

Advanced Sampled-Data Control (038798)

lecture no. 6

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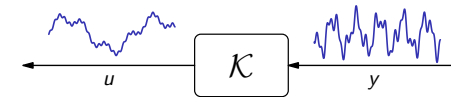
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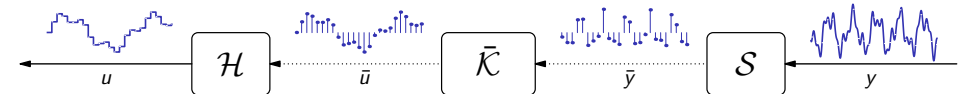
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Sampled-data approximation problem

Given an analog system (controller?)



find its “good” approximation of the form



We may consider the following 4 situations:

1. S and H are fixed, find $\bar{K}$
2. H is fixed, find $\bar{K}$ and S (or just S , $\bar{K}$ can always be absorbed in it)
3. S is fixed, find $\bar{K}$ and H (or just H , $\bar{K}$ can always be absorbed in it)
4. find S , $\bar{K}$, and H (or just S and H)

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Content

Sampled-data L_2 approximations for Type II and III problems

Sampled-data L_2 approximations for Type IV problem: bad news

SVD and Schmidt decomposition

Sampled-data L_2 approximations for Type IV problem: good news

Sampling Theorems

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Type II

Assume that $K \in L_\infty \cap L_2$ and $H \in L_\infty$ are given. The problem is:

$$\underset{S \in L_\infty \cap L_2}{\text{minimize}} \|K - HS\|_2$$

A key fact is that

- HS is a subspace of L_2 over all $S \in L_2$,

so we can use the Projection Theorem and end up with the normal equation $H^*H S_{\text{opt}} = H^*K$. If H^*H is invertible,

$$S_{\text{opt}} = \underbrace{(H^*H)^{-1}}_{\bar{K}_2} \underbrace{H^*K}_{S_2}.$$

The optimal cost is then

$$\|\mathcal{G}_e\|_2^2 = \|K\|_2^2 - \|H(H^*H)^{-1}H^*K\|_2^2$$

(here $H(H^*H)^{-1}H^*$ is the orthogonal projection onto the image of H).

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Type II: computations

Here is where the lifting is useful:

- calculate the frequency response of $\mathcal{H}^*\mathcal{H}$ by either of

$$\bar{K}_2^{-1}(\mathbf{e}^{j\theta}) = \int_0^h [\check{\phi}(\mathbf{e}^{j\theta}; \tau)]' \check{\phi}(\mathbf{e}^{j\theta}; \tau) d\tau = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Phi'(-j\omega_i) \Phi(j\omega_i)$$

where $\check{\phi} = \mathcal{L}_h \phi$, $\Phi = \mathfrak{F}\{\phi\}$, and $\omega_i := \frac{\theta}{h} + \frac{2\pi}{h}i$;

- because $\mathcal{H}^* = \mathcal{S}_{\text{idl}} \mathcal{F}_\phi^*$, the sampling function $\psi_2(t)$ of $\mathcal{S}_2 = \mathcal{H}^* \mathcal{K}$ is the inverse Fourier transform of

$$\Psi_2(j\omega) = \Phi'(-j\omega)K(j\omega).$$

Type II: optimal performance

Computing $\|\mathcal{K}\|_2$ is standard, so consider

$$\begin{aligned}\|\mathcal{H}\mathcal{S}_{\text{opt}}\|_2^2 &= \langle \mathcal{H}\mathcal{S}_{\text{opt}}, \mathcal{H}\mathcal{S}_{\text{opt}} \rangle_2 = \langle \mathcal{H}^*\mathcal{H}, \mathcal{S}_{\text{opt}}\mathcal{S}_{\text{opt}}^* \rangle_2 \\ &= \langle \mathcal{H}^*\mathcal{H}, (\mathcal{H}^*\mathcal{H})^{-1}\mathcal{H}^*\mathcal{K}\mathcal{K}^*\mathcal{H}(\mathcal{H}^*\mathcal{H})^{-1} \rangle_2 \\ &= \langle (\mathcal{H}^*\mathcal{H})^{-1}, \mathcal{H}^*\mathcal{K}\mathcal{K}^*\mathcal{H} \rangle_2\end{aligned}$$

Both $\bar{\mathcal{K}}_2 = (\mathcal{H}^* \mathcal{H})^{-1}$ and $\bar{\mathcal{G}}_2 := \mathcal{H}^* \mathcal{K} \mathcal{K}^* \mathcal{H}$ are conventional discrete-time systems whose (self-adjoint) frequency responses we know how to compute. Thus,

$$\begin{aligned}\|\mathcal{H}_{\text{Sopt}}\|_2^2 &= \frac{1}{2\pi\hbar} \int_{-\pi}^{\pi} \langle \bar{K}_2(e^{j\theta}), \bar{G}_2(e^{j\theta}) \rangle_{\text{HS}} d\theta \\ &= \frac{1}{2\pi\hbar} \int_{-\pi}^{\pi} \text{tr}(\bar{G}_2(e^{j\theta}) \bar{K}_2(e^{j\theta})) d\theta,\end{aligned}$$

which is computable (although might not be easy w/o additional tools)...

