

Advanced Sampled-Data Control (038798)

lecture no. 5

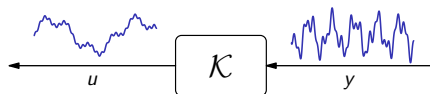
Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT

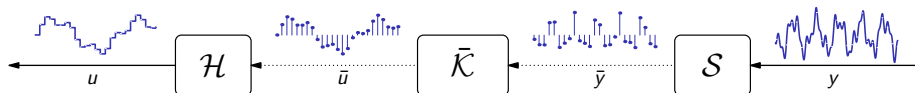


Sampled-data approximation problem

Given an analog system (controller?)



find its “good” approximation of the form



We may consider the following 4 situations:

1. S and $\mathcal{H}$ are fixed, find $\bar{\mathcal{K}}$
2. $\mathcal{H}$ is fixed, find $\bar{\mathcal{K}}$ and S (or just S , $\bar{\mathcal{K}}$ can always be absorbed in it)
3. S is fixed, find $\bar{\mathcal{K}}$ and $\mathcal{H}$ (or just $\mathcal{H}$, $\bar{\mathcal{K}}$ can always be absorbed in it)
4. find S , $\bar{\mathcal{K}}$, and $\mathcal{H}$ (or just S and $\mathcal{H}$)

Content

L_2 non-causal model-matching

Lifted systems in frequency domain

L_2 norm of sampled-data systems

Content

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General model matching problem

The problem of finding a stable $\mathcal{K}$ minimizing a size of

$$\mathcal{G}_e = \mathcal{G}_1 - \mathcal{G}_2 \mathcal{K} \mathcal{G}_3$$

for given $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ under constraint that the error system $\mathcal{G}_e$ is stable is called the **model matching** problem.

Classification: we say that GMMP is

- one-sided if either $\mathcal{G}_2 = I$ or $\mathcal{G}_3 = I$
- two-sided if $\mathcal{G}_2$ is not right invertible and $\mathcal{G}_3$ is not left invertible by a stable $\mathcal{K}$

$L_2(j\mathbb{R})$ space of matrix-valued functions

The space $L_2^{p \times m}(j\mathbb{R})$ is defined as

$$L_2^{p \times m}(j\mathbb{R}) := \left\{ G : j\mathbb{R} \rightarrow \mathbb{C}^{p \times m} \mid \|G\|_2 := \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|G(j\omega)\|_F^2 d\omega \right)^{1/2} < \infty \right\}$$

where $\|\cdot\|_F$ stands for the Frobenius matrix norm. $L_2(j\mathbb{R})$ is a Hilbert space, with the inner product

$$\langle G_1, G_2 \rangle_2 = \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}([G_2(j\omega)]' G_1(j\omega)) d\omega.$$

Remark: A nonzero constant $G(j\omega)$ does *not* belong to $L_2(j\mathbb{R})$, so

$$G \in L_\infty(j\mathbb{R}) \not\Rightarrow G \in L_2(j\mathbb{R}).$$

L_2 space of continuous-time LTI system

Define the space of LTI systems whose frequency response, $G(j\omega)$, belongs to $L_2(j\mathbb{R})$ by L_2 . This is a Hilbert space with

$$\langle \mathcal{G}_1, \mathcal{G}_2 \rangle := \langle G_1, G_2 \rangle_2 = \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}([G_2(j\omega)]' G_1(j\omega)) d\omega$$

and

$$\|\mathcal{G}\|_2 := \|G\|_2 = \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|G(j\omega)\|_F^2 d\omega \right)^{1/2}.$$

Important properties:

- $T\mathcal{Q} \in L_2$ and $\mathcal{Q}T \in L_2$ whenever $\mathcal{Q} \in L_2$ and $T \in L_\infty$
- for all $\mathcal{Q}_1, \mathcal{Q}_2 \in L_2$ and $T_1, T_2 \in L_\infty$,

$$\langle \mathcal{Q}_1, T_1 \mathcal{Q}_2 T_2 \rangle_2 = \langle T_1' \mathcal{Q}_1 T_2', \mathcal{Q}_2 \rangle_2$$

(just because $\text{tr}(M_1 M_2) = \text{tr}(M_2 M_1)$)

L₂ space of continuous-time LTI system

Define the space of LTI systems whose frequency response, $G(j\omega)$, belongs to $L_2(j\mathbb{R})$ by L_2 . This is a Hilbert space with

$$\langle \mathcal{G}_1, \mathcal{G}_2 \rangle := \langle G_1, G_2 \rangle_2 = \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}([G_2(j\omega)]' G_1(j\omega)) d\omega$$

and

$$\|\mathcal{G}\|_2 := \|G\|_2 = \left(\frac{1}{2\pi} \int_{\mathbb{R}} \|G(j\omega)\|_F^2 d\omega \right)^{1/2}.$$

