

Advanced Sampled-Data Control (038798) lecture no. 7

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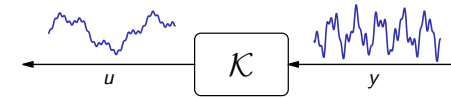
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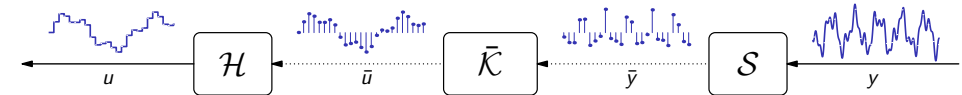
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Causal sampled-data approximation problem

Given a **causal** analog system (controller?)



find its “good” **causal** approximation of the form



We may again consider the following 4 situations:

1. S and H are fixed, find $\bar{K}$
2. H is fixed, find $\bar{K}$ and S (or just S , $\bar{K}$ can always be absorbed in it)
3. S is fixed, find $\bar{K}$ and H (or just H , $\bar{K}$ can always be absorbed in it)
4. find S , $\bar{K}$, and H (or just S and H)

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Content

H_2 model-matching

Lifted systems in z domain

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Sampled-data H_2 approximations for Type III problems

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General causal model matching problem

The problem of finding a stable and causal K minimizing a size of

$$\mathcal{G}_e = \mathcal{G}_1 - \mathcal{G}_2 K \mathcal{G}_3$$

for given causal $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ under constraint that the error system $\mathcal{G}_e$ is stable is called the **causal model matching** problem.

Why is it different: Causality implies

- impulse responses with support in either $\mathbb{R}^+$ or $\mathbb{Z}^+$

for involved systems. The Fourier transform is no longer useful here, there is no intuitive way to characterize such signals in the frequency domain. So

- Laplace / z -transforms should be used instead,

as belonging to $\mathbb{R}^+$ or $\mathbb{Z}^+$ determines the RoC, which, in turn, translates to the analyticity of transformed functions in either $\mathbb{C}_\alpha$ or $\mathbb{C} \setminus \mathbb{D}_\alpha$.

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$H_2(\mathbb{C}_0)$ space of matrix-valued functions

The space $H_2^{p \times m}(\mathbb{C}_0)$, or just H_2 , is defined as

$$H_2^{p \times m}(\mathbb{C}_0) := \left\{ G : \mathbb{C}_0 \rightarrow \mathbb{C}^{p \times m} \mid G(s) \text{ is analytic in } \mathbb{C}_0 \text{ and} \right. \\ \left. \text{s.t. } \|G\|_{H_2} := \sup_{\xi > 0} \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|G(\xi + j\omega)\|_F^2 d\omega \right)^{1/2} < \infty \right\}$$

For every $G \in H_2$, the function $\tilde{G}(j\omega) = \lim_{\xi \downarrow 0} G(\xi + j\omega)$ exists for almost all ω , belongs to $L_2(j\mathbb{R})$, and $\|\tilde{G}\|_2 = \|G\|_{H_2}$. It is customary to identify G with $\tilde{G}$ and regard

- H_2 as a closed subspace of $L_2(j\mathbb{R})$,

so H_2 is a Hilbert space with the inner product

$$\langle G_1, G_2 \rangle_2 = \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}([G_2(j\omega)]' G_1(j\omega)) d\omega$$

and the norm $\|G\|_2^2 = \langle G, G \rangle_2$.

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Paley-Wiener

The Laplace transform is an isomorphic mapping $L_{2+}(\mathbb{R}) \rightarrow H_2$ preserving sizes,

$$\underbrace{\int_{\mathbb{R}^+} \|g(t)\|_F^2 dt}_{\|g\|_2^2} = \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} \|G(j\omega)\|_F^2 d\omega}_{\|G\|_2^2},$$

and angles,

$$\underbrace{\int_{\mathbb{R}^+} \text{tr}([g_2(t)]' g_1(t)) dt}_{\langle g_1, g_2 \rangle_2} = \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}([G_2(j\omega)]' G_1(j\omega)) d\omega}_{\langle G_1, G_2 \rangle_2}.$$

As $L_2(\mathbb{R}) = L_{2+}(\mathbb{R}) \oplus L_{2-}(\mathbb{R})$, we can present

$$L_2(j\mathbb{R}) = H_2 \oplus H_2^\perp,$$

where $H_2^\perp := H_2(\mathbb{C} \setminus \bar{\mathbb{C}}_0)$ contains the Laplace transforms of $L_{2-}(\mathbb{R}) \dots$

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H_2 space of continuous-time LTI system

Define the space of LTI systems with transfer functions in $H_2(\mathbb{C}_0)$ by H_2 . This is a Hilbert space with

$$\langle \mathcal{G}_1, \mathcal{G}_2 \rangle := \langle G_1, G_2 \rangle_2 = \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}([G_2(j\omega)]' G_1(j\omega)) d\omega$$

and

$$\|\mathcal{G}\|_2 := \|G\|_2 = \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|G(j\omega)\|_F^2 d\omega \right)^{1/2}.$$

Important properties:

- $\mathcal{T}H_2 \subset H_2$ iff $\mathcal{T} \in H_\infty$
- $\mathcal{T}H_2 = H_2$ iff $\mathcal{T}, \mathcal{T}^{-1} \in H_\infty$
- all properties inherited from L_2 , e.g. $\langle \mathcal{G}_1, \mathcal{T}_1 \mathcal{G}_2 \mathcal{T}_2 \rangle_2 = \langle \mathcal{T}_1^* \mathcal{G}_1 \mathcal{T}_2^*, \mathcal{G}_2 \rangle_2$

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$H_2(\mathbb{C} \setminus \bar{\mathbb{D}}_1)$ space of matrix-valued functions

The space $H_2^{p \times m}(\mathbb{C} \setminus \bar{\mathbb{D}}_1)$, or just H_2 , is defined as

$$H_2^{p \times m}(\mathbb{C} \setminus \bar{\mathbb{D}}_1) := \left\{ \bar{G} : \mathbb{C} \setminus \bar{\mathbb{D}}_1 \rightarrow \mathbb{C}^{p \times m} \mid \bar{G}(z) \text{ is analytic in } \mathbb{C} \setminus \bar{\mathbb{D}}_1 \text{ and} \right. \\ \left. \text{s.t. } \|\bar{G}\|_{H_2} := \sup_{\rho > 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \|\bar{G}(\rho e^{j\theta})\|_F^2 d\theta \right)^{1/2} < \infty \right\}$$

For every $\bar{G} \in H_2$, the function $\tilde{G}(e^{j\theta}) = \lim_{\rho \downarrow 1} \bar{G}(\rho e^{j\theta})$ exists for almost all θ , belongs to $L_2(\mathbb{T})$, and $\|\tilde{G}\|_2 = \|\bar{G}\|_{H_2}$. It is customary to identify $\bar{G}$ with $\tilde{G}$ and regard

- H_2 as a closed subspace of $L_2(\mathbb{T})$,

so H_2 is a Hilbert space with the inner product

$$\langle \bar{G}_1, \bar{G}_2 \rangle_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{tr}([\bar{G}_2(e^{j\theta})]' \bar{G}_1(e^{j\theta})) d\theta$$

and the norm $\|\bar{G}\|_2^2 = \langle \bar{G}, \bar{G} \rangle_2$.

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Paley-Wiener

The z-transform is an isomorphic mapping $\ell_{2+}(\mathbb{Z}) \rightarrow H_2$ preserving sizes,

$$\underbrace{\sum_{i \in \mathbb{Z}^+} \|\bar{g}[i]\|_F^2}_{\|\bar{g}\|_2^2} = \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} \|\bar{G}(e^{j\theta})\|_F^2 d\theta}_{\|\bar{G}\|_2^2},$$

and angles,

$$\underbrace{\sum_{i \in \mathbb{Z}^+} \text{tr}((\bar{g}_2[i])' \bar{g}_1[i])}_{\langle \bar{g}_1, \bar{g}_2 \rangle_2} = \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{tr}([\bar{G}_2(e^{j\theta})]' \bar{G}_1(e^{j\theta})) d\theta}_{\langle \bar{G}_1, \bar{G}_2 \rangle_2}.$$

As $\ell_2(\mathbb{Z}) = \ell_{2+}(\mathbb{Z}) \oplus \ell_{2-}(\mathbb{Z})$, we can present

$$L_2(\mathbb{T}) = H_2 \oplus H_2^\perp,$$

where $H_2^\perp := H_2(\mathbb{D}_1)$ contains the z-transforms of $\ell_{2-}(\mathbb{Z})$...