Type II: example

Let $K(j\omega) = \frac{1}{j\omega+1}$ and $\mathcal{H} = \mathcal{H}_{\text{FOH}}$ with $\phi(t) = (1 - \frac{|t|}{h})\mathbb{1}_{[-h,h]}(t)$. Then

$$\begin{aligned}\bar{K}_2^{-1}(e^{j\theta}) &= \int_0^h \left[\frac{\tau}{h} e^{j\theta} + \left(1 - \frac{\tau}{h}\right) \right]' \left[\frac{\tau}{h} e^{j\theta} + \left(1 - \frac{\tau}{h}\right) \right] d\tau \\ &= \frac{(4 + e^{j\theta} + e^{-j\theta})h}{6} = \frac{(2 + \cos(\theta))h}{3}\end{aligned}$$

(invertible). Its inverse is a discrete system with the impulse response

$$\bar{k}_2[i] = \frac{\sqrt{3}}{h} (-2 + \sqrt{3})^{|i|} =$$

Then,

$$\Psi_2(j\omega) = \frac{2(1 - \cos(\omega h))}{\omega^2 h} \frac{1}{j\omega + 1} =$$

which is the Fourier transform of the impulse response of $\mathcal{F}_\phi^* \mathcal{K} = \mathcal{F}_\phi \mathcal{K}$.

Type II: example (contd)

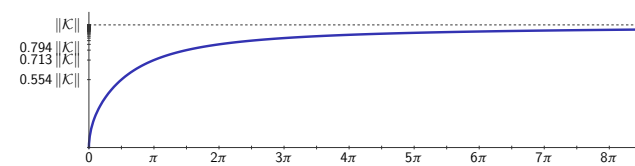
With *some* extra effort, it can be shown that

$$\bar{G}_2(e^{j\theta}) = \frac{2h(h^2 - 3) + h(h^2 + 6)\cos(\theta)}{3h^2} - \frac{2}{h^2} \frac{(1 - \cos(\theta))^2 \sinh(h)}{\cos(\theta) - \cosh(h)}$$

and then

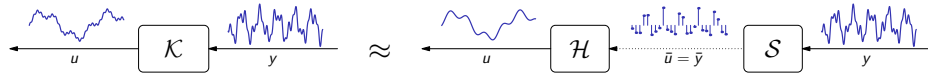
$$\|\mathcal{H}\mathcal{S}_{\text{opt}}\|_2^2 = \frac{h^3 - 6(\sqrt{3} - 1)h - 24 + 6e^{-h}}{h^4} + \frac{18}{h^4} \frac{\sqrt{3} + 6e^{-h} - \sqrt{3}e^{-2h}}{1 + 4e^{-h} + e^{-2h}}$$

Taking into account that $\|\mathcal{K}\|_2^2 = \frac{1}{2}$, the optimal cost is



which expectably vanishes as $h \rightarrow 0$ and approaches $\|\mathcal{K}\|_2$ as $h \rightarrow \infty$.

Type II: optimization for a given y



Consider the problem of

$$\underset{S \in L_\infty \cap L_2}{\text{minimize}} \|\mathcal{K}y - \mathcal{H}Sy\|_2$$

for some given $\mathcal{K} \in L_\infty$ and $y \in L_2$. It is readily seen that

- $\mathcal{H}Sy$ is still a subspace of L_2 ,

so we can use the Projection Theorem and end up with the normal equation $\mathcal{H}^* \mathcal{H} S_{\text{opt}} y y^* = \mathcal{H}^* \mathcal{K} y y^*$. But if some S_0 solves the L_2 normal equation, $\mathcal{H}^* \mathcal{H} S_0 = \mathcal{H}^* \mathcal{K}$, it verifies the one with y as well. In other words,

- if the L_2 optimal solution exists, it is also optimal for every $y \in L_2$.

A consequence of this is that the **L_2 optimal solution is also L_∞ optimal** (as it minimizes the error for the worst-case y as well).

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Type III

Assume that $\mathcal{K} \in L_\infty \cap L_2$ and $\mathcal{S} \in L_\infty$ are given. The problem is:

$$\underset{\mathcal{H} \in L_\infty \cap L_2}{\text{minimize}} \|\mathcal{K} - \mathcal{H}\mathcal{S}\|_2$$

A key fact again is that

- $\mathcal{H}\mathcal{S}$ is a subspace of L_2 over all $\mathcal{H} \in L_2$,

so we can use the Projection Theorem and end up with the normal equation $\mathcal{H}_{\text{opt}} \mathcal{S} \mathcal{S}^* = \mathcal{K} \mathcal{S}^*$. If $\mathcal{S} \mathcal{S}^*$ is invertible,

$$\mathcal{H}_{\text{opt}} = \underbrace{\mathcal{K} \mathcal{S}^*}_{\mathcal{H}_3} \underbrace{(\mathcal{S} \mathcal{S}^*)^{-1}}_{\bar{\mathcal{K}}_3}.$$

The optimal cost is then

$$\|\mathcal{G}_e\|_2^2 = \|\mathcal{K}\|_2^2 - \|\mathcal{K} \mathcal{S}^* (\mathcal{S} \mathcal{S}^*)^{-1} \mathcal{S}\|_2^2$$

(here $\mathcal{S}^* (\mathcal{S} \mathcal{S}^*)^{-1} \mathcal{S}$ is the orthogonal projection onto the co-image of $\mathcal{S}$).