Important properties:

- $\mathcal{T}\mathcal{G} \in L_2$ and $\mathcal{G}\mathcal{T} \in L_2$ whenever $\mathcal{G} \in L_2$ and $\mathcal{T} \in L_\infty$
- for all $\mathcal{G}_1, \mathcal{G}_2 \in L_2$ and $\mathcal{T}_1, \mathcal{T}_2 \in L_\infty$,

$$\langle \mathcal{G}_1, \mathcal{T}_1 \mathcal{G}_2 \mathcal{T}_2 \rangle_2 = \langle \mathcal{T}_1^* \mathcal{G}_1 \mathcal{T}_2^*, \mathcal{G}_2 \rangle_2$$

(just because $\text{tr}(M_1 M_2) = \text{tr}(M_2 M_1)$)

L₂ system norm: interpretation

Time domain interpretations (deterministic and stochastic):

- by Parseval,

$$\|G\|_2^2 = \int_{\mathbb{R}} \|g(t)\|_F^2 dt \quad \text{and} \quad \langle G_1, G_2 \rangle_2 = \int_{\mathbb{R}} \text{tr}([g_2(t)]' g_1(t)) dt$$

i.e. $\|\mathcal{G}\|_2^2$ equals the **energy of its impulse response** of $\mathcal{G}$ (average, in a sense)

- it can be shown that $\|\mathcal{G}\|_2^2$ equals the **variance of the response** of $\mathcal{G}$ to a **white noise** process having unit intensity (i.e. unit spectral density)

Remark: $\mathcal{G} \in L_2$ does *not* guarantee stability. As a counterexample, pick $\mathcal{G}$ with the impulse response $g(t) = \text{sinc}_h(t)\mathbb{1}(t)$ (truncated ideal low pass).

$L_2(\mathbb{T})$ space of matrix-valued functions

The space $L_2^{p \times m}(\mathbb{T})$ is defined as

$$L_2^{p \times m}(\mathbb{T}) := \left\{ \bar{G} : \mathbb{T} \rightarrow \mathbb{C}^{p \times m} \mid \|\bar{G}\|_2 := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \|\bar{G}(e^{j\theta})\|_F^2 d\theta \right)^{1/2} < \infty \right\}$$

It is also a Hilbert space, with the inner product

$$\langle \bar{G}_1, \bar{G}_2 \rangle_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{tr}([\bar{G}_2(e^{j\theta})]' \bar{G}_1(e^{j\theta})) d\theta.$$

Remark: Any nonzero constant $\bar{G}(e^{j\theta})$ does belong to $L_2(\mathbb{T})$. In fact,

$$\bar{G} \in L_{\infty}(\mathbb{T}) \implies \bar{G} \in L_2(\mathbb{T})$$

(the integration path is finite and $\|M\|_F \geq \|M\|$).

L₂ space of discrete-time LTI system

Define the space of LTI systems whose frequency response, $\bar{G}(e^{j\theta})$, belongs to $L_2(\mathbb{T})$ by L_2 . This is a Hilbert space with

$$\langle \bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2 \rangle := \langle \bar{G}_1, \bar{G}_2 \rangle_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{tr}([\bar{G}_2(e^{j\theta})]' \bar{G}_1(e^{j\theta})) d\theta$$

and

$$\|\bar{\mathcal{G}}\|_2 := \|\bar{G}\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \|\bar{G}(e^{j\theta})\|_F^2 d\theta \right)^{1/2}.$$

And we again have that

- $\bar{\mathcal{T}}\bar{\mathcal{G}} \in L_2$ and $\bar{\mathcal{G}}\bar{\mathcal{T}} \in L_2$ whenever $\bar{\mathcal{G}} \in L_2$ and $\bar{\mathcal{T}} \in L_\infty$
- for all $\bar{\mathcal{G}}_1, \bar{\mathcal{G}}_2 \in L_2$ and $\bar{\mathcal{T}}_1, \bar{\mathcal{T}}_2 \in L_\infty$,

$$\langle \bar{\mathcal{G}}_1, \bar{\mathcal{T}}_1 \bar{\mathcal{G}}_2 \bar{\mathcal{T}}_2 \rangle_2 = \langle \bar{\mathcal{T}}_1^* \bar{\mathcal{G}}_1 \bar{\mathcal{T}}_2^*, \bar{\mathcal{G}}_2 \rangle_2$$

and the same time-domain interpretations as in the continuous-time case.