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H_2 space of discrete-time LTI system

Define the space of LTI systems with transfer functions in $H_2(\mathbb{C} \setminus \bar{\mathbb{D}}_1)$ by H_2 . This is a Hilbert space with

$$\langle \bar{G}_1, \bar{G}_2 \rangle := \langle \bar{G}_1, \bar{G}_2 \rangle_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{tr}([\bar{G}_2(e^{j\theta})]' \bar{G}_1(e^{j\theta})) d\theta$$

and

$$\|\bar{G}\|_2 := \|\bar{G}\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \|\bar{G}(e^{j\theta})\|_F^2 d\theta \right)^{1/2}.$$

Important properties:

- $\bar{T}H_2 \subset H_2$ iff $\bar{T} \in H_\infty$
- $\bar{T}H_2 = H_2$ iff $\bar{T}, \bar{T}^{-1} \in H_\infty$
- all properties inherited from L_2 , e.g. $\langle \bar{G}_1, \bar{T}_1 \bar{G}_2 \bar{T}_2 \rangle_2 = \langle \bar{T}_1^* \bar{G}_1 \bar{T}_2^*, \bar{G}_2 \rangle_2$

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H_2 space: examples

$L_2(j\mathbb{R}) / L_2(\mathbb{T})$ functions that are not in H_2 :

- $G(s) = \frac{2}{s^2-1} \notin H_2$: not analytic in $\mathbb{C}_0$ (pole at $s = 1$);
- $G(s) = \frac{e^s}{s+1} \notin H_2$: because

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{|e^{\xi+j\omega}|^2}{|1+\xi+j\omega|^2} d\omega = \frac{e^{2\xi}}{2(1+\xi)}$$

is unbounded on $\xi > 0$;

- $\bar{G}(z) = \frac{2}{z(z+2)} \notin H_2$: not analytic in $\mathbb{C} \setminus \bar{\mathbb{D}}_1$ (pole at $z = -2$);
- $\bar{G}(z) = z \notin H_2$: because

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |\rho e^{j\theta}|^2 d\theta = \rho^2$$

is unbounded on $\rho > 1$.

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H_2 space: examples (contd)

Decompositions $H_2 \oplus H_2^\perp$:

- if $G \in L_2(j\mathbb{R})$ and rational, partial fraction expansion does the trick:

$$G(s) = \frac{2}{s^2-1} = \frac{2}{(s-1)(s+1)} = -\frac{1}{s+1} + \frac{1}{s-1};$$

- if $G \in L_2(j\mathbb{R})$ is irrational, it might be less obvious:

$$G(s) = \frac{e^s}{s+1} = \frac{e^{-1}}{s+1} + \frac{e^s - e^{-1}}{s+1};$$

- if $\bar{G} \in L_2(\mathbb{T})$ and rational, partial fraction expansion is again our tool:

$$\bar{G}(z) = \frac{2}{z(z+2)} = \frac{1}{z} - \frac{1}{z+2},$$

$$\bar{G}(z) = \frac{z^4 + z^3 + z^2 + z + 1}{(2z+1)(z+2)} = \frac{\frac{11}{24}}{2z+1} + \frac{15}{8} - \frac{\frac{11}{3}}{z+2} - \frac{3}{4}z + \frac{1}{2}z^2.$$

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General H_2 model matching problem

Assume that $\mathcal{G}_1 \in H_2$ and that $\mathcal{G}_2, \mathcal{G}_3 \in H_\infty$. The problem is:

$$\underset{\mathcal{K} \in H_2}{\text{minimize}} \quad \|\mathcal{G}_1 - \mathcal{G}_2 \mathcal{K} \mathcal{G}_3\|_2$$

(and we usually hope that the optimal $\mathcal{K}_{\text{opt}} \in H_\infty$ as well). A key property here is that

- if $\mathcal{K}$ is an arbitrary H_2 system, $\mathcal{G}_2 \mathcal{K} \mathcal{G}_3$ is a subspace of H_2 , so we can use the Projection Theorem and obtain the optimality condition

$$\langle \mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^* - \mathcal{G}_2^* \mathcal{G}_2 \mathcal{K}_{\text{opt}} \mathcal{G}_3 \mathcal{G}_3^*, \mathcal{K} \rangle_2 = 0, \quad \forall \mathcal{K} \in H_2$$

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General H_2 model matching problem: required alternations

Two nontrivial differences from the L_2 case:

1. In general, $\mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^* - \mathcal{G}_2^* \mathcal{G}_2 \mathcal{K}_{\text{opt}} \mathcal{G}_3 \mathcal{G}_3^* \notin H_2$ (it belongs to L_2). Hence, the solvability condition requires

$$\mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^* - \mathcal{G}_2^* \mathcal{G}_2 \mathcal{K}_{\text{opt}} \mathcal{G}_3 \mathcal{G}_3^* \in H_2^\perp$$

instead of the normal equation. This is a minor alternation though.

2. In general, $\mathcal{G}_2^* \mathcal{G}_2$ and $\mathcal{G}_3 \mathcal{G}_3^*$ are invertible in L_∞ , but not in H_∞ . Hence,

$$\mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^* - \mathcal{G}_2^* \mathcal{G}_2 \mathcal{K}_{\text{opt}} \mathcal{G}_3 \mathcal{G}_3^* = \mathcal{G}_0$$

might not be solvable in $\mathcal{K}_{\text{opt}} \in H_2$ for all $\mathcal{G}_0 \in H_2^\perp$. This requires nontrivial alternations.

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Spectral factorization

A workaround is to factorize, if possible,

$$\mathcal{G}_2^* \mathcal{G}_2 = \mathcal{W}_2^* \mathcal{W}_2 \quad \text{and} \quad \mathcal{G}_3 \mathcal{G}_3^* = \mathcal{W}_3 \mathcal{W}_3^*$$

for $\mathcal{W}_2, \mathcal{W}_2^{-1} \in H_\infty$ and $\mathcal{W}_3, \mathcal{W}_3^{-1} \in H_\infty$ (**spectral factorization**). With this factorization, the solvability condition reads

$$\langle \mathcal{W}_2^{-*} \mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^* \mathcal{W}_3^{-*} - \mathcal{W}_2 \mathcal{K}_{\text{opt}} \mathcal{W}_3, \mathcal{W}_2 \mathcal{K} \mathcal{W}_3 \rangle_2 = 0, \quad \forall \mathcal{K} \in H_2$$

or, $\mathcal{K}_{\text{opt}} = \mathcal{W}_2^{-1} \mathcal{Q}_{\text{opt}} \mathcal{W}_3^{-1}$, where $\mathcal{Q}_{\text{opt}}$ verifies

$$\langle \mathcal{W}_2^{-*} \mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^* \mathcal{W}_3^{-*} - \mathcal{Q}_{\text{opt}}, \mathcal{Q} \rangle_2 = 0, \quad \forall \mathcal{Q} \in H_2$$

This yields

$$\mathcal{Q}_{\text{opt}} = \text{proj}_{H_2} \mathcal{W}_2^{-*} \mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^* \mathcal{W}_3^{-*},$$

which merely cancels out the $H_2^\perp$ part of $\mathcal{W}_2^{-*} \mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^* \mathcal{W}_3^{-*}$.