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Type III: computations

Here is where the lifting is useful, again:

- calculate the frequency response of $\mathcal{H}^* \mathcal{H}$ by either of

$$\bar{\mathcal{K}}_3^{-1}(\mathrm{e}^{\mathrm{j}\theta}) = \int_0^h \check{\psi}(\mathrm{e}^{\mathrm{j}\theta}; \sigma) [\check{\psi}(\mathrm{e}^{\mathrm{j}\theta}; \sigma)]' \mathrm{d}\sigma = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Psi(\mathrm{j}\omega_i) \Psi'(-\mathrm{j}\omega_i)$$

where $\check{\psi} = \mathcal{L}_h \psi$, $\Psi = \mathfrak{F}\{\psi\}$, and $\omega_i := \frac{\theta}{h} + \frac{2\pi}{h}i$;

- because $\mathcal{S}^* = \mathcal{F}_\psi^* \mathcal{H}_{\text{imp}}$, the hold function $\phi_3(t)$ of $\mathcal{H}_3 = \mathcal{K} \mathcal{S}^*$ is the inverse Fourier transform of

$$\Phi_3(\mathrm{j}\omega) = \mathcal{K}(\mathrm{j}\omega) \Psi'(-\mathrm{j}\omega).$$

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Type III: optimal performance

Computing $\|\mathcal{K}\|_2$ is standard, so consider

$$\begin{aligned} \|\mathcal{H}_{\text{opt}} \mathcal{S}\|_2^2 &= \langle \mathcal{H}_{\text{opt}} \mathcal{S}, \mathcal{H}_{\text{opt}} \mathcal{S} \rangle_2 = \langle \mathcal{H}_{\text{opt}}^* \mathcal{H}_{\text{opt}}, \mathcal{S} \mathcal{S}^* \rangle_2 \\ &= \langle (\mathcal{S} \mathcal{S}^*)^{-1} \mathcal{S} \mathcal{K}^* \mathcal{K} \mathcal{S}^* (\mathcal{S} \mathcal{S}^*)^{-1}, \mathcal{S} \mathcal{S}^* \rangle_2 \\ &= \langle \mathcal{S} \mathcal{K}^* \mathcal{K} \mathcal{S}^*, (\mathcal{S} \mathcal{S}^*)^{-1} \rangle_2 \end{aligned}$$

Both $\bar{\mathcal{K}}_3 = (\mathcal{S} \mathcal{S}^*)^{-1}$ and $\bar{\mathcal{G}}_3 := \mathcal{S} \mathcal{K}^* \mathcal{K} \mathcal{S}^*$ are conventional discrete-time systems whose (self-adjoint) frequency responses we know how to compute. Thus,

$$\begin{aligned} \|\mathcal{H} \mathcal{S}_{\text{opt}}\|_2^2 &= \frac{1}{2\pi h} \int_{-\pi}^{\pi} \langle \bar{\mathcal{G}}_3(\mathrm{e}^{\mathrm{j}\theta}), \bar{\mathcal{K}}_3(\mathrm{e}^{\mathrm{j}\theta}) \rangle_{\text{HS}} \mathrm{d}\theta \\ &= \frac{1}{2\pi h} \int_{-\pi}^{\pi} \text{tr}(\bar{\mathcal{K}}_3(\mathrm{e}^{\mathrm{j}\theta}) \bar{\mathcal{G}}_3(\mathrm{e}^{\mathrm{j}\theta})) \mathrm{d}\theta, \end{aligned}$$

which is computable (although might not be easy w/o additional tools)...

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Type III: example

Let $K(j\omega) = \mathbb{1}_{[-\omega_N, \omega_N]}(\omega)$ and $\psi(t) = \frac{1}{h} \text{sinc}_h(t)$, meaning $\mathcal{F}_\psi = \mathcal{K} = \mathcal{F}_{\text{ilp}}$, where $\mathcal{F}_{\text{ilp}}$ is the ideal low-pass filter with a bandwidth of ω_N . Then

$$\bar{K}_3^{-1}(e^{j\theta}) = \frac{1}{h} \sum_{i \in \mathbb{Z}} |\Psi(j\omega_i)|^2 = \frac{1}{h} |\Psi(j\omega_0)|^2 = \frac{1}{h}$$

(invertible). Furthermore, $\Phi_3(j\omega) = \mathbb{1}_{[-\omega_N, \omega_N]}(\omega)$, so that

$$\Phi_{\text{opt}}(\omega) = h \mathbb{1}_{[-\omega_N, \omega_N]}(\omega) \implies \phi_{\text{opt}}(t) = \text{sinc}_h(t),$$

which implies that $\mathcal{H}_{\text{opt}} = \mathcal{H}_{\text{sinc}}$, which we already know from the Sampling Theorem. Then, $\bar{G}_3(e^{j\theta}) = \frac{1}{h}$, so that

$$\|\mathcal{H}_{\text{opt}} \mathcal{S}\|_2^2 = \frac{1}{2\pi h} \int_{-\pi}^{\pi} \bar{K}_3(e^{j\theta}) \bar{G}_3(e^{j\theta}) d\theta = \frac{1}{h}.$$

But $\|\mathcal{K}\|_2^2 = \frac{1}{h}$, so $\|\mathcal{K} - \mathcal{H}_{\text{opt}} \mathcal{S}\|_2 = 0$. Therefore, $\mathcal{H}_{\text{opt}} \mathcal{S} = \mathcal{K}$ and, strangely enough, the sample-and-hold circuit $\mathcal{H}_{\text{sinc}} \mathcal{S}_{\text{idl}} \mathcal{F}_{\text{ilp}}$ is **time invariant**, gee!