General model matching problem in L_2

Assume that $\mathcal{G}_1 \in L_\infty \cap L_2$ and that $\mathcal{G}_2, \mathcal{G}_3 \in L_\infty$. The problem is:

$$\underset{\mathcal{K} \in L_\infty \cap L_2}{\text{minimize}} \|\mathcal{G}_1 - \mathcal{G}_2 \mathcal{K} \mathcal{G}_3\|_2$$

Some comments:

- no causality constraints is imposed on $\mathcal{K}$
- $L_\infty \cap L_2$ implies both stability and Hilbert space properties
- may be extended to unstable $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ (requires “stabilizability”)

Preliminary: the Projection Theorem

Let $\mathbb{H}$ be a Hilbert space (with inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and norm $\|\cdot\|_{\mathbb{H}}$) and let $\mathbb{H}_g$ be a *subspace* of $\mathbb{H}$. Given $f \in \mathbb{H}$, we say that $g_o \in \mathbb{H}_g$ is its best approximation in $\mathbb{H}_g$ if

$$\|f - g_o\|_{\mathbb{H}} \leq \|f - g\|_{\mathbb{H}}, \quad \forall g \in \mathbb{H}_g.$$

The Projection Theorem says that

- if the best approximation of f in $\mathbb{H}_g$ exists, it is unique

– g_o is the best approximation of f in $\mathbb{H}_g$ iff $f - g_o \perp \mathbb{H}_g$, i.e.

$$\langle f - g_o, g \rangle_{\mathbb{H}} = 0, \quad \forall g \in \mathbb{H}_g \quad \text{or}$$

- the norm of the best approximation error $\|f - g_o\|_{\mathbb{H}}^2 = \|f\|_{\mathbb{H}}^2 - \|g_o\|_{\mathbb{H}}^2$
(by Pythagoras, $\|f\|_{\mathbb{H}}^2 = \|(f - g_o) + g_o\|_{\mathbb{H}}^2 = \|f - g_o\|_{\mathbb{H}}^2 + \|g_o\|_{\mathbb{H}}^2$)

Preliminary: the Projection Theorem

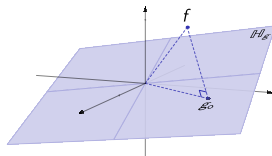
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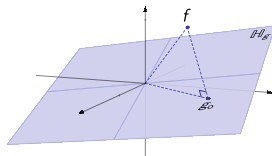
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Solution of GMMP

Just notice that

- if $\mathcal{K}$ is an arbitrary L_2 system, $\mathcal{G}_2 \mathcal{K} \mathcal{G}_3$ is a subspace of L_2

Thus, if the optimal $\mathcal{K}$ exists, it must satisfy

$$\langle \mathcal{G}_1 - \mathcal{G}_2 \mathcal{K}_{\text{opt}} \mathcal{G}_3, \mathcal{G}_2 \mathcal{K} \mathcal{G}_3 \rangle_2 = 0, \quad \forall \mathcal{K} \in L_2$$

$$\Updownarrow$$

$$\langle \mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^* - \mathcal{G}_2^* \mathcal{G}_2 \mathcal{K}_{\text{opt}} \mathcal{G}_3 \mathcal{G}_3^*, \mathcal{K} \rangle_2 = 0, \quad \forall \mathcal{K} \in L_2$$

which yields the **normal equation**

$$\mathcal{G}_2^* \mathcal{G}_2 \mathcal{K}_{\text{opt}} \mathcal{G}_3 \mathcal{G}_3^* = \mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^*.$$

If $\mathcal{G}_2^* \mathcal{G}_2$ and $\mathcal{G}_3 \mathcal{G}_3^*$ are invertible, the unique solution to the L_2 GMMP is

$$\mathcal{K}_{\text{opt}} = (\mathcal{G}_2^* \mathcal{G}_2)^{-1} \mathcal{G}_2^* \mathcal{G}_1 \mathcal{G}_3^* (\mathcal{G}_3 \mathcal{G}_3^*)^{-1}.$$

Frequency-domain solution of GMMP

By definition,

$$\begin{aligned}\langle \mathcal{G}_1, \mathcal{T} \mathcal{G}_2 \rangle &= \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}([T(j\omega) G_2(j\omega)]' G_1(j\omega)) d\omega \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \text{tr}([G_2(j\omega)]' [T(j\omega)]' G_1(j\omega)) d\omega = \langle \mathcal{T}^* \mathcal{G}_1, \mathcal{G}_2 \rangle.\end{aligned}$$

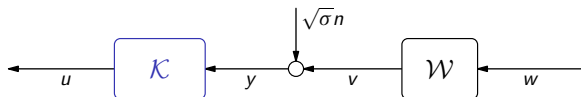
Thus,

- $\mathcal{T}^*$ is the system with the frequency response $[T(j\omega)]' = T'(-j\omega)$ and the frequency response of the optimal solution is

$$\begin{aligned}K_{\text{opt}}(j\omega) &= (G_2'(-j\omega) G_2(j\omega))^{-1} \\ &\quad \times G_2'(-j\omega) G_1(j\omega) G_3'(-j\omega) \\ &\quad \times (G_3(j\omega) G_3'(-j\omega))^{-1},\end{aligned}$$

which is (relatively) easy to compute.