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Optimal solution

Thus, the optimal $\mathcal{K}$ is

$$\mathcal{K}_{\text{opt}} = \mathcal{W}_2^{-1} \mathcal{Q}_{\text{opt}} \mathcal{W}_3^{-1},$$

where $\mathcal{Q}_{\text{opt}} = \text{proj}_{H_2} \mathcal{W}_2^{-*} \mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^* \mathcal{W}_3^{-*}$.

The optimal performance is then $\|\mathcal{G}_e\|_2^2 = \|\mathcal{G}_1\|_2^2 - \|\mathcal{G}_2 \mathcal{K}_{\text{opt}} \mathcal{G}_3\|_2^2$, but

$$\begin{aligned} \|\mathcal{G}_2 \mathcal{K}_{\text{opt}} \mathcal{G}_3\|_2^2 &= \langle \mathcal{G}_2 \mathcal{W}_2^{-1} \mathcal{Q}_{\text{opt}} \mathcal{W}_3^{-1} \mathcal{G}_3, \mathcal{G}_2 \mathcal{W}_2^{-1} \mathcal{Q}_{\text{opt}} \mathcal{W}_3^{-1} \mathcal{G}_3 \rangle_2 \\ &= \langle \mathcal{W}_2^{-*} \mathcal{G}_2^* \mathcal{G}_2 \mathcal{W}_2^{-1} \mathcal{Q}_{\text{opt}} \mathcal{W}_3^{-1} \mathcal{G}_3 \mathcal{G}_3^* \mathcal{W}_3^{-*}, \mathcal{Q}_{\text{opt}} \rangle_2 \\ &= \langle \mathcal{Q}_{\text{opt}}, \mathcal{Q}_{\text{opt}} \rangle_2 = \|\mathcal{Q}_{\text{opt}}\|_2^2. \end{aligned}$$

Hence,

$$\|\mathcal{G}_e\|_2^2 = \|\mathcal{G}_1\|_2^2 - \|\mathcal{Q}_{\text{opt}}\|_2^2.$$

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Transfer function of the adjoint system: continuous time

By definition,

$$\begin{aligned}\langle \mathcal{S}_1, \mathcal{G}\mathcal{S}_2 \rangle_2 &= \int_{\mathbb{R}} \text{tr} \left(\left[\int_{\mathbb{R}} g(t, s) s_2(s) ds \right]' s_1(t) \right) dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \text{tr} (s_2'(s) g'(t, s) s_1(t)) ds dt \\ &= \int_{\mathbb{R}} \text{tr} (s_2'(t) \int_{\mathbb{R}} g'(s, t) s_1(s)) ds dt \\ &= \langle \mathcal{G}^* \mathcal{S}_1, \mathcal{S}_2 \rangle_2.\end{aligned}$$

Thus, $\mathcal{G}^*$ has the impulse response $g'(s, t)$. In the causal LTI case,

– $\mathcal{G}^*$ has the impulse response $g'(-t)$, supported in $\mathbb{R}^-$ (i.e. anti-causal). Its transfer function, with RoC in a left-half plane, is then

$$G^*(s) = \int_{\mathbb{R}} g'(-t) e^{-st} dt = \left[\int_{\mathbb{R}} g(t) e^{-(s)t} dt \right]' = [G(-s)]' =: \mathbf{\color{red}G}^{\sim}(s),$$

which is called the **conjugate transfer function**.

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Transfer function of the adjoint system: discrete time

By definition,

$$\begin{aligned}\langle \bar{\mathcal{S}}_1, \bar{\mathcal{G}}\bar{\mathcal{S}}_2 \rangle_2 &= \sum_{i \in \mathbb{Z}} \text{tr} \left(\left[\sum_{j \in \mathbb{Z}} \bar{g}[i, j] \bar{s}_2[j] \right]' \bar{s}_1[i] \right) \\ &= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \text{tr} (\bar{s}_2'[j] \bar{g}'[i, j] \bar{s}_1[i]) = \sum_{i \in \mathbb{Z}} \text{tr} \left(\bar{s}_2'[i] \sum_{j \in \mathbb{Z}} \bar{g}'[j, i] \bar{s}_1[j] \right) \\ &= \langle \bar{\mathcal{G}}^* \bar{\mathcal{S}}_1, \bar{\mathcal{S}}_2 \rangle_2.\end{aligned}$$

Thus, $\bar{\mathcal{G}}^*$ has the impulse response $\bar{g}'[j, i]$. In the causal LTI case,

– $\bar{\mathcal{G}}^*$ has the impulse response $\bar{g}'[-i]$, supported in $\mathbb{Z}^-$ (i.e. anti-causal). Its transfer function, with RoC in the exterior of a disk, is

$$\bar{G}^*(z) = \sum_{i \in \mathbb{Z}} \bar{g}'[-i] z^{-i} = \left[\sum_{i \in \mathbb{Z}} \bar{g}'[i] \left(\frac{1}{z}\right)^{-i} \right]' = [\bar{G}(\frac{1}{z})]' =: \mathbf{\color{red}\bar{G}}^{\sim}(z),$$

which is called the **conjugate transfer function**.

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s-domain solution

Spectral factorizations:

$$G_2^{\sim}(s) G_2(s) = W_2^{\sim}(s) W_2(s) \quad \text{and} \quad G_3(s) G_3^{\sim}(s) = W_3(s) W_3^{\sim}(s)$$

for $W_2, W_2^{-1}, W_3, W_3^{-1} \in H_{\infty}$ (can be done whenever neither $G_2^{\sim}(s) G_2(s)$ nor $G_3(s) G_3^{\sim}(s)$ has poles or zeros in $j\mathbb{R} \cup j\infty$);

Transfer function of the optimal solution:

$$K_{\text{opt}}(s) = W_2^{-1}(s) Q_{\text{opt}}(s) W_3^{-1}(s),$$

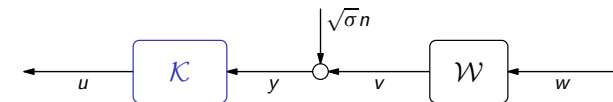
where $Q_{\text{opt}}(s) = \text{proj}_{H_2} [W_2^{\sim}(s)]^{-1} G_2^{\sim}(s) G_1(s) G_3^{\sim}(s) [W_3^{\sim}(s)]^{-1}$, with the projection done via truncating the part analytic in $\mathbb{C} \setminus \bar{\mathbb{C}}_0$.

Optimal cost:

$$\|G_e\|_2^2 = \|G_1\|_2^2 - \|Q_{\text{opt}}\|_2^2.$$

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Example: filtering out noise



The problem is to recover transmitted signal v from its noisy measurement. Here $\mathcal{W}$ is a weight. The error system is then

$$\mathcal{G}_e = \begin{bmatrix} \mathcal{W} & 0 \end{bmatrix} - \mathcal{K} \begin{bmatrix} \mathcal{W} & \sqrt{\sigma} I \end{bmatrix}$$

The spectral factorization to solve:

$$\sigma + W(s) W^{\sim}(s) = W_3(s) W_3^{\sim}(s)$$

for a bistable $W_3(s)$. The optimal filter:

$$K_{\text{opt}}(s) = \text{proj}_{H_2} (W(s) W^{\sim}(s) [W_3^{\sim}(s)]^{-1}) W_3^{-1}(s).$$

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Example: filtering out noise (contd)

Let $W(s) = \frac{1}{s+1}$, so that $W^\sim(s) = \frac{1}{-s+1}$ and then

$$\sigma + W(s)W^\sim(s) = \frac{-\sigma s^2 + \sigma + 1}{-s^2 + 1} = \underbrace{\frac{\sqrt{\sigma}s - \sqrt{\sigma+1}}{s-1}}_{W_3^\sim(s)} \underbrace{\frac{\sqrt{\sigma}s + \sqrt{\sigma+1}}{s+1}}_{W_3(s)}$$

Then,

$$\frac{W(s)W^\sim(s)}{W_3^\sim(s)} = \frac{1}{(s+1)(-\sqrt{\sigma}s + \sqrt{\sigma+1})} = \underbrace{\frac{1}{\sqrt{\sigma+1} + \sqrt{\sigma}}}_{\in H_2} - \underbrace{\frac{\sqrt{\sigma}}{\sqrt{\sigma+1} + \sqrt{\sigma}}}_{\in H_2^\perp}$$

and we end up with the causal optimal filter

$$K_{\text{opt}}(s) = \frac{1}{\sqrt{\sigma+1} + \sqrt{\sigma}} \frac{1}{\sqrt{\sigma}s + \sqrt{\sigma+1}}$$

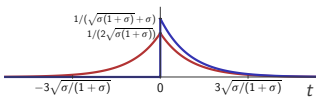
and the optimal cost $\|G_e\|_2^2 = \frac{1}{2} \left(1 - \left(\frac{1}{\sqrt{\sigma+1} + \sqrt{\sigma}} \right)^2 \right) = \sqrt{\sigma}(\sqrt{\sigma+1} - \sqrt{\sigma})$.