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Sampled-data L_2 approximations for Type II and III problems

Sampled-data L_2 approximations for Type IV problem: bad news

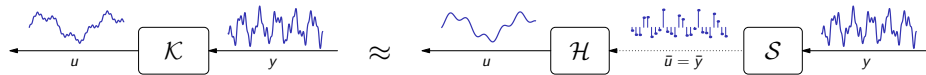
SVD and Schmidt decomposition

Sampled-data L_2 approximations for Type IV problem: good news

Sampling Theorems

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Type IV



Assume that $\mathcal{K} \in L_\infty \cap L_2$ is given. The problem is:

$$\begin{aligned} & \underset{\mathcal{S}, \mathcal{H} \in L_\infty}{\text{minimize}} && \|\mathcal{K} - \mathcal{H} \mathcal{S}\|_2 \\ & \text{subject to} && \dim \bar{u} = \dim \bar{y} \leq r \end{aligned}$$

(not well defined if the dimension of discrete signals is unconstrained). But

- is the set of all admissible $\mathcal{K}_{\text{SHC}} := \mathcal{H} \mathcal{S}$ a subspace of L_2 ?

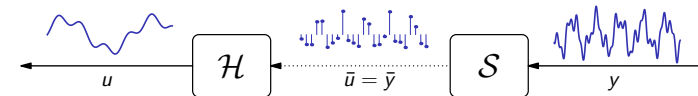
To answer this question, we should first understand, what

- constraints on the operator $y \mapsto u$

are imposed by the structure $\mathcal{H} \mathcal{S}$?

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Characterizing sampled-data systems



Intuitively the sample-and-hold structure $\mathcal{H} \mathcal{S}$ should somehow

- constraint the richness of the mapping $y \mapsto u$,

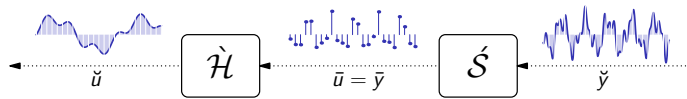
perhaps $u(t)$ cannot have an arbitrary waveform and not all components of $y(t)$ can affect the output. In this respect, it appears natural to look at

- some kind of rank deficiency of the operator $\mathcal{K}_{\text{SHC}} = \mathcal{H} \mathcal{S}$,

as the cardinality of the set of all $\bar{y}$'s should be smaller than the cardinality of the set of all y 's and u 's. However, all these sets are still infinite, so rank deficiency arguments might not be immediate. . . **Lifting helps here!**

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Characterizing sampled-data systems in the lifted domain



Let $\mathcal{S}$ and $\mathcal{H}$ be shift-invariant converters with sampling and hold functions $\psi(t) \in \mathbb{R}^{n_{\bar{y}} \times n_y}$ and $\phi(t) \in \mathbb{R}^{n_u \times n_{\bar{y}}}$, respectively. The kernel of the lifted frequency response of the sample-and-hold circuit of the form $\mathcal{K}_{\text{SHC}} = \mathcal{H}\mathcal{S}$ is

$$\check{K}_{\text{SHC}}(e^{j\theta}; \tau, \sigma) = \check{\Phi}(e^{j\theta}; \tau) e^{-j\theta} \check{\Psi}(e^{j\theta}; h - \sigma),$$

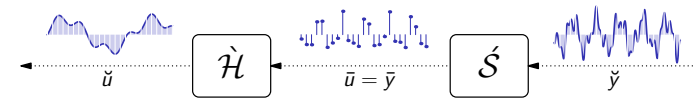
where $\check{\Phi} = \mathfrak{F}\{\check{\phi}\}$ and $\check{\Psi} = \mathfrak{F}\{\check{\psi}\}$. The **separability** of the parts containing τ and σ is important here, as it implies that

- the **rank** of the operator $\check{K}_{\text{SHC}}(e^{j\theta}) : \{[0, h] \rightarrow \mathbb{F}^{n_y}\} \rightarrow \{[0, h] \rightarrow \mathbb{F}^{n_u}\}$ is **at most $n_{\bar{y}}$** .

It turns out that this property is **defining** in characterizing sample-and-hold circuits.

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The Rank Theorem



Theorem (Rank Theorem)

Let $\mathcal{K} \in L_\infty$ and suppose that the kernel $\check{K}(e^{j\theta}; \tau, \sigma)$ of its lifted frequency response is piecewise continuous. Then $\mathcal{K}$ is a sampled-data system if and only if there is $r \in \mathbb{N}$ such that $\text{rank } \check{F}(e^{j\theta}) \leq r, \forall \theta \in [-\pi, \pi]$. In this case there are implementations of $\mathcal{K} = \mathcal{H}\mathcal{S}$ with $\dim \bar{y} = r$.

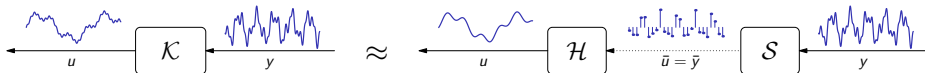
Proof.

Sufficiency we already saw. Necessity requires more involved technicalities, so (lucky you) its proof is omitted. $\square$

Remark: As its rank is finite, $\check{K}_{\text{SHC}}$ is a Hilbert-Schmidt operator whenever it is bounded.

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Type IV (contd)



Assume that $\mathcal{K} \in L_\infty \cap L_2$ is given. The problem is:

$$\begin{aligned} & \underset{\mathcal{S}, \mathcal{H} \in L_\infty}{\text{minimize}} && \|\mathcal{K} - \mathcal{H}\mathcal{S}\|_2 \\ & \text{subject to} && \dim \bar{u} = \dim \bar{y} \leq r \end{aligned}$$

(not well defined if the dimension of discrete signals is unconstrained). But

- is the set of all admissible $\mathcal{K}_{\text{SHC}} := \mathcal{H}\mathcal{S}$ a subspace of L_2 ?