Example: filtering out noise



The problem is to recover transmitted signal v from its noisy measurement. Here $\mathcal{W}$ is a weight. The error system is then

$$\mathcal{G}_e = \begin{bmatrix} \mathcal{W} & 0 \end{bmatrix} - \mathcal{K} \begin{bmatrix} \mathcal{W} & \sqrt{\sigma}I \end{bmatrix}$$

Normal equation, which is $K_{\text{opt}}(j\omega)(\sigma + |W(j\omega)|^2) = |W(j\omega)|^2$, is solvable if $\sigma + |W(j\omega)|^2$ is invertible in L_∞ and the solution is then

$$K_{\text{opt}}(j\omega) = \frac{|W(j\omega)|^2}{\sigma + |W(j\omega)|^2}.$$

The optimal cost:

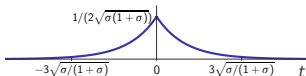
$$\|\mathcal{G}_e\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |W(j\omega)|^2 d\omega - \frac{1}{2\pi} \int_{\mathbb{R}} \frac{|W(j\omega)|^4}{\sigma + |W(j\omega)|^2} d\omega.$$

Example: filtering out noise (contd)

Signal with $w(t) = e^{-t}\mathbb{1}(t)$:

$$W(j\omega) = \frac{1}{1+j\omega} \implies K_{\text{opt}}(j\omega) = \frac{1}{1+\sigma+\sigma\omega^2} \in L_{\infty}(j\mathbb{R}) \cap L_2(j\mathbb{R})$$

so

$$k_{\text{opt}}(t) = \frac{1}{2\sqrt{\sigma(1+\sigma)}} e^{-\sqrt{1+1/\sigma}|t|} =$$


Signal with $w(t) = \mathbb{1}(t)$: let's apply the formulae mechanically

$$W(j\omega) = \frac{1}{j\omega} \implies K_{\text{opt}}(j\omega) = \frac{1}{1+\sigma\omega^2} \in L_{\infty}(j\mathbb{R}) \cap L_2(j\mathbb{R})$$

so

$$k_{\text{opt}}(t) = \frac{1}{2\sqrt{\sigma}} e^{-|t|/\sqrt{\sigma}} =$$

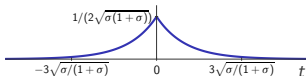
With this choice $G_s(j\omega) = - \begin{bmatrix} \frac{j\sigma\omega}{1+\sigma\omega^2} & \frac{\sqrt{\sigma}}{1+\sigma\omega^2} \end{bmatrix} \in L_{\infty}$, so it does work...

Example: filtering out noise (contd)

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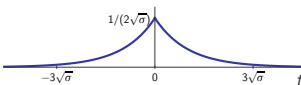
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With this choice $G_e(j\omega) = - \begin{bmatrix} \frac{j\sigma\omega}{1+\sigma\omega^2} & \frac{\sqrt{\sigma}}{1+\sigma\omega^2} \end{bmatrix} \in L_{\infty}$, so it does work...

Bottom line

- L_2 system norm as a performance measure
- the Projection Theorem as the solution tool
- frequency-domain treatment as a solution facilitator

Content

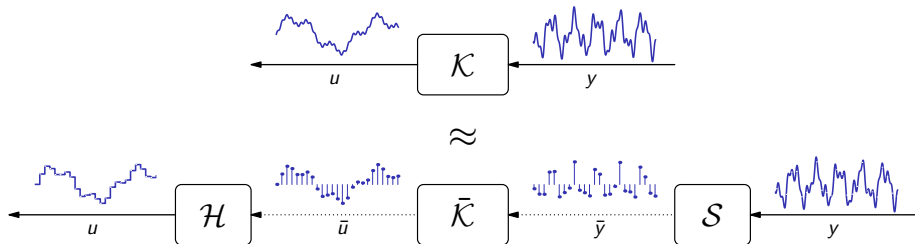
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Lifted systems in frequency domain

L_2 norm of sampled-data systems

Error system in sampled-data problem

Our approximation problem



can be cast as GMMP with $\mathcal{G}_e = \mathcal{K} - \mathcal{H}\bar{\mathcal{K}}\mathcal{S}$ or even

$$\mathcal{G}_e = \mathcal{W}_e \mathcal{K} \mathcal{W}_y - \mathcal{W}_e \mathcal{H} \bar{\mathcal{K}} \mathcal{S} \mathcal{W}_y$$

if spectral properties of y and errors of interest are accounted for. This is a

- periodic system

(if all components are shift invariant) and we need a measure of its size and frequency-domain tools. Lifting can help here...