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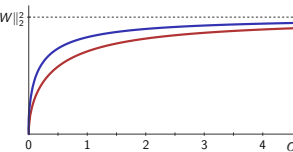
Example: filtering out noise (contd)

Compare causal and non-causal solutions:

— optimal filter:

$$\left. \begin{aligned} k_{\text{opt},c}(t) &= \frac{1}{\sqrt{\sigma(1+\sigma)} + \sigma} e^{-\sqrt{1+1/\sigma}t} \mathbb{1}(t) \\ k_{\text{opt},nc}(t) &= \frac{1}{2\sqrt{\sigma(1+\sigma)}} e^{-\sqrt{1+1/\sigma}|t|} \end{aligned} \right\} :$$


— optimal cost:

$$\left. \begin{aligned} \|G_{e,c}\|_2^2 &= \frac{\sqrt{\sigma}}{\sqrt{\sigma+1} + \sqrt{\sigma}} \\ \|G_{e,nc}\|_2^2 &= \frac{\sqrt{\sigma}}{2\sqrt{\sigma+1}} \end{aligned} \right\} :$$


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Transfer function of $\mathcal{L}_h \mathcal{G} \mathcal{L}_h^{-1}$

Let $\mathcal{G}$ be an h -periodic continuous-time system with the impulse response $g(t, s)$. Its lifting $\check{\mathcal{G}}$ is shift invariant and in the lifted z -domain, in RoC,

$$\check{Y}(z) = \check{\mathcal{G}}(z) \check{U}(z) \quad \text{or} \quad \check{Y}(z; \tau) = \int_0^h \check{\mathcal{G}}(z; \tau, \sigma) \check{U}(z; \sigma) d\sigma$$

with the operator $\check{\mathcal{G}}(z) : \{[0, h) \rightarrow \mathbb{F}^{n_u}\} \rightarrow \{[0, h) \rightarrow \mathbb{F}^{n_y}\}$ called the **lifted transfer function** and the function $\check{\mathcal{G}}(z; \tau, \sigma)$, which is the lifted z -transform of $g(t, s)$ with respect to its “ t ” variable, is the *kernel* of $\check{\mathcal{G}}(z)$.

If $\mathcal{G}$ is time invariant, with the impulse response $g(t)$,

$$\check{Y}(z; \tau) = \int_0^h \check{\mathcal{G}}(z; \tau - \sigma) \check{U}(z; \sigma) d\sigma,$$

where¹ $\check{\mathcal{G}}(z; \tau)$ is the z -transform $\mathcal{Z}\{\check{g}\}$ of the lifting of $g(t)$.

¹With some abuse of notation, by $\check{g}[i](\tau)$ for $\tau < 0$ we understand $\check{g}[i-1](h+\tau)$, and then $\check{\mathcal{G}}(z; \tau) = z^{-1} \check{\mathcal{G}}(z; h+\tau)$ whenever $\tau \in [-h, 0)$.

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Example: stable first-order system

Let $\mathcal{G}$ be the LTI system having the impulse response $g(t) = e^{-at}\mathbb{1}(t)$ for some $a > 0$. Its lifted transfer function kernel (with $\tau \in [-h, h]$) is

$$\begin{aligned}\check{G}(z; \tau) &= \sum_{i \in \mathbb{Z}} e^{-a(ih+\tau)} \mathbb{1}(ih+\tau) z^{-i} = e^{-a\tau} \mathbb{1}(\tau) + e^{-a\tau} \sum_{i \in \mathbb{N}} e^{-ahi} z^{-i} \\ &= e^{-a\tau} \mathbb{1}(\tau) + \frac{e^{-a(\tau+h)}}{z - e^{-ah}} = e^{-a\tau} \frac{\mathbb{1}(\tau)z + e^{-ah}\mathbb{1}(-\tau)}{z - e^{-ah}}.\end{aligned}$$

(the sum over $\mathbb{N}$ is the standard, not lifted, z-transform of $e^{-ahi}\mathbb{1}[i-1]$). Thus, in the lifted z-domain

$$\check{Y}(z; \tau) = \int_0^\tau e^{-a(\tau-\sigma)} \check{U}(z; \sigma) d\sigma + \frac{e^{-a\tau}}{z - e^{-ah}} \int_0^h e^{-a(h-\sigma)} \check{U}(z; \sigma) d\sigma.$$

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Lifted transfer functions of $\mathcal{SL}_h^{-1}$ and $\mathcal{L}_h\mathcal{H}$

Let $\mathcal{S}$ and $\mathcal{H}$ be shift-invariant sampler and hold, with the sampling and hold functions $\psi(t)$ and $\phi(t)$, respectively, and a sampling period h . In the lifted z-domain, in their RoC, they act as

$$\bar{Y}(z) = \check{\Psi}(z) \check{Y}(z) \quad \text{or} \quad \bar{Y}(z) = z^{-1} \int_0^h \check{\Psi}(z; h-\sigma) \check{Y}(z; \sigma) d\sigma$$

and

$$\check{U}(z) = \check{\Phi}(z) \bar{U}(z) \quad \text{or} \quad \check{U}(z; \tau) = \check{\Phi}(z; \tau) \bar{U}(z).$$

The operators

$$\check{\Psi}(z) : \{[0, h] \rightarrow \mathbb{F}^{n_y}\} \rightarrow \mathbb{F}^{n_y} \quad \text{and} \quad \check{\Phi}(z) : \mathbb{F}^{n_u} \rightarrow \{[0, h] \rightarrow \mathbb{F}^{n_u}\}$$

are called the lifted transfer functions of $\check{\mathcal{S}}$ and $\check{\mathcal{H}}$, respectively (their kernels are the z-transforms $\check{\Psi}(z)$ and $\check{\Phi}(z)$ of the lifted sampling and hold functions $\check{\psi}$ and $\check{\phi}$).

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Example: sampler with $\psi(t) = ae^{-at}\mathbb{1}(t)$

Consider $\mathcal{S} = \mathcal{S}_{\text{idl}}\mathcal{F}_a$, where $\mathcal{F}_a$ is the first-order low-pass having the transfer function $F_a(s) = \frac{a}{s+a}$ (its impulse response $f_a(t) = ae^{-at}\mathbb{1}(t)$). In this case

$$\check{\Psi}(z; \sigma) = \check{F}_a(z; \sigma) = ae^{-a\sigma} \frac{\mathbb{1}(\sigma)z + e^{-ah}\mathbb{1}(-\sigma)}{z - e^{-ah}},$$

so the sampler acts in the lifted z-domain as

$$\begin{aligned}\bar{Y}(z) &= z^{-1} a \int_0^h e^{-a(h-\sigma)} \frac{\mathbb{1}(h-\sigma)z + e^{-ah}\mathbb{1}(\sigma-h)}{z - e^{-ah}} \check{U}(z; \sigma) d\sigma \\ &= \frac{a}{z - e^{-ah}} \int_0^h e^{-a(h-\sigma)} \check{U}(z; \sigma) d\sigma.\end{aligned}$$