Bad news: the answer is negative, just because

$$\text{rank}(O_1 + O_2) \neq \text{rank}(O_1) + \text{rank}(O_2),$$

and the Projection Theorem arguments do not apply.

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Sampling Theorems

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Matrix SVD

For every $M \in \mathbb{F}^{p \times m}$ there are unitary $U \in \mathbb{F}^{p \times p}$ and $V \in \mathbb{F}^{m \times m}$ (i.e. such that $U'U = I$ and $V'V = I$) and a diagonal $\Sigma = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_{\min\{p,m\}}\}$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$ such that

$$M = U \Sigma V'.$$

The real numbers σ_i are called the **singular values** of M and the columns u_i and v_i of U and V , respectively, are called its **singular vectors**.

Alternatively (rank-one decomposition),

$$M = \sum_i \sigma_i u_i v_i', \quad \sigma_1 \geq \sigma_2 \geq \dots \geq 0,$$

where $\{v_i\}$ and $\{u_i\}$ are **orthonormal** bases for $\mathbb{F}^m$ and $\mathbb{F}^p$, respectively. In this case, $y = Mw$ reads $y = \sum_i \sigma_i \langle w, v_i \rangle u_i$.

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Counterpart for operators on abstract Hilbert spaces

Let $O : \mathbb{H}_i \rightarrow \mathbb{H}_o$, for some Hilbert spaces $\mathbb{H}_i$ and $\mathbb{H}_o$, be compact. Then

$$O : \zeta \mapsto \sum_i \sigma_i \langle \zeta, \psi_i \rangle_{\mathbb{H}_i} \phi_i \quad (\text{Schmidt decomposition}),$$

where $\{\psi_i\}$ and $\{\phi_i\}$, called the **Schmidt pairs** of O , are **orthonormal** bases for $\mathbb{H}_i$ and $\mathbb{H}_o$, respectively, and real numbers $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$ are called the **singular values** (or *s-numbers*) of O .

Some history:

- proposed by Erhard Schmidt in 1907 for integral operators
- independent of matrix SVD
- term *singular values* coined by Émile Picard in 1910

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Low-rank approximation via SVD

Schmidt (1907) proved (for the Hilbert-Schmidt norm) that

$$\min_{\text{rank } O_r \leq r} \|O - O_r\|_{\text{HS}} = \left(\sum_{i>r} \sigma_i^2 \right)^{1/2} \quad \text{with } O_{r,\text{opt}} : \zeta \mapsto \sum_{i=1}^r \sigma_i \langle \zeta, \psi_i \rangle_{\mathbb{H}_i} \phi_i$$

Weyl (1912) then proved (for the induced norm) that

$$\min_{\text{rank } O_r \leq r} \|O - O_r\|_{\mathbb{H}_i \rightarrow \mathbb{H}_o} = \sigma_{r+1} \quad \text{with the same } O_{r,\text{opt}}$$

Powerful results with plenty of applications:

- control
- signal / image processing
- physics
- ...

We shall see that it's powerful in sampled-data approximation as well.

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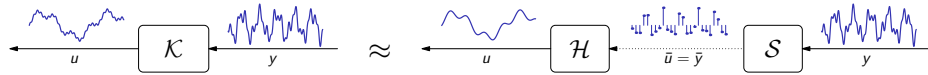
SVD and Schmidt decomposition

Sampled-data L_2 approximations for Type IV problem: good news

Sampling Theorems

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Type IV



The problem

$$\begin{aligned} & \underset{S, \mathcal{H} \in L_\infty}{\text{minimize}} && \|\mathcal{K} - \mathcal{H}S\|_2 \\ & \text{subject to} && \dim \bar{u} = \dim \bar{y} \leq r \end{aligned}$$

can be reformulated in the lifted frequency domain as

$$\begin{aligned} & \underset{\check{K}_{\text{SHC}} \in L_2(\mathbb{T})}{\text{minimize}} && \|\check{K} - \check{K}_{\text{SHC}}\|_2 \\ & \text{subject to} && \text{rank } \check{K}_{\text{SHC}} \leq r \end{aligned}$$

for a given $\check{K} \in L_2^{\mathbb{L} \rightarrow \mathbb{L}}(\mathbb{T})$ and solved via Schmidt decomposition.

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Type IV: formal solution

Bring in a (frequency-dependent) Schmidt decomposition of $\check{K}$, so that

$$\check{U}(e^{j\theta}) = \sum_i \sigma_i(e^{j\theta}) \langle \check{Y}(e^{j\theta}), \check{\Psi}_i(e^{j\theta}) \rangle_2 \check{\Phi}_i(e^{j\theta})$$

for some $\sigma_1(e^{j\theta}) \geq \sigma_2(e^{j\theta}) \geq \dots \geq 0$ and orthonormal bases $\{\check{\Psi}_i(e^{j\theta})\}$ and $\{\check{\Phi}_i(e^{j\theta})\}$ for $\mathbb{L}^{n_y}$ and $\mathbb{L}^{n_u}$, respectively, at each $\theta \in [-\pi, \pi]$. The optimal solution to the L_2 approximation problem is then

$$\begin{aligned} \check{U}(e^{j\theta}; \tau) &= \sum_{i \leq r} \sigma_i(e^{j\theta}) \langle \check{Y}(e^{j\theta}), \check{\Psi}_i(e^{j\theta}) \rangle_2 \check{\Phi}_i(e^{j\theta}; \tau) \\ &= \underbrace{[\alpha_1 \check{\Phi}_1 \quad \dots \quad \alpha_r \check{\Phi}_r]}_{\text{hold}}(e^{j\theta}; \tau) \int_0^h \underbrace{\begin{bmatrix} \frac{\sigma_1}{\alpha_1} \check{\Psi}_1^* \\ \vdots \\ \frac{\sigma_r}{\alpha_r} \check{\Psi}_r^* \end{bmatrix}}_{\text{sampler}}(e^{j\theta}; \sigma) \check{Y}(e^{j\theta}; \sigma) d\sigma \end{aligned}$$

for any $\alpha_i(e^{j\theta}) \neq 0$, so it boils down to finding a Schmidt decomposition. . .