Lifted frequency response

Apply the lifted Fourier transform to the lifted convolution $\check{y} = \check{g} * \check{u}$:

$$\begin{aligned}
 \check{Y}(e^{j\theta}; \tau) &= \sum_{i \in \mathbb{Z}} \check{y}[i](\tau) e^{-j\theta i} \\
 &= \sum_{i \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} \int_0^h g((i-j)h + \tau, \sigma) \check{u}[j](\sigma) d\sigma \right) e^{-j\theta i} \\
 &= \sum_{i \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} \int_0^h g(ih + \tau, \sigma) \check{u}[j](\sigma) d\sigma \right) e^{-j\theta(i+j)} \\
 &= \int_0^h \sum_{i \in \mathbb{Z}} g(ih + \tau, \sigma) e^{-j\theta i} \sum_{j \in \mathbb{Z}} \check{u}[j](\sigma) e^{-j\theta j} d\sigma \\
 &= \int_0^h \underbrace{\sum_{i \in \mathbb{Z}} g(ih + \tau, \sigma) e^{-j\theta i}}_{\check{G}(e^{j\theta}; \tau, \sigma)} \check{U}(e^{j\theta}; \sigma) d\sigma
 \end{aligned}$$

Lifted frequency response (contd)

Assuming that the lifted Fourier transform of $g(t, s)$ with respect to its “ t ” variable exists, introduce $\check{G}(e^{j\theta}) : \{[0, h) \rightarrow \mathbb{F}^{n_u}\} \rightarrow \{[0, h) \rightarrow \mathbb{F}^{n_y}\}$ via

$$\check{G}(e^{j\theta}) : \check{\zeta} \mapsto \int_0^h \left(\sum_{i \in \mathbb{Z}} g(ih + \tau, \sigma) e^{-j\theta i} \right) \check{\zeta}(\sigma) d\sigma.$$

Then the lifted convolution reads

$$\check{Y}(e^{j\theta}) = \check{G}(e^{j\theta}) \check{U}(e^{j\theta}) \quad \text{or} \quad \check{Y}(e^{j\theta}; \tau) = \int_0^h \check{G}(e^{j\theta}; \tau, \sigma) \check{U}(e^{j\theta}; \sigma) d\sigma$$

in the lifted frequency domain, i.e.

- convolution again becomes a “product.”

The operator $\check{G}(e^{j\theta})$ is called the **lifted frequency response** and can be also interpreted in terms of the response of $\check{G}$ to a harmonic input (we’ll discuss it later on in the course). The function $\check{G}(e^{j\theta}; \tau, \sigma)$ is the *kernel* of $\check{G}(e^{j\theta})$.

Lifted frequency response of continuous-time LTI systems

If $\mathcal{G}$ is LTI, then $g(t, s) = g(t - s)$ and then

$$\check{Y}(e^{j\theta}; \tau) = \int_0^h \sum_{i \in \mathbb{Z}} g(ih + \tau - \sigma) e^{-j\theta i} \check{U}(e^{j\theta}; \sigma) d\sigma.$$

Thus, the relation $\check{Y}(e^{j\theta}) = \check{G}(e^{j\theta}) \check{U}(e^{j\theta})$ reads

$$\check{Y}(e^{j\theta}; \tau) = \int_0^h \check{G}(e^{j\theta}; \tau - \sigma) \check{U}(e^{j\theta}; \sigma) d\sigma,$$

where the kernel of the lifted frequency response is

$$\check{G}(e^{j\theta}; \tau) = \sum_{i \in \mathbb{Z}} g(ih + \tau) e^{-j\theta i} = \sum_{i \in \mathbb{Z}} \check{g}[i](\tau) e^{-j\theta i} = \mathfrak{F}\{\check{g}\}$$

is the lifted Fourier transform of the impulse response $g(t)$ of $\mathcal{G}$.

Example: stable first-order system

Let $\mathcal{G}$ be the LTI system having the impulse response $g(t) = e^{-at}\mathbb{1}(t)$ for some $a > 0$. Its lifted frequency response kernel is

$$\begin{aligned}\check{G}(e^{j\theta}; \tau) &= \sum_{i \in \mathbb{Z}} e^{-a(ih+\tau)} \mathbb{1}(ih + \tau) e^{-j\theta i} = e^{-a\tau} \mathbb{1}(\tau) + e^{-a\tau} \sum_{i \in \mathbb{N}} e^{-(ah+j\theta)i} \\ &= e^{-a\tau} \mathbb{1}(\tau) + \frac{1}{e^{j\theta} - e^{-ah}} e^{-a(\tau+h)}.\end{aligned}$$

(the sum over $\mathbb{N}$ is the standard, not lifted, DTFT of $e^{-ahi} \bar{\mathbb{1}}[i-1]$). Thus, in the lifted frequency domain it acts as

$$\check{Y}(e^{j\theta}; \tau) = \int_0^\tau e^{-a(\tau-\sigma)} \check{U}(e^{j\theta}; \sigma) d\sigma + \frac{e^{-a\tau}}{e^{j\theta} - e^{-ah}} \int_0^h e^{-a(h-\sigma)} \check{U}(e^{j\theta}; \sigma) d\sigma.$$