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Example: zero-order hold

Consider $\mathcal{H}_{\text{ZOH}}$ with the hold function $\phi_{\text{ZOH}}(t) = \mathbb{1}_{[0, h]}(t)$. In this case

$$\begin{aligned}\check{\Phi}_{\text{ZOH}}(z; \tau) &= \sum_{i \in \mathbb{Z}} \check{\phi}_{\text{ZOH}}[i](\tau) z^{-i} = \check{\phi}_{\text{ZOH}}[i](\tau) z^{-i} \Big|_{i=0} \\ &= 1\end{aligned}$$

and the hold acts in the lifted z-domain as

$$\check{U}(z; \tau) = \bar{U}(z).$$

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Lifted H_2 and H_∞ spaces

Lifted-domain spaces

$$H_\infty^{\mathbb{L} \rightarrow \mathbb{L}}, \quad H_\infty^{\mathbb{L} \rightarrow \mathbb{F}}, \quad H_\infty^{\mathbb{F} \rightarrow \mathbb{L}}, \quad H_2^{\mathbb{L} \rightarrow \mathbb{L}}, \quad H_2^{\mathbb{L} \rightarrow \mathbb{F}}, \quad \text{and} \quad H_2^{\mathbb{F} \rightarrow \mathbb{L}}$$

are defined like their pure discrete counterparts, with obvious alternations:

- Frobenius matrix norm is replaced by Hilbert-Schmidt operator norm;
- spectral matrix norm is replaced by the induced operator norm;

and less obvious alternations:

- the spaces $H_\infty^{\mathbb{L} \rightarrow \mathbb{L}}$ and $H_2^{\mathbb{L} \rightarrow \mathbb{L}}$ consist of lifted systems whose impulse response operator $\check{G}[i] = 0$ whenever $i < 0$, this is not sufficient for causality (see Lecture 4, slide 27);
- the norm on H_2 is scaled by $\frac{1}{h}$ (inherited from the lifted $L_2(\mathbb{T})$).

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Conjugate transfer functions in the lifted domain

Repeating familiar arguments, lifted conjugate transfer functions, which are the transfer functions of adjoint systems, are (at each z where they exist)

$\check{G}^\sim(z)$ is an operator $\mathbb{L}^{n_u} \rightarrow \mathbb{L}^{n_y}$, whose kernel

$$\check{G}^\sim(z; \tau, \sigma) = \check{G}'(\frac{1}{z}; \sigma, \tau) \quad \text{or} \quad \check{G}^\sim(z; \tau) = \check{G}'(\frac{1}{z}; -\tau) = z \check{G}'(\frac{1}{z}; h - \tau)$$

$\check{\Psi}^\sim(z)$ is an operator $\mathbb{F}^{n_{\bar{y}}} \rightarrow \mathbb{L}^{n_y}$, whose kernel

$$\check{\Psi}^\sim(z; \tau) = z \check{\Psi}'(\frac{1}{z}; h - \tau)$$

$\check{\Phi}^\sim(z)$ is an operator $\mathbb{L}^{n_u} \rightarrow \mathbb{F}^{n_{\bar{u}}}$, whose kernel

$$\check{\Phi}^\sim(z; \sigma) = z \check{\Phi}'(\frac{1}{z}; h - \sigma)$$

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Transfer function of $\bar{\mathcal{G}}_I = \mathcal{S}\mathcal{H}$

Let $\mathcal{S}$ and $\mathcal{H}$ be generalized sampler and hold with the sampling and hold functions $\psi(t)$ and $\phi(t)$, respectively. Then

$$\bar{\mathcal{G}}_I(z) = \check{\Psi}(z)\check{\Phi}(z) = z^{-1} \int_0^h \check{\Psi}(z; h - \sigma) \check{\Phi}(z; \sigma) d\sigma$$

By the Key Lifting Formula (with $s_i := \frac{\ln z}{h} + \frac{2\pi}{h}i$)

$$\check{\Psi}(z; \sigma) = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Psi(s_i) e^{s_i \sigma} \quad \text{and} \quad \check{\Phi}(z; \sigma) = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Phi(s_i) e^{s_i \sigma},$$

so that

$$\begin{aligned} \check{\Psi}(z; h - \sigma) \check{\Phi}(z; \sigma) &= \frac{1}{h^2} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \Psi(s_i) \Phi(s_j) e^{s_i h} e^{(s_i - s_j) \sigma} \\ &= \frac{1}{h^2} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \Psi(s_i) \Phi(s_j) z e^{j 2\pi(i-j)\sigma/h} \end{aligned}$$

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Transfer function of $\bar{\mathcal{G}}_I = \mathcal{S}\mathcal{H}$ (contd)

Hence,

$$\begin{aligned} \bar{\mathcal{G}}_I(z) &= z^{-1} \int_0^h \frac{1}{h^2} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \Psi(s_i) \Phi(s_j) z e^{j 2\pi(i-j)\sigma/h} d\sigma \\ &= \frac{1}{h^2} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \Psi(s_i) \Phi(s_j) \int_0^h e^{j 2\pi(i-j)\sigma/h} d\sigma = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Psi(s_i) \Phi(s_i) \end{aligned}$$

(by $\int_0^h e^{j 2\pi i \sigma/h} d\sigma = h \bar{\delta}[i]$). We thus end up with two alternative forms of the transfer function of $\mathcal{S}\mathcal{H}$:

$$\bar{\mathcal{G}}_I(z) = z^{-1} \int_0^h \check{\Psi}(z; h - \sigma) \check{\Phi}(z; \sigma) d\sigma = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Psi(s_i) \Phi(s_i)$$

The first one of them is especially useful. . .

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Transfer function of $\bar{\mathcal{G}}_I = \mathcal{SH}$: special cases

If $\bar{\mathcal{G}}_I = \mathcal{H}^* \mathcal{H}$,

$$\bar{G}_I(z) = \Phi^\sim(z) \Phi(z) = \int_0^h \check{\Phi}'(\tfrac{1}{z}; \tau) \check{\Phi}(z; \tau) d\tau = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Phi^\sim(s_i) \Phi(s_i)$$

where $\check{\Phi} = \mathfrak{Z}\{\mathcal{L}_h \phi\}$ and $\check{\Phi}'$ here is the transpose (not complex-conjugate).

If $\bar{\mathcal{G}}_I = \mathcal{SS}^*$,

$$\bar{G}_I(z) = \Psi(z) \Psi^\sim(z) = \int_0^h \check{\Psi}(z; \sigma) \check{\Psi}'(\tfrac{1}{z}; \sigma) d\sigma = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Psi(s_i) \Psi^\sim(s_i)$$

where $\check{\Psi} = \mathfrak{Z}\{\mathcal{L}_h \psi\}$ and $\check{\Psi}'$ here is the transpose (not complex-conjugate).

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Sampled-data H_2 approximations for Type III problems

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Type I

Assume that $\mathcal{K} \in H_\infty \cap H_2$, $\mathcal{S} \in H_\infty$ and $\mathcal{H} \in H_\infty$ are given. The problem is:

$$\underset{\bar{\mathcal{K}} \in H_\infty \cap H_2}{\text{minimize}} \|\mathcal{K} - \mathcal{H} \bar{\mathcal{K}} \mathcal{S}\|_2$$

It can be reformulated in the lifted domain as

$$\underset{\check{\mathcal{K}} \in H_\infty \cap H_2}{\text{minimize}} \|\check{\mathcal{K}} - \check{\mathcal{H}} \check{\mathcal{K}} \check{\mathcal{S}}\|_2$$

for given $\check{\mathcal{S}} \in H_\infty$, $\check{\mathcal{H}} \in H_\infty$, and causal $\check{\mathcal{K}} \in H_\infty \cap H_2$. This problem is LTI and $\check{\mathcal{H}} \check{\mathcal{K}} \check{\mathcal{S}}$ is a subspace of H_2 , so we get the frequency-domain optimality condition:

$$\langle \check{\mathcal{K}} - \Phi \bar{\mathcal{K}}_{\text{opt}} \check{\Psi}, \Phi \bar{\mathcal{K}} \check{\Psi} \rangle_2 = 0, \quad \forall \bar{\mathcal{K}} \in H_2$$

(and we hope that the resulting $\bar{\mathcal{K}}_{\text{opt}} \in H_\infty$ as well).