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Frequency response operator revisited

Let $\mathcal{K} \in L_\infty \cap L_2$ be an LTI SISO system $y \mapsto u$ with the impulse response $k(t)$. The kernel of its lifted frequency response operator is then

$$\check{K}(e^{j\theta}; \tau) = \mathfrak{F}\{\check{k}\} = \frac{1}{h} \sum_{i \in \mathbb{Z}} K(j\omega_i) e^{j\omega_i \tau}$$

(by the Key Lifting Formula). Then

$$\begin{aligned} \check{U}(e^{j\theta}; \tau) &= \int_0^h \frac{1}{h} \sum_{i \in \mathbb{Z}} K(j\omega_i) e^{j\omega_i(\tau-\sigma)} \check{Y}(e^{j\theta}; \sigma) d\sigma \\ &= \sum_{i \in \mathbb{Z}} |K(j\omega_i)| e^{j \arg K(j\omega_i)} \int_0^h \frac{1}{\sqrt{h}} e^{-j\omega_i \sigma} \check{Y}(e^{j\theta}; \sigma) d\sigma \frac{1}{\sqrt{h}} e^{j\omega_i \tau} \\ &= \sum_{i \in \mathbb{Z}} |K(j\omega_i)| \left\langle \check{Y}(e^{j\theta}; \sigma), \frac{1}{\sqrt{h}} e^{j(\omega_i \sigma + \alpha_i)} \right\rangle_2 \frac{1}{\sqrt{h}} e^{j(\omega_i \tau + \beta_i)} \end{aligned}$$

whenever $\beta_i - \alpha_i = \arg K(j\omega_i)$.

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Frequency response operator revisited (contd)

The next thing to notice is that

$$\{e_{i,\alpha}(e^{j\theta})\} := \left\{ \frac{1}{\sqrt{h}} e^{j(\omega_i \tau + \alpha_i)} \right\}$$

constitute an orthonormal basis for $\mathbb{L} := L_2[0, h)$ for any choice of $\alpha_i \in \mathbb{R}$. Thus,

$$\check{K}(e^{j\theta}) : \zeta \mapsto \sum_{i \in \mathbb{Z}} |K(j\omega_i)| \langle \zeta, e_{i,\alpha}(e^{j\theta}) \rangle_2 e_{i,\beta}(e^{j\theta})$$

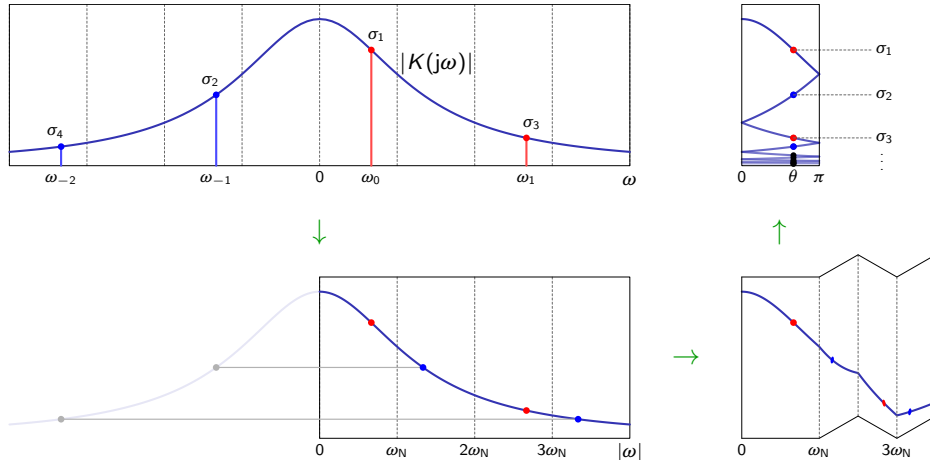
is a Schmidt decomposition of $\check{K}(e^{j\theta})$ with the

- singular values $|K(j\omega_i)|$, where $\omega_i = \frac{\theta}{h} + \frac{2\pi}{h} i$
- Schmidt pairs $\{e_{i,\alpha}(e^{j\theta})\}$ and $\{e_{i,\beta}(e^{j\theta})\}$ with $\beta_i - \alpha_i = \arg K(j\omega_i)$

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Singular values of $\check{K}(e^{j\theta})$

Graphical interpretation (folding):



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Type IV: more concrete solution

