatrix} = \left[\begin{array}{c|cc} A + L_y C_y & B + L_y D_y & L_y \\ \hline C_v & 0 & 0 \\ R_y C_y & R_y D_y & R_y \end{array} \right]$$

so that $G_y = M_y^{-1} N_y$ and $G_v = N_v - M_v G_y$. Then a stable K stabilizes the error system iff

$$K(s) = -M_v(s) + Q(s)M_y(s)$$

for some $Q \in H_\infty$ (and with $Q \in H_2 \implies K \in H_2$), in which case

$$G_e(s) = N_v(s) - Q(s)N_y(s).$$

This is now a one-sided H_2 model matching problem, whose solution verifies

$$\langle N_v - Q_{\text{opt}} N_y, Q N_y \rangle_2 = \langle N_v N_y^\sim - Q_{\text{opt}} N_y N_y^\sim, Q \rangle_2 = 0 \quad \forall Q \in H_2.$$

Choice of L_y and R_y

The solution is greatly simplified if we

- choose L_y and R_y so that $N_y(s)$ is co-inner, i.e. $N_y(s)N_y^\sim(s) = I$, which is the case (cf. $W^{-1}G$ in Slide 20) with

$$L_y = -(YC'_y + BD'_y)(D_y D'_y)^{-1} \quad \text{and} \quad R_y = (D_y D'_y)^{-1/2},$$

where $Y \geq 0$ is the stabilizing solution to the CARE

$$\begin{aligned} 0 &= AY + YA' + BB' - (YC'_y + BD'_y)(D_y D'_y)^{-1}(C_y Y + D_y B') \\ &= (A + L_y C_y)Y + Y(A + L_y C_y)' + (B + L_y D_y)(B + L_y D_y)' \end{aligned}$$

In this case, the solution verifies

$$\langle N_v N_y^\sim - Q_{\text{opt}}, Q \rangle_2 = 0 \quad \forall Q \in H_2,$$

so that $Q_{\text{opt}} = \text{proj}_{H_2} N_v N_y^\sim$.

Calculating $N_v N_y^\sim$ et cetera

By the formula of Slide 9 and because $(B + L_y D_y)D_y' + Y C_y' = 0$,

$$\begin{aligned} N_v(s) N_y^\sim(s) &= \left[\begin{array}{c|c} \frac{A + L_y C_y}{C_v} & 0 \\ \hline & 0 \end{array} \right] R_y' + \left[\begin{array}{c|c} \frac{A + L_y C_y}{C_y} & Y C_v' \\ \hline & 0 \end{array} \right]^\sim R_y' \\ &= \left[\begin{array}{c|c} \frac{A + L_y C_y}{C_y} & Y C_v' \\ \hline & 0 \end{array} \right]^\sim R_y' \in H_2^\perp. \end{aligned}$$

Therefore, $Q_{\text{opt}} = \text{proj}_{H_2} N_v N_y^\sim = 0$, so that

$$K_{\text{opt}}(s) = -M_v(s) = - \left[\begin{array}{c|c} \frac{A + L_y C_y}{C_v} & L_y \\ \hline & 0 \end{array} \right]$$

(Kalman filter) and

$$\|G_e\|_2^2 = \|N_v\|_2^2 = \left\| \left[\begin{array}{c|c} \frac{A + L_y C_y}{C_v} & B + L_y D_y \\ \hline & 0 \end{array} \right] \right\|_2^2 = \text{tr}(C_v Y C_v'),$$

because Y is the controllability Gramian of $N_v(s)$.