Lifted Parseval

It can be shown that the lifted Fourier transform preserves sizes,

$$\|\check{f}\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|\check{F}(e^{j\theta})\|_{\mathbb{L}}^2 d\theta =: \|\check{F}\|_2^2,$$

and angles,

$$\langle \check{f}_1, \check{f}_2 \rangle_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle \check{F}_1(e^{j\theta}), \check{F}_2(e^{j\theta}) \rangle_{\mathbb{L}} d\theta =: \langle \check{F}_1, \check{F}_2 \rangle_2,$$

where the inner product on $\mathbb{L} := L_2[0, h)$ is naturally defined as

$$\langle \check{F}_1(e^{j\theta}), \check{F}_2(e^{j\theta}) \rangle_{\mathbb{L}} := \int_0^h [\check{F}_2(e^{j\theta}; \tau)]' \check{F}_1(e^{j\theta}; \tau) d\tau$$

(consider the proof a homework assignment).

Adjoint system in the frequency domain

Consider an LTI $\check{G}$ with the frequency response $\check{G}(e^{j\theta})$. Then

$$\begin{aligned}
 \langle \check{G}\check{u}, \check{y} \rangle_2 &= \langle \check{G}\check{U}, \check{Y} \rangle_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle \check{G}(e^{j\theta})\check{U}(e^{j\theta}), \check{Y}(e^{j\theta}) \rangle_{\mathbb{L}} d\theta \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^h [\check{Y}(e^{j\theta}; \tau)]' \int_0^h \check{G}(e^{j\theta}; \tau, \sigma) \check{U}(e^{j\theta}; \sigma) d\sigma d\tau d\theta \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^h \int_0^h [[\check{G}(e^{j\theta}; \tau, \sigma)]' \check{Y}(e^{j\theta}; \tau)]' \check{U}(e^{j\theta}; \sigma) d\sigma d\tau d\theta \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\int_0^h \int_0^h [\check{G}(e^{j\theta}; \tau, \sigma)]' \check{Y}(e^{j\theta}; \tau) d\tau \right]' \check{U}(e^{j\theta}; \sigma) d\sigma d\theta \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\int_0^h \int_0^h [\check{G}(e^{j\theta}; \sigma, \tau)]' \check{Y}(e^{j\theta}; \sigma) d\sigma \right]' \check{U}(e^{j\theta}; \tau) d\tau d\theta \\
 &= \langle \check{U}, \check{G}^* \check{Y} \rangle_2 = \langle \check{u}, \check{G}^* \check{y} \rangle_2
 \end{aligned}$$

Frequency response of adjoint systems

Thus, the adjoint system acts in the frequency domain as

$$\check{U}(e^{j\theta}; \tau) = \int_0^h [\check{G}(e^{j\theta}; \sigma, \tau)]' \check{Y}(e^{j\theta}; \sigma) d\sigma,$$

which corresponds to a system with the frequency response $\check{G}'(e^{-j\theta}; \sigma, \tau)$.

If the continuous-time $\mathcal{G}$ is LTI, the adjoint system $\check{\mathcal{G}}^*$ acts in the frequency domain as

$$\check{U}(e^{j\theta}; \tau) = \int_0^h [\check{G}(e^{j\theta}; \sigma - \tau)]' \check{Y}(e^{j\theta}; \sigma) d\sigma,$$

which corresponds to a system with the frequency response $\check{G}'(e^{-j\theta}; -\tau)$.

Lifted frequency response of A/D and D/A converters

Repeating arguments used in the continuous-time case, lifted sampler and hold in the frequency domain act as

$$\bar{Y}(e^{j\theta}) = \bar{\Psi}(e^{j\theta}) \check{Y}(e^{j\theta}) \quad \text{or} \quad \bar{Y}(e^{j\theta}) = e^{-j\theta} \int_0^h \bar{\Psi}(e^{j\theta}; h - \sigma) \check{Y}(e^{j\theta}; \sigma) d\sigma$$

and

$$\check{U}(e^{j\theta}) = \check{\Phi}(e^{j\theta}) \bar{U}(e^{j\theta}) \quad \text{or} \quad \check{U}(e^{j\theta}; \tau) = \check{\Phi}(e^{j\theta}; \tau) \bar{U}(e^{j\theta})$$