Let $|K(j\omega)| = |K(-j\omega)|$, so only $\theta \in [0, \pi]$ may be examined, and let $\mathbb{I}_r(\theta)$ be the set of r indices i corresponding to the r largest $|K(j\omega_i)|$ at each θ . The solution of the Type IV approximation problem yields then

$$\begin{aligned} \check{U}(e^{j\theta}; \tau) &= \sum_{i \in \mathbb{I}_r(\theta)} |K(j\omega_i)| \langle \check{Y}(e^{j\theta}), e_{i,\alpha}(e^{j\theta}) \rangle_2 e_{i,\beta}(e^{j\theta}) \\ &= \sum_{i \in \mathbb{I}_r(\theta)} |K(j\omega_i)| \left\langle \check{Y}(e^{j\theta}; \sigma), \frac{1}{\sqrt{h}} e^{j(\omega_i \sigma + \alpha_i)} \right\rangle_2 \frac{1}{\sqrt{h}} e^{j(\omega_i \tau + \beta_i)} \\ &= \sum_{i \in \mathbb{I}_r(\theta)} |K(j\omega_i)| e^{j \arg K(j\omega_i)} \left\langle \check{Y}(e^{j\theta}; \sigma), \frac{1}{h} e^{j\omega_i \sigma} \right\rangle_2 e^{j\omega_i \tau} \\ &= \sum_{i \in \mathbb{I}_r(\theta)} e^{j\omega_i \tau} K(j\omega_i) \int_0^h \frac{1}{h} e^{-j\omega_i \sigma} \check{Y}(e^{j\theta}; \sigma) d\sigma \\ &= \sum_{i \in \mathbb{I}_r(\theta)} e^{j\omega_i \tau} K(j\omega_i) e^{-j\theta} \int_0^h \frac{1}{h} e^{j\omega_i(h-\sigma)} \check{Y}(e^{j\theta}; \sigma) d\sigma \end{aligned}$$

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Type IV: more concrete solution (contd)

Each term,

$$\check{U}_i(e^{j\theta}; \tau) = e^{j\omega_i \tau} K(j\omega_i) e^{-j\theta} \int_0^h \frac{1}{h} e^{j\omega_i(h-\sigma)} \check{Y}(e^{j\theta}; \sigma) d\sigma$$

is a sample-and-hold circuit of the form $\mathcal{H}_i \bar{\mathcal{K}}_i \mathcal{S}_i$, where

- $\mathcal{S}_i$ has the lifted frequency response kernel $\frac{1}{h} e^{j\omega_i \sigma}$,
- $\bar{\mathcal{K}}(e^{j\theta}) = K(j\omega_i)$,
- $\mathcal{H}_i$ has the lifted frequency response kernel $e^{j\omega_i \sigma}$.

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Schmidt pairs with monotonically decreasing $|K(j\omega)|$

To find sampling and hold functions of $\mathcal{S}_i$ and $\mathcal{H}_i$, let us figure out what signal $\xi_i(t)$ produces $\check{\Xi}_i(e^{j\theta}; \tau) = e^{j\omega_i \tau}$. By the Key Lifting Formula,

$$e^{j\omega_i \tau} = \frac{1}{h} \sum_{j \in \mathbb{Z}} \Xi_i(j\omega_j) e^{j\omega_j \tau}, \quad \forall \theta \in [0, \pi]$$

whose solution is the frequency response of the **ideal bandpass filter**,

$$\Xi_i(j\omega) = h \mathbb{1}_{[\omega_N i, \omega_N(i+1)]}(|\omega|),$$

so

$$\xi_i(t) = (i+1) \text{sinc}_h((i+1)t) - i \text{sinc}_h(it) =$$

Thus,

- $\mathcal{S}_i$ has the sampling function $\psi_i(t) = \frac{1}{h} \xi_i(t)$,
- $\mathcal{H}_i$ has the hold function $\phi_i(t) = \xi_i(t)$.

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Type IV: optimal performance

At each frequency $\theta \in [0, \pi]$

$$\|\check{K}(e^{j\theta}) - \check{K}_{\text{SHC}}(e^{j\theta})\|_{\text{HS}}^2 = \sum_{i \notin \mathbb{I}_r(\theta)} |K(\omega_i)|^2$$

so that

$$\|\mathcal{K} - \mathcal{K}_{\text{SHC}}\|_2^2 = \frac{1}{\pi h} \int_0^\pi \sum_{i \notin \mathbb{I}_r(\theta)} |K(\omega_i)|^2 d\theta$$

If $|K(j\omega)|$ is a monotonically decreasing function, we end up with

$$\begin{aligned} \|\mathcal{K} - \mathcal{K}_{\text{SHC}}\|_2^2 &= \frac{1}{\pi h} \int_0^\pi \sum_{i > r} |K(\frac{\theta}{h} + \frac{\pi}{h}i)|^2 d\theta = \frac{1}{\pi h} \sum_{i > r} \int_0^\pi |K(\frac{\theta}{h} + \frac{\pi}{h}i)|^2 d\theta \\ &= \frac{1}{\pi} \sum_{i > r} \int_{(i-1)\omega_N}^{i\omega_N} |K(\omega)|^2 d\omega = \frac{1}{\pi} \int_{r\omega_N}^\infty |K(\omega)|^2 d\omega \end{aligned}$$

which is the L_2 norm of the unmatched “tail.”

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Content

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Sampled-data L_2 approximations for Type IV problem: bad news

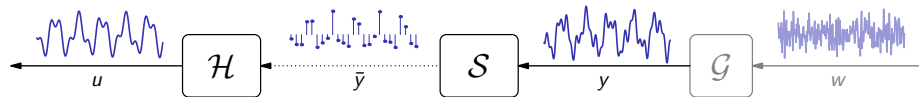
SVD and Schmidt decomposition

Sampled-data L_2 approximations for Type IV problem: good news

Sampling Theorems

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Sampled signal reconstruction problem



The problem is to

- reconstruct y from its sampled measurements $\bar{y}$,

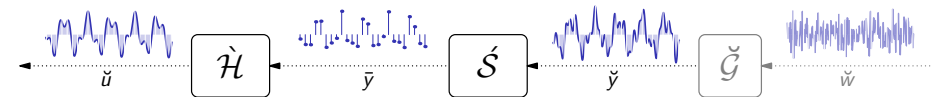
i.e. to end up with $u \approx y$. The *signal generator* $\mathcal{G}$ shapes properties of y by generating it from a structureless normalized (white) w .