Suboptimal solutions

It follows from the condition $\langle N_v - Q_{\text{opt}} N_y, Q N_y \rangle_2 = \langle N_v, Q N_y \rangle_2 = 0$ that

$$\|N_v - Q N_y\|_2^2 = \|N_v\|_2^2 + \|Q N_y\|_2^2 = \|N_v\|_2^2 + \|Q\|_2^2, \quad \forall Q \in H_2$$

(by Pythagoras and co-inner N_y). Thus, given $\gamma > \|N_v\|_2$, all γ -suboptimal estimators are

$$\begin{aligned} K(s) &= -M_v(s) + Q(s)M_y(s) \\ &= -\left[\frac{A + L_y C_y}{C_v} \middle| \frac{L_y}{0} \right] + Q(s) (D_y D_y')^{-1/2} \left[\frac{A + L_y C_y}{C_y} \middle| \frac{L_y}{I} \right] \end{aligned}$$

for any Q such that $\|Q\|_2^2 \leq \gamma^2 - \text{tr}(C_v Y C_v')$. This parametrization can be extended to time-varying estimators (with an appropriate norm definition).

$$K = -\left[\frac{A + L_y C_y}{C_v} \middle| \frac{L_y}{0} \right] + Q (D_y D_y')^{-1/2} \left[\frac{A + L_y C_y}{C_y} \middle| \frac{L_y}{I} \right]$$

for any Q such that $\|Q\|_2^2 \leq \gamma^2 - \text{tr}(C_v Y C_v')$.

Suboptimal solutions

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$$\|N_v - Q N_y\|_2^2 = \|N_v\|_2^2 + \|Q N_y\|_2^2 = \|N_v\|_2^2 + \|Q\|_2^2, \quad \forall Q \in H_2$$

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$$\begin{aligned} K(s) &= -M_v(s) + Q(s)M_y(s) \\ &= -\left[\frac{A + L_y C_y}{C_v} \middle| \frac{L_y}{0} \right] + Q(s) (D_y D'_y)^{-1/2} \left[\frac{A + L_y C_y}{C_y} \middle| \frac{L_y}{I} \right] \end{aligned}$$

for any Q such that $\|Q\|_2^2 \leq \gamma^2 - \text{tr}(C_v Y C'_v)$. This parametrization can be extended to *time-varying* estimators (with an appropriate norm definition):

$$\mathcal{K} = -\left[\frac{A + L_y C_y}{C_v} \middle| \frac{L_y}{0} \right] + \mathcal{Q} (D_y D'_y)^{-1/2} \left[\frac{A + L_y C_y}{C_y} \middle| \frac{L_y}{I} \right]$$

for any $\mathcal{Q}$ such that $\|\mathcal{Q}\|_2^2 \leq \gamma^2 - \text{tr}(C_v Y C'_v)$.

Content

State-space machinery

Continuous-time H_2 estimation

Discrete-time H_2 estimation

Discrete-time H_2 estimation

Let

$$\begin{bmatrix} \bar{G}_v(z) \\ \bar{G}_y(z) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_v & D_v \\ C_y & D_y \end{array} \right]$$

where (C_y, A) is detectable and $\begin{bmatrix} A - zI & B \\ C_y & D_y \end{bmatrix}$ has full row rank $\forall z \in \mathbb{T}$. We aim at designing $\bar{K} \in H_\infty \cap H_2$ that stabilizes

$$\bar{G}_e(z) = \bar{G}_v(z) - \bar{K}(z)\bar{G}_y(z)$$

and minimizes

$$\|\bar{G}_e\|_2 = \|\bar{G}_v - \bar{K}\bar{G}_y\|_2.$$

This would be a one-sided H_2 model matching problem if $\bar{G}_v$ and $\bar{G}_y$ were stable. So the first step is to regain stability.