Here the lifted frequency responses are operators as well:

$$\bar{\Psi}(e^{j\theta}) : \{[0, h) \rightarrow \mathbb{F}^{n_y}\} \rightarrow \mathbb{F}^{n_{\bar{y}}}, \quad \text{with} \quad \bar{\Psi}(e^{j\theta}; \sigma) = \sum_{i \in \mathbb{Z}} \psi(ih + \sigma) e^{-j\theta i}$$

$$\check{\Phi}(e^{j\theta}) : \mathbb{F}^{n_{\bar{u}}} \rightarrow \{[0, h) \rightarrow \mathbb{F}^{n_u}\}, \quad \text{with} \quad \check{\Phi}(e^{j\theta}; \tau) = \sum_{i \in \mathbb{Z}} \phi(ih + \tau) e^{-j\theta i}$$

Note that $\bar{\Psi}(e^{j\theta}; \sigma) = \check{\psi}(e^{j\theta}; \sigma)$ (the lifted Fourier transforms of $\mathcal{L}_h \psi$) and $\check{\Phi}(e^{j\theta}; \tau) = \check{\phi}(e^{j\theta}; \tau)$ (the lifted Fourier transforms of $\mathcal{L}_h \phi$).

Example: ideal sampler

Consider $\mathcal{S}_{\text{idl}}$ having the sampling function $\psi(s) = \delta(s)$. Then

$$\hat{\Psi}(e^{j\theta}; \sigma) = \check{\Psi}(e^{j\theta}; \sigma) = \sum_{i \in \mathbb{Z}} \delta(ih + \sigma) e^{-j\theta i} = \delta(\sigma),$$

so the sampler acts in the lifted frequency domain as

$$\bar{Y}(e^{j\theta}) = e^{-j\theta} \int_0^h \delta(h - \sigma) \check{Y}(e^{j\theta}; \sigma) d\sigma = e^{-j\theta} \check{Y}(e^{j\theta}; h) = \check{Y}(e^{j\theta}; 0).$$

Example: averaging sampler

Consider $\mathcal{S}_{\text{avg}}$ having the sampling function $\psi(s) = \frac{1}{h} \mathbb{1}_{[0,h]}(s)$. We already saw that in this case

$$\Psi(e^{j\theta}; \sigma) = \check{\Psi}(e^{j\theta}; \sigma) = \frac{1}{h},$$

so the sampler acts in the lifted frequency domain as

$$\bar{Y}(e^{j\theta}) = \frac{e^{-j\theta}}{h} \int_0^h \check{Y}(e^{j\theta}; \sigma) d\sigma.$$

Example: zero-order hold

Consider $\mathcal{H}_{\text{ZOH}}$ with the hold function $\phi(t) = \mathbb{1}_{[0,h]}(t)$. Again, we know that

$$\dot{\Phi}(e^{j\theta}; \tau) = \check{\Phi}(e^{j\theta}; \tau) = 1$$

and the hold acts in the lifted frequency domain as

$$\check{U}(e^{j\theta}; \tau) = \bar{U}(e^{j\theta}).$$

Example: first-order hold

Consider $\mathcal{H}_{\text{FOH}}$ with the hold function $\psi(t) = (1 - \frac{|t|}{h}) \mathbb{1}_{[-h,h]}(t)$. Then, see Lecture 4,

$$\Phi(e^{j\theta}; \tau) = \phi(e^{j\theta}; \tau) = \frac{\tau}{h} e^{j\theta} + \left(1 - \frac{\tau}{h}\right)$$

and the hold acts in the lifted frequency domain as

$$\check{U}(e^{j\theta}; \tau) = \left(\frac{\tau}{h} e^{j\theta} + \frac{h - \tau}{h}\right) \bar{U}(e^{j\theta}).$$

Example: sample-and-hold circuits

Let $\mathcal{G} = \mathcal{H}_{\text{ZOH}} \mathcal{S}_{\text{avg}}$. Its lifted impulse response operator is the “product” (or the cascade, more precisely) of those of $\mathcal{H}_{\text{ZOH}}$ and $\mathcal{S}_{\text{avg}}$. Thus,

$$\check{Y}(e^{j\theta}; \tau) = \frac{e^{-j\theta}}{h} \int_0^h \check{U}(e^{j\theta}; \sigma) d\sigma.$$

Remark: Can also be derived via the impulse response of $\mathcal{G}$, which is

$$g(t, s) = \mathbb{1}_{[i_t-1, i_t]}(s), \quad \text{where } i_t := \lfloor t/h \rfloor.$$

Let $\mathcal{G} = \mathcal{H}_{\text{ZOH}} \mathcal{S}_{\text{avg}}$. Again, combining its lifted components we get:

$$\check{Y}(e^{j\theta}; \tau) = \frac{1}{h} \left(\frac{\tau}{h} + \frac{h-\tau}{h} e^{-j\theta} \right) \int_0^h \check{U}(e^{j\theta}; \sigma) d\sigma.$$