Formulation: Let $\mathcal{G} \in L_\infty \cap L_2$ be given,

$$\begin{aligned} &\underset{\mathcal{S}, \mathcal{H} \in L_\infty}{\text{minimize}} && \|(I - \mathcal{H}\mathcal{S})\mathcal{G}\|_2 \\ &\text{subject to} && \dim \bar{y} \leq r \end{aligned}$$

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Lifted-domain reformulation



The problem is to

$$\begin{aligned} &\underset{\check{\mathcal{G}}_{\text{SHC}} \in L_2(\mathbb{T})}{\text{minimize}} && \|(I - \check{\mathcal{G}}_{\text{SHC}})\check{\mathcal{G}}\|_2 \\ &\text{subject to} && \text{rank } \check{\mathcal{G}}_{\text{SHC}} \leq r \end{aligned}$$

for a given $\check{\mathcal{G}} \in L_2^{\mathbb{L} \rightarrow \mathbb{L}}(\mathbb{T})$. It can be solved via the Schmidt decomposition of $\check{\mathcal{G}}$, again.

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Low-rank weighted identity approximation via SVD

Let $O : \mathbb{H}_i \rightarrow \mathbb{H}_o$, for some Hilbert spaces $\mathbb{H}_i$ and $\mathbb{H}_o$, be compact. Bring in its Schmidt decomposition

$$O : \zeta \mapsto \sum_i \sigma_i \langle \zeta, \psi_i \rangle_{\mathbb{H}_i} \phi_i.$$

Then

$$\min_{\text{rank } l_r \leq r} \|(I - l_r)O\|_{\text{HS}} = \left(\sum_{i > r} \sigma_i^2 \right)^{1/2} \quad \text{with } l_{r,\text{opt}} : \zeta \mapsto \sum_{i=1}^r \langle \zeta, \phi_i \rangle \phi_i$$

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Lifted-domain solution

Bring in the Schmidt decomposition

$$\check{G}(e^{j\theta}) : \zeta \mapsto \sum_{i \in \mathbb{Z}} |G(\omega_i)| \langle \zeta, e_{i,\alpha}(e^{j\theta}) \rangle_2 e_{i,\beta}(e^{j\theta}),$$

where $e_{i,\alpha}(e^{j\theta}) = \frac{1}{\sqrt{h}} e^{j(\omega_i \tau + \alpha_i)}$ and $e_{i,\beta}(e^{j\theta}) = \frac{1}{\sqrt{h}} e^{j(\omega_i \tau + \beta_i)}$ for whatever $\beta_i - \alpha_i = \arg G(j\omega_i)$; since we need only $e_{i,\beta}$, pick $\beta_i = 0$. Now, denote by $\mathbb{I}_r(\theta)$ the set of r indices i corresponding to the r largest $|G(j\omega_i)|$ at each $\theta \in [0, \pi]$. The resulting optimal reconstructor acts then as

$$\check{U}(e^{j\theta}) = \sum_{i \in \mathbb{I}_r(\theta)} e^{j\omega_i \tau} e^{-j\theta} \int_0^h \frac{1}{h} e^{j\omega_i(h-\sigma)} \check{Y}(e^{j\theta}; \sigma) d\sigma$$

(the same as before modulo independence of the singular values of $\check{G}(e^{j\theta})$).

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Time-domain solution, monotonically decreasing $|G(j\omega)|$

In this case $\mathbb{I}_r = \{0, -1, 1, -2, 2, \dots\}$ and the optimal solution

$$\mathcal{G}_{\text{SHC,opt}} = \sum_{i \in \mathbb{Z}_{0..r-1}} \mathcal{H}_i \mathcal{S}_i,$$

where

- $\mathcal{S}_i$ has the sampling function $\psi_i(t) = \frac{1}{h} \xi_i(t)$,
- $\mathcal{H}_i$ has the hold function $\phi_i(t) = \xi_i(t)$,

and $\xi_i(t) = (i+1) \text{sinc}_h((i+1)t) - i \text{sinc}_h(it)$ is the impulse response of the ideal band-pass filter of the frequency band $|\omega| \in [\omega_N i, \omega_N(i+1)]$.

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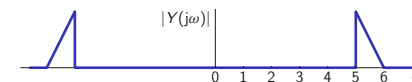
When is the perfect reconstruction possible?

The immediate answer is

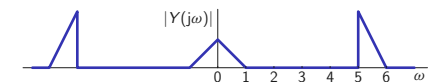
- if $G(j\omega_i) \neq 0$ only at at most r different $i \in \mathbb{Z}$.

This result is exhaustive and includes

- the WKS formulation (bandlimited within $[-r\omega_N, r\omega_N]$)
- more exotic cases, like



perfectly reconstructable with $\omega_N = 1$



perfectly reconstructable with $\omega_N = 2$

- less evident situations from the Sampling Theorem point of view, like $\mathcal{G} = \mathcal{H}_{\mathcal{G}} \mathcal{S}_{\mathcal{G}} \dots$

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