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z -domain solution

Spectral factorizations:

$$\Phi^\sim(z) \Phi(z) = \bar{W}_\phi^\sim(z) \bar{W}_\phi(z) \quad \text{and} \quad \Psi(z) \Psi^\sim(z) = \bar{W}_\psi(z) \bar{W}_\psi^\sim(z)$$

for $\bar{W}_\phi, \bar{W}_\phi^{-1}, \bar{W}_\psi, \bar{W}_\psi^{-1} \in H_\infty$, both are **pure discrete** factorizations;

Transfer function of the optimal solution:

$$\bar{\mathcal{K}}_{\text{opt}}(z) = \bar{W}_\phi^{-1}(z) \bar{Q}_{\text{opt}}(z) \bar{W}_\psi^{-1}(z),$$

where

$$\bar{Q}_{\text{opt}}(z) = \text{proj}_{H_2} [\bar{W}_\phi^\sim(z)]^{-1} \Phi^\sim(z) \check{\mathcal{K}}(z) \Psi^\sim(z) [\bar{W}_\psi^\sim(z)]^{-1}$$

and the projection done via truncating the part analytic in $\mathbb{D}_1$.

Optimal cost: $\|\check{\mathcal{G}}_e\|_2^2 = \|\check{\mathcal{K}}\|_2^2 - \frac{1}{h} \|\bar{Q}_{\text{opt}}\|_2^2$ (because $\|\bar{Q}_{\text{opt}}\|_2$ is not scaled by $\frac{1}{h}$ itself).

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Example: the problem

Consider the feedback control system with

$$P(s) = \frac{k_p(s + \alpha)}{s(s + 1)} \quad \text{and} \quad C(s) = \frac{s + 1}{s + \alpha}$$

The problem is to

- digitally redesign $\mathcal{C}$ as $\mathcal{H}_{\text{ZOH}} \bar{\mathcal{C}} \mathcal{S}_{\text{idl}}$ for some stable $\bar{\mathcal{C}}$.

We cannot minimize $\|\mathcal{C} - \mathcal{H}_{\text{ZOH}} \bar{\mathcal{C}} \mathcal{S}_{\text{idl}}\|$, because $\mathcal{S}_{\text{idl}}$ is unbounded on $L_2(\mathbb{R})$. Rather, consider minimizing

$$\|\mathcal{K} - \mathcal{H}_{\text{ZOH}} \bar{\mathcal{C}} \mathcal{S}_1\|_2,$$

where $K(s) = C(s) \frac{1}{s+1} = \frac{1}{s+\alpha}$ and $\mathcal{S}_1$ has $\psi_1(t) = e^{-t} \mathbb{1}(t)$ (i.e. consider a part of the plant as an antialiasing filter). This problem fits our setup.

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Example: spectral factorizations

First, we already saw that $\check{\Phi}_{\text{ZOH}}(z; \tau) = 1$, so that

$$\check{\Phi}_{\text{ZOH}}(z) \check{\Phi}_{\text{ZOH}}(z) = \int_0^h \check{\Phi}'_{\text{ZOH}}(\frac{1}{z}; \tau) \check{\Phi}_{\text{ZOH}}(z; \tau) d\tau = h$$

and $\bar{W}_\phi(z) = \sqrt{h}$ (easy). Then, $\check{\Psi}_1 = e^{-\sigma} \frac{\mathbb{1}(\sigma)z + e^{-h}\mathbb{1}(-\sigma)}{z - e^{-h}}$, so that

$$\begin{aligned} \check{\Psi}_1(z) \check{\Psi}_1^\sim(z) &= \int_0^h \check{\Psi}_1(z; \sigma) \check{\Psi}_1^\sim(\frac{1}{z}; \sigma) d\sigma \\ &= \int_0^h \frac{e^{-\sigma} z}{z - e^{-h}} \frac{e^{-\sigma}}{1 - e^{-h} z} d\sigma = \frac{(1 - e^{-2h})z}{2(z - e^{-h})(1 - e^{-h}z)} \\ &= \underbrace{\frac{\sqrt{1 - e^{-2h}}z}{\sqrt{2}(z - e^{-h})}}_{\bar{W}_\psi(z)} \underbrace{\frac{\sqrt{1 - e^{-2h}}}{\sqrt{2}(1 - e^{-h}z)}}_{\bar{W}_\psi^\sim(z)} \end{aligned}$$

(not too hard in this case either).

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Example: finding $\bar{Q}_{\text{opt}}(z)$

First,

$$\check{\Phi}_{\text{ZOH}}(z) \check{K}(z) \check{\Psi}_1^\sim(z) = \check{\Phi}_{\text{ZOH}}(z) \check{\Phi}_a(z),$$

where $\mathcal{H}_a$ is the hold with the hold function

$$\phi_a(t) = \frac{1}{1 + \alpha} (e^{-\alpha t} \mathbb{1}(t) + e^t \mathbb{1}(-t)),$$

which is the inverse Laplace transform of

$$\Phi_a(s) = K(s) \left(\frac{1}{s+1} \right)^\sim = \frac{1}{s+\alpha} \frac{1}{-s+1} = \frac{1}{1+\alpha} \left(\frac{1}{s+\alpha} - \frac{1}{s-1} \right)$$

(assuming that $\phi_a \in L_2(\mathbb{R})$). Hence,

$$\check{\Phi}_a(z; \tau) = \frac{1}{1 + \alpha} \left(\frac{e^{-\alpha \tau} z}{z - e^{-\alpha h}} + \frac{e^\tau e^{-h} z}{1 - e^{-h} z} \right).$$

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Example: finding $\bar{Q}_{\text{opt}}(z)$ (contd)

Then

$$\begin{aligned} \check{\Phi}_{\text{ZOH}}(z) \check{K}(z) \check{\Psi}_1^\sim(z) &= z^{-1} \int_0^h z \check{\Phi}_{\text{ZOH}}(\frac{1}{z}; \tau) \check{\Phi}_a(z; \tau) d\tau = \int_0^h \check{\Phi}_a(z; \tau) d\tau \\ &= \frac{1}{1 + \alpha} \left(\frac{\frac{1 - e^{-\alpha h}}{\alpha} z}{z - e^{-\alpha h}} + \frac{(1 - e^{-h})z}{1 - e^{-h}z} \right) \end{aligned}$$

and

$$\begin{aligned} \frac{\check{\Phi}_{\text{ZOH}}(z) \check{K}(z) \check{\Psi}_1^\sim(z)}{\bar{W}_\phi^\sim(z) \bar{W}_\psi^\sim(z)} &= \frac{\sqrt{2}}{\sqrt{h}(1 + \alpha)} \left(\frac{\frac{1 - e^{-\alpha h}}{\alpha} z}{z - e^{-\alpha h}} + \frac{(1 - e^{-h})z}{1 - e^{-h}z} \right) \frac{1 - e^{-h}z}{\sqrt{1 - e^{-2h}}} \\ &= \frac{\sqrt{2}(1 - e^{-\alpha h})(1 - e^{-(1+\alpha)h})}{\sqrt{h}\sqrt{1 - e^{-2h}}\alpha(1 + \alpha)} \frac{z}{z - e^{-\alpha h}} \\ &\quad - \frac{\sqrt{2}e^{-h}(1 - e^{-\alpha h} + (e^h - 1)\alpha)}{\sqrt{h}\sqrt{1 - e^{-2h}}\alpha(1 + \alpha)} z, \end{aligned}$$

where the **last term** is the $H_2^\perp$ part of it, which remains after truncation.