Stabilization

Let L_y be any matrix such that $A + L_y C_y$ is Schur and $\det R_y \neq 0$. Define

$$\begin{bmatrix} \bar{N}_v(z) & \bar{M}_v(z) \\ \bar{N}_y(z) & \bar{M}_y(z) \end{bmatrix} = \left[\begin{array}{c|cc} A + L_y C_y & B + L_y D_y & L_y \\ \hline C_v + S_y C_y & D_v + S_y D_y & S_y \\ R_y C_y & R_y D_y & R_y \end{array} \right]$$

so that $\bar{G}_y = \bar{M}_y^{-1} \bar{N}_y$ and $\bar{G}_v = \bar{N}_v - \bar{M}_v \bar{G}_y$. A stable $\bar{K}$ stabilizes the error system iff

$$\bar{K}(z) = -\bar{M}_v(z) + \bar{Q}(z) \bar{M}_y(z)$$

for some $\bar{Q} \in H_\infty$ (and with $\bar{Q} \in H_2 \implies \bar{K} \in H_2$), in which case

$$\bar{G}_e(z) = \bar{N}_v(z) - \bar{Q}(z) \bar{N}_y(z).$$

This is now a one-sided H_2 model matching problem, whose solution verifies

$$\langle \bar{N}_v - \bar{Q}_{\text{opt}} \bar{N}_y, \bar{Q} \bar{N}_y \rangle_2 = \langle \bar{N}_v \bar{N}_y^\sim - \bar{Q}_{\text{opt}} \bar{N}_y \bar{N}_y^\sim, \bar{Q} \rangle_2 = 0 \quad \forall \bar{Q} \in H_2.$$

Choice of L_y and R_y

The solution is greatly simplified if we

- choose L_y and R_y so that $\bar{N}_y(z)$ is co-inner, i.e. $\bar{N}_y(z)\bar{N}_y^\sim(z) = I$, which is the case (cf. $\bar{W}^{-1}\bar{G}$ in Slide 22) with

$$L_y = -(BD_y' + AYCY)(D_yD_y' + C_yYC_y')^{-1} \quad \text{and} \quad R_y = (D_yD_y' + C_yYC_y')^{-1/2}$$

where $Y \geq 0$ is the stabilizing solution to the CARE

$$\begin{aligned} Y &= AYA' + BB' - (BD_y' + AYCY')(D_yD_y' + C_yYC_y')^{-1}(D_yB' + C_yYA') \\ &= (A + L_yC_y)Y(A + L_yC_y)' + (B + L_yD_y)(B + L_yD_y)' \end{aligned}$$

In this case, the solution verifies

$$\langle \bar{N}_v \bar{N}_y^\sim - \bar{Q}_{\text{opt}}, \bar{Q} \rangle_2 = 0 \quad \forall \bar{Q} \in H_2,$$

so that $\bar{Q}_{\text{opt}} = \text{proj}_{H_2} \bar{N}_v \bar{N}_y^\sim$.

Calculating $\bar{N}_v \bar{N}_y^\sim$ et cetera

By the formula of Slide 11 and since $(B + L_y D_y) D_y' + (A + L_y C_y) Y C_y' = 0$,

$$\begin{aligned}\bar{N}_v(z) \bar{N}_y^\sim(z) &= \left[\begin{array}{c|c} \frac{A + L_y C_y}{C_v} & 0 \\ \hline & D_a \end{array} \right] R_y' + \left[\begin{array}{c|c} \frac{A + L_y C_y}{C_y} & B_a \\ \hline & 0 \end{array} \right]^\sim R_y' \\ &= D_a R_y' + \left[\begin{array}{c|c} \frac{A + L_y C_y}{C_y} & B_a \\ \hline & 0 \end{array} \right]^\sim R_y'\end{aligned}$$

where $B_a := (B + L D_y)(D_v + S_y D_y)' + (A + L C_y) Y (C_v + S_y C_y)'$ and

$$D_a := S_y (D_y D_y' + C_y Y C_y') + D_v D_y' + C_v Y C_y'.$$

To repeat the steps of the continuous-time design, pick

$$S_y = -(D_v D_y' + C_v Y C_y')(D_y D_y' + C_y Y C_y')^{-1},$$

which yields $D_a = 0$ and, consequently, $\bar{Q}_{\text{opt}} = \text{proj}_{H_2} \bar{N}_v \bar{N}_y^\sim = 0$.

Calculating $\bar{N}_v \bar{N}_y^\sim$ et cetera (contd)

Thus, the discrete Kalman filter is

$$\bar{K}_{\text{opt}}(z) = -\bar{M}_v(z) = - \left[\begin{array}{c|c} \frac{A + L_y C_y}{C_v} & \frac{L_y}{0} \end{array} \right] - S_y \left[\begin{array}{c|c} \frac{A + L_y C_y}{C_y} & \frac{L_y}{I} \end{array} \right]$$

and $\|\bar{G}_e\|_2^2 = \|\bar{N}_v\|_2^2$. To calculate the latter, denote $T_y := S_y R_y^{-1}$, so that

$$\bar{N}_v(z) = \left[\begin{array}{c|c} \frac{A + L_y C_y}{C_v} & \frac{B + L_y D_y}{D_v} \end{array} \right] + T_y \bar{N}_y(z) =: \bar{N}_0(z) + T_y \bar{N}_y(z)$$

Now,

$$\begin{aligned} \|\bar{N}_0\|_2^2 &= \|\bar{N}_v - T_y \bar{N}_y\|_2^2 = \|\bar{N}_v\|_2^2 + \|T_y \bar{N}_y\|_2^2 - 2 \operatorname{Re} \langle \bar{N}_v \bar{N}_y^\sim, T_y \rangle \\ &= \|\bar{N}_v\|_2^2 + \|T_y\|_2^2, \end{aligned}$$

because $\bar{N}_y(z)$ is co-inner and $\bar{N}_v \bar{N}_y^\sim \in H_2^\perp$ and $T_y \in H_2$ (static). Hence,

$$\|\bar{G}_e\|_2^2 = \|\bar{N}_0\|_2^2 - \|T_y\|_2^2 = \operatorname{tr}(D_v D_v' + C_v Y C_v' - S_y R^{-2} S_y').$$

Discrete-time strictly causal H_2 estimation

Consider the same H_2 filtering problem as before, but now

- imposing an additional constraint that K is **strictly causal**.

Stability still requires

$$\bar{K}(z) = -\bar{M}_v(z) + \bar{Q}(z)\bar{M}_y(z)$$

with the corresponding model matching with the error system

$$\bar{G}_e(z) = \bar{N}_v(z) - \bar{Q}(z)\bar{N}_y(z).$$

Two questions to be addressed:

1. Under what condition on $\bar{Q}$ the resulting $\bar{K}$ is strictly causal?
2. Could this condition be incorporated into the minimization of $\|\bar{G}_e\|_2$?

Strict causality and stabilization

The transfer function of any causal system LTI system $\mathcal{G}$ is of the form

$$\bar{G}(z) = \bar{g}[0] + \bar{g}[1]z^{-1} + \bar{g}[2]z^{-2} + \dots$$

and it is strictly causal iff its feedthrough term $\bar{g}[0] = \bar{G}(\infty) = 0$. Now,

$$\begin{aligned}\bar{K}(z) &= -(\bar{m}_v[0] + \bar{m}_v[1]z^{-1} + \dots) \\ &\quad + (\bar{q}[0] + \bar{q}[1]z^{-1} + \dots)(\bar{m}_y[0] + \bar{m}_y[1]z^{-1} + \dots) \\ &= -\bar{m}_v[0] + \bar{q}[0]\bar{m}_y[0] + (\times)z^{-1}\end{aligned}$$

Thus, $\bar{K}$ is strictly causal iff

$$\bar{q}[0] = \bar{m}_v[0](\bar{m}_y[0])^{-1} = S_y R_y^{-1}.$$

It is convenient to pick $S_y = 0$, in which case $\bar{M}_v$ is strictly causal and

- $\bar{K}$ is strictly causal iff $\bar{Q}$ is strictly causal.

We have a space for that...