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Example: finding $\bar{C}_{\text{opt}}(z)$

Thus,

$$\bar{Q}_{\text{opt}} = \frac{\sqrt{2}(1 - e^{-\alpha h})(1 - e^{-(1+\alpha)h})}{\sqrt{h}\sqrt{1 - e^{-2h}}\alpha(1 + \alpha)} \frac{z}{z - e^{-\alpha h}}$$

and, therefore,

$$\begin{aligned}\bar{C}_{\text{opt}}(z) &= \frac{\sqrt{2}(1 - e^{-\alpha h})(1 - e^{-(1+\alpha)h})}{h\sqrt{1 - e^{-2h}}\alpha(1 + \alpha)} \frac{z}{z - e^{-\alpha h}} \frac{\sqrt{2}(z - e^{-h})}{\sqrt{1 - e^{-2h}}z} \\ &= \frac{2(1 - e^{-\alpha h})(1 - e^{-(1+\alpha)h})}{h(1 - e^{-2h})\alpha(1 + \alpha)} \frac{z - e^{-h}}{z - e^{-\alpha h}}\end{aligned}$$

This is a first-order (unless $\alpha = 1$, expectably) system, whose pole and zero match those of $C(s)$. Clearly, $\bar{C}_{\text{opt}} \in H_{\infty}$ (lucky we). If $C(s)$ is static,

$$\lim_{\alpha \rightarrow 1} \bar{C}_{\text{opt}}(z) = \frac{1 - e^{-h}}{h}$$

and the gain decreases as h grows (unlike Tustin).

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Example: optimal cost

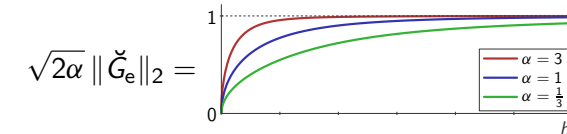
Some lengthy computations yield

$$\|\bar{Q}_{\text{opt}}\|_2^2 = \frac{2(1 - e^{-\alpha h})(1 - e^{-(1+\alpha)h})^2}{(1 - e^{-2h})(1 + e^{-\alpha h})h\alpha^2(1 + \alpha)^2},$$

from which

$$\|\check{G}_e\|_2 = \sqrt{\frac{1}{2\alpha} - \frac{2(1 - e^{-\alpha h})(1 - e^{-(1+\alpha)h})^2}{(1 - e^{-2h})(1 + e^{-\alpha h})h^2\alpha^2(1 + \alpha)^2}}.$$

If $\alpha = 1$ (static $C(s)$), $\|\check{G}_e\|_2 = \sqrt{\frac{1}{2} - \frac{(1 - e^{-h})^2}{2h^2}}$. The normalized, by $\|K\|_2$, performance



(deteriorates quicker for a lead $C(s)$, perhaps because of a higher $|C(\infty)|$).

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Example: closed-loop stability

Remarkably (the method itself guarantees nothing), closed-loop stability is preserved for most h if the gain k_p is “sufficiently small.” For instance, if $\alpha = 1$, the closed-loop pole is

$$z_p = 1 - k(1 - e^{-h})$$

and it is stable iff $0 < k < \frac{2}{1 - e^{-h}}$ (with $k \in (0, 2)$, the closed-loop system is stable for all $h > 0$). By comparison, the redesigned $\bar{C}(z) = 1$ (e.g. Tustin, step-invariant, etc) yields the closed-loop pole at

$$z_p = 1 - kh$$

and it is stable iff $0 < k < \frac{2}{h}$ (stable for arbitrarily large $h > 0$ only if $k \downarrow 0$).

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Type III

Assume that $\mathcal{K} \in H_\infty \cap H_2$ and $\mathcal{S} \in H_\infty$ are given. The problem is:

$$\underset{\mathcal{H} \in H_\infty \cap H_2}{\text{minimize}} \|\mathcal{K} - \mathcal{H}\mathcal{S}\|_2$$

It can be reformulated in the lifted domain as

$$\underset{\check{\mathcal{H}} \in H_\infty \cap H_2}{\text{minimize}} \|\check{\mathcal{K}} - \check{\mathcal{H}}\check{\mathcal{S}}\|_2$$

for given $\check{\mathcal{S}} \in H_\infty$ and causal $\check{\mathcal{K}} \in H_\infty \cap H_2$. This problem is LTI and $\check{\mathcal{H}}\check{\mathcal{S}}$ is a subspace of H_2 , so we get the frequency-domain optimality condition:

$$\langle \check{\mathcal{K}} - \check{\Phi}_{\text{opt}}\check{\Psi}, \check{\Phi}\check{\Psi} \rangle_2 = 0, \quad \forall \check{\Phi} \in H_2$$

(and we hope that the resulting $\check{\Phi}_{\text{opt}} \in H_\infty$ as well).

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z-domain solution

Spectral factorization:

$$\check{\Psi}(z)\check{\Psi}^\sim(z) = \bar{W}_\psi(z)\bar{W}_\psi^\sim(z)$$

for $\bar{W}_\psi, \bar{W}_\psi^{-1} \in H_\infty$, which is the same as in Type I (pure discrete);

Transfer function of the optimal solution:

$$\check{\Phi}_{\text{opt}}(z) = \check{Q}_{\text{opt}}(z)\bar{W}_\psi^{-1}(z),$$

where

$$\check{Q}_{\text{opt}}(z) = \text{proj}_{H_2} [\check{\mathcal{K}}(z)\check{\Psi}^\sim(z)[\bar{W}_\psi^\sim(z)]^{-1}]$$

and the projection done via truncating the part analytic outside $\bar{\mathbb{D}}_1$.

Optimal cost: $\|\check{G}_e\|_2^2 = \|\check{\mathcal{K}}\|_2^2 - \|\check{Q}_{\text{opt}}\|_2^2$.

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Example: the problem

The same as in the Type I case: consider the feedback control system with

$$P(s) = \frac{k_p(s + \alpha)}{s(s + 1)} \quad \text{and} \quad C(s) = \frac{s + 1}{s + \alpha}$$

The problem is to

- digitally redesign $\mathcal{C}$ as $\mathcal{H}\bar{\mathcal{C}}\mathcal{S}_{\text{idl}}$ for some stable $\mathcal{H}$ and $\bar{\mathcal{C}}$.

We cannot minimize $\|\mathcal{C} - \mathcal{H}\bar{\mathcal{C}}\mathcal{S}_{\text{idl}}\|$, because $\mathcal{S}_{\text{idl}}$ is unbounded on $L_2(\mathbb{R})$.

Rather, consider minimizing

$$\|\mathcal{K} - \mathcal{H}\bar{\mathcal{C}}\mathcal{S}_1\|_2,$$

where $K(s) = C(s)\frac{1}{s+1} = \frac{1}{s+\alpha}$ and $\mathcal{S}_1$ has $\psi_1(t) = e^{-t}\mathbb{1}(t)$ (i.e. consider a part of the plant as an antialiasing filter). This problem fits our setup after absorbing $\bar{\mathcal{C}}$ into $\mathcal{H}$.

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Example: spectral factorization

Exactly as in the Type I case:

$$\check{\Psi}_1(z)\check{\Psi}_1^\sim(z) = \underbrace{\frac{\sqrt{1 - e^{-2h}z}}{\sqrt{2}(z - e^{-h})}}_{\bar{W}_\psi(z)} \underbrace{\frac{\sqrt{1 - e^{-2h}}}{\sqrt{2}(1 - e^{-h}z)}}_{\bar{W}_\psi^\sim(z)}$$

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Example: finding $\check{Q}_{\text{opt}}(z)$

We already saw that $\check{K}(z)\check{\psi}_1^{\sim}(z) = \check{\Phi}_a(z)$, where the kernel of $\check{\Phi}_a(z)$ is

$$\check{\Phi}_a(z; \tau) = \frac{1}{1+\alpha} \left(\frac{e^{-\alpha\tau} z}{z - e^{-\alpha h}} + \frac{e^{\tau} e^{-h} z}{1 - e^{-h} z} \right).$$

Then

$$\begin{aligned} \frac{(\check{K}\check{\psi}_1^{\sim})(z; \tau)}{\check{W}_{\psi}^{\sim}(z)} &= \frac{\sqrt{2}}{1+\alpha} \left(\frac{e^{-\alpha\tau} z}{z - e^{-\alpha h}} + \frac{e^{\tau} e^{-h} z}{1 - e^{-h} z} \right) \frac{1 - e^{-h} z}{\sqrt{1 - e^{-2h}}} \\ &= e^{-\alpha\tau} \frac{\sqrt{2}(1 - e^{-(1+\alpha)h})}{\sqrt{1 - e^{-2h}}(1+\alpha)} \frac{z}{z - e^{-\alpha h}} \\ &\quad - (e^{\tau} - e^{-\alpha\tau}) \frac{\sqrt{2}e^{-h}}{\sqrt{1 - e^{-2h}}(1+\alpha)} z \end{aligned}$$

and the **last term** is the $H_2^{\perp}$ part of it, which remains after truncation.