The subset of strictly causal systems in H_2 is clearly a *subspace*, defined as

$$z^{-1}H_2^{p \times m} := \{G \in H_2^{p \times m} \mid zG(z) \in H_2^{p \times m}\}$$

i.e. it comprises H_2 functions that decay at least as fast as $\frac{1}{|z|}$ as $|z|$ grows.
The z -transform is an isomorphic mapping $\bar{\mathcal{U}}\ell_{2+}(\mathbb{Z}) \rightarrow z^{-1}H_2$.

Thus, the solution to the strictly causal estimation problem must satisfy

$$\langle \bar{N}_x - \bar{Q}_{\text{opt}} \bar{N}_y, \bar{Q} \bar{N}_y \rangle_2 = \langle \bar{N}_x \bar{N}_y^* - \bar{Q}_{\text{opt}} \bar{N}_y \bar{N}_y^*, \bar{Q} \rangle_2 = 0 \quad \forall \bar{Q} \in z^{-1}H_2,$$

and it is still advantageous to render $\bar{N}_y(z)$ co-inner, so that the optimality condition reads $\langle \bar{N}_x \bar{N}_y^* - \bar{Q}_{\text{opt}}, \bar{Q} \rangle_2 = 0, \forall \bar{Q} \in z^{-1}H_2$, from which

$$\bar{Q}_{\text{opt}} = \text{proj}_{z^{-1}H_2} \bar{N}_x \bar{N}_y^*,$$

which should eliminate the part of the impulse response in $\mathbb{Z} \setminus \mathbb{N}_0$.

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The subset of strictly causal systems in H_2 is clearly a *subspace*, defined as

$$z^{-1}H_2^{p \times m} := \{G \in H_2^{p \times m} \mid zG(z) \in H_2^{p \times m}\}$$

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Thus, the solution to the strictly causal estimation problem must satisfy

$$\langle \bar{N}_v - \bar{Q}_{\text{opt}} \bar{N}_y, \bar{Q} \bar{N}_y \rangle_2 = \langle \bar{N}_v \bar{N}_y^\sim - \bar{Q}_{\text{opt}} \bar{N}_y \bar{N}_y^\sim, \bar{Q} \rangle_2 = 0 \quad \forall \bar{Q} \in z^{-1}H_2.$$

and it is still advantageous to render $\bar{N}_y(z)$ co-inner, so that the optimality condition reads $\langle \bar{N}_v \bar{N}_y^\sim - \bar{Q}_{\text{opt}}, \bar{Q} \rangle_2 = 0$, $\forall \bar{Q} \in z^{-1}H_2$, from which

$$\bar{Q}_{\text{opt}} = \text{proj}_{z^{-1}H_2} \bar{N}_v \bar{N}_y^\sim,$$

which should eliminate the part of the impulse response in $\mathbb{Z} \setminus \mathbb{N}$.

Calculating $\bar{N}_v \bar{N}_y^\sim$ et cetera

We already know that

$$\bar{N}_v(z) \bar{N}_y^\sim(z) = \left(D_a + \left[\frac{A + L_y C_y}{C_y} \middle| \frac{B_a}{0} \right]^\sim \right) R_y',$$

which is orthogonal to $z^{-1}H_2$. Hence, $\bar{Q}_{\text{opt}} = \text{proj}_{z^{-1}H_2} \bar{N}_v \bar{N}_y^\sim = 0$ and we recover all results for the causal case, modulo $S_y = 0$. Namely, the optimal strictly causal estimator has the transfer function

$$\bar{K}_{\text{opt}}(z) = -\bar{M}_v(z) = - \left[\frac{A + L_y C_y}{C_v} \middle| \frac{L_y}{0} \right]$$

and the optimal cost

$$\|\bar{G}_e\|_2^2 = \|\bar{N}_v\|_2^2 = \text{tr}(D_v D_v' + C_v Y C_v').$$

The increase of the cost with respect to the causal case is

$$\text{tr}((D_v D_v' + C_v Y C_v')(D_y D_y' + C_y Y C_y')^{-1}(D_y D_y' + C_y Y C_y')).$$