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Example: finding $\check{\Phi}_{\text{opt}}(z)$

Thus, $\check{Q}_{\text{opt}}(z)$ is the transfer function of a hold with the kernel

$$\check{Q}_{\text{opt}}(z; \tau) = e^{-\alpha\tau} \frac{\sqrt{2}(1 - e^{-(1+\alpha)h})}{\sqrt{1 - e^{-2h}}(1+\alpha)} \frac{z}{z - e^{-\alpha h}}$$

and, therefore, $\check{\Phi}_{\text{opt}}(z)$ is a hold with the kernel

$$\begin{aligned} \check{\Phi}_{\text{opt}}(z; \tau) &= e^{-\alpha\tau} \frac{\sqrt{2}(1 - e^{-(1+\alpha)h})}{\sqrt{1 - e^{-2h}}(1+\alpha)} \frac{z}{z - e^{-\alpha h}} \frac{\sqrt{2}(z - e^{-h})}{\sqrt{1 - e^{-2h}} z} \\ &= e^{-\alpha\tau} \frac{2(1 - e^{-(1+\alpha)h})}{(1 - e^{-2h})(1+\alpha)} \frac{z - e^{-h}}{z - e^{-\alpha h}}, \end{aligned}$$

which is also an H_{∞} systems. If $C(s)$ is static,

$$\lim_{\alpha \rightarrow 1} \check{\Phi}_{\text{opt}}(z; \tau) = e^{-\alpha\tau}.$$

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Example: peeling off $\check{\Phi}_{\text{opt}}(z)$

One way to return to time domain may be via literal inverse z-transform:

$$\begin{aligned} \check{\phi}_{\text{opt}}[i](\tau) &= \mathfrak{Z}^{-1}\{\check{\Phi}_{\text{opt}}(z; \tau)\} = e^{-\alpha\tau} \frac{2(1 - e^{-(1+\alpha)h})}{(1 - e^{-2h})(1+\alpha)} \mathfrak{Z}^{-1}\left\{ \frac{z - e^{-h}}{z - e^{-\alpha h}} \right\} \\ &= \frac{2(1 - e^{-(1+\alpha)h})}{(1 - e^{-2h})(1+\alpha)} \left(e^{-\alpha(ih+\tau)} \mathbb{1}[i] - e^{-\alpha(ih+\tau)} e^{(\alpha-1)h} \mathbb{1}[i-1] \right) \end{aligned}$$

from which

$$\phi_{\text{opt}}(t) = \frac{2(1 - e^{-(1+\alpha)h})}{(1 - e^{-2h})(1+\alpha)} \left(e^{-\alpha t} \mathbb{1}(t) - e^{-\alpha t} e^{(\alpha-1)h} \mathbb{1}(t-h) \right).$$

It is simplified if $\alpha = 1$, in which case $\phi_{\text{opt}}(t) = e^{-\alpha t} \mathbb{1}_{[0,h)}(t)$ (static hold).

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Example: peeling off $\check{\Phi}_{\text{opt}}(z)$ (contd)

Another possibility is to split

$$\mathcal{H}_{\text{opt}} = \mathcal{H}_{\text{stat}} \bar{\mathcal{C}}_{\text{opt}},$$

where $\bar{\mathcal{C}}_{\text{opt}}$ is a discrete system with the transfer function

$$\bar{\mathcal{C}}_{\text{opt}}(z) = \frac{2(1 - e^{-(1+\alpha)h})}{(1 - e^{-2h})(1+\alpha)} \frac{z - e^{-h}}{z - e^{-\alpha h}}$$

and $\mathcal{H}_{\text{stat}}$ is the static hold with the hold function

$$\phi_{\text{stat}}(t) = e^{-\alpha t} \mathbb{1}_{[0,h)}(t)$$

(the kernel of its lifted transfer function $\check{\Phi}_{\text{stat}}(z; \tau) = e^{-\alpha\tau}$). Note that this hold **imitates** the impulse response of $\mathcal{K}$.

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Example: optimal cost

We can present $\dot{Q}_{\text{opt}} = \dot{\mathcal{H}}_{\text{stat}} \bar{Q}_{\text{opt}}$, where

$$\bar{Q}_{\text{opt}}(z) = \frac{\sqrt{2}(1 - e^{-(1+\alpha)h})}{\sqrt{1 - e^{-2h}}(1 + \alpha)} \frac{z}{z - e^{-\alpha h}}.$$

Then

$$\|\dot{Q}_{\text{opt}}\|_2^2 = \langle \dot{\Phi}_{\text{stat}} \bar{Q}_{\text{opt}}, \dot{\Phi}_{\text{stat}} \bar{Q}_{\text{opt}} \rangle_2 = \langle \dot{\Phi}_{\text{stat}}^{\sim} \dot{\Phi}_{\text{stat}} \bar{Q}_{\text{opt}}, \bar{Q}_{\text{opt}} \rangle_2,$$

We know that

$$\dot{\Phi}_{\text{stat}}^{\sim}(z) \dot{\Phi}_{\text{stat}}(z) = \int_0^h e^{-2\alpha\tau} d\tau = \frac{1 - e^{-2\alpha h}}{2\alpha}.$$

Thus,

$$\|\dot{Q}_{\text{opt}}\|_2^2 = \frac{1 - e^{-2\alpha h}}{2\alpha h} \|\bar{Q}_{\text{opt}}\|_2^2$$

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Example: optimal cost (contd)

Now

$$\|\bar{Q}_{\text{opt}}\|_2^2 = \frac{2(1 - e^{-(1+\alpha)h})^2}{(1 - e^{-2h})(1 - e^{-2\alpha h})(1 + \alpha)^2},$$

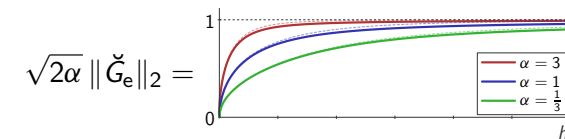
so that

$$\|\dot{Q}_{\text{opt}}\|_2^2 = \frac{(1 - e^{-(1+\alpha)h})^2}{\alpha h(1 - e^{-2h})(1 + \alpha)^2}$$

from which

$$\|\check{G}_e\|_2 = \sqrt{\frac{1}{2\alpha} - \frac{(1 - e^{-(1+\alpha)h})^2}{\alpha h(1 - e^{-2h})(1 + \alpha)^2}}.$$

The normalized, by $\|K\|_2$, performance



The improvement over the ZOH case exists, but not very visible. . .

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Example: closed-loop stability

Curiously, the discretized closed-loop system with this controller is 1-order and the closed-loop pole is

$$z_p = \frac{1 - 2k_p + \alpha + e^{-h}(1 + 2e^{-\alpha h}k_p + \alpha)}{(1 + e^{-h})(1 + \alpha)}$$

The closed-loop system is stable iff

$$0 < k < \frac{(1 + e^{-h})(1 + \alpha)}{1 - e^{-(1+\alpha)h}} \xrightarrow{\alpha \rightarrow 1} \frac{2}{1 - e^{-h}}.$$

This bound is not always less restrictive than that for the fixed hold.

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Bottom line

- causality is accounted for in terms of analyticity and boundedness in certain regions of s - and z -domains
- the Projection Theorem is still the solution tool (accompanied by the spectral factorization)
- model matching as a digital redesign method
- calculations quickly get messy $\implies$ new tools are needed (state space)

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