Example: sample-and-hold circuits

Let $\mathcal{G} = \mathcal{H}_{\text{ZOH}}\mathcal{S}_{\text{avg}}$. Its lifted impulse response operator is the “product” (or the cascade, more precisely) of those of $\mathcal{H}_{\text{ZOH}}$ and $\mathcal{S}_{\text{avg}}$. Thus,

$$\check{Y}(\mathrm{e}^{\mathrm{j}\theta}; \tau) = \frac{\mathrm{e}^{-\mathrm{j}\theta}}{h} \int_0^h \check{U}(\mathrm{e}^{\mathrm{j}\theta}; \sigma) \mathrm{d}\sigma.$$

Remark: Can also be derived via the impulse response of $\mathcal{G}$, which is

$$g(t, s) = \mathbb{1}_{[i_t-1, i_t]}(s), \quad \text{where } i_t := \lfloor t/h \rfloor.$$

Let $\mathcal{G} = \mathcal{H}_{\text{FOH}}\mathcal{S}_{\text{avg}}$. Again, combining its lifted components we get:

$$\check{Y}(\mathrm{e}^{\mathrm{j}\theta}; \tau) = \frac{1}{h} \left(\frac{\tau}{h} + \frac{h-\tau}{h} \mathrm{e}^{-\mathrm{j}\theta} \right) \int_0^h \check{U}(\mathrm{e}^{\mathrm{j}\theta}; \sigma) \mathrm{d}\sigma.$$

Frequency response of adjoint sampler and hold

We already know (Lecture 2) that adjoint of a generalized sampler with the sampling function $\psi(t)$ is a generalized hold with the hold function $\psi'(-t)$. Its lifted frequency response is then $[\dot{\Psi}(e^{j\theta})]^* : \mathbb{F}^{n_{\bar{y}}} \rightarrow \{[0, h) \rightarrow \mathbb{F}^{n_y}\}$ with

$$\begin{aligned} [\dot{\Psi}(e^{j\theta}; \tau)]^* &=: \dot{\Psi}(e^{j\theta}; \tau) = \sum_{i \in \mathbb{Z}} \phi'(-ih - \tau) e^{-j\theta i} = \sum_{i \in \mathbb{Z}} \phi'(ih - \tau) e^{j\theta i} \\ &= e^{j\theta} \dot{\Psi}'(e^{-j\theta}; h - \tau) \end{aligned}$$

Likewise, the frequency response of the adjoint of a generalized hold with the hold function $\phi(t)$ is a generalized sampler with the sampling function $\phi'(-t)$. Its lifted frequency response $[\dot{\Phi}(e^{j\theta})]^* : \{[0, h) \rightarrow \mathbb{F}^{n_u}\} \rightarrow \mathbb{F}^{n_u}$ is

$$\begin{aligned} [\dot{\Phi}(e^{j\theta}; \sigma)]^* &=: \dot{\Psi}(e^{j\theta}; \sigma) = \sum_{i \in \mathbb{Z}} \psi'(-ih - \sigma) e^{-j\theta i} = \sum_{i \in \mathbb{Z}} \psi'(ih - \sigma) e^{j\theta i} \\ &= e^{j\theta} \dot{\Psi}'(e^{-j\theta}; h - \sigma) \end{aligned}$$

Frequency response of adjoint sampler and hold

We already know (Lecture 2) that adjoint of a generalized sampler with the sampling function $\psi(t)$ is a generalized hold with the hold function $\psi'(-t)$. Its lifted frequency response is then $[\acute{\Psi}(e^{j\theta})]^* : \mathbb{F}^{n_{\bar{y}}} \rightarrow \{[0, h) \rightarrow \mathbb{F}^{n_y}\}$ with

$$\begin{aligned} [\acute{\Psi}(e^{j\theta}; \tau)]^* &=: \grave{\Psi}(e^{j\theta}; \tau) = \sum_{i \in \mathbb{Z}} \phi'(-ih - \tau) e^{-j\theta i} = \sum_{i \in \mathbb{Z}} \phi'(ih - \tau) e^{j\theta i} \\ &= e^{j\theta} \acute{\Psi}'(e^{-j\theta}; h - \tau) \end{aligned}$$

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$$\begin{aligned} [\acute{\Phi}(e^{j\theta}; \sigma)]^* &=: \acute{\Psi}(e^{j\theta}; \sigma) = \sum_{i \in \mathbb{Z}} \psi'(-ih - \sigma) e^{-j\theta i} = \sum_{i \in \mathbb{Z}} \psi'(ih - \sigma) e^{j\theta i} \\ &= e^{j\theta} \grave{\Phi}'(e^{-j\theta}; h - \sigma) \end{aligned}$$

Frequency response of $\bar{\mathcal{G}}_I = \mathcal{SH}$

Let $\mathcal{S}$ and $\mathcal{H}$ be generalized sampler and hold with the sampling and hold functions $\psi(t)$ and $\phi(t)$, respectively. Then

$$\begin{aligned}\bar{G}_I(e^{j\theta}) &= \dot{\Psi}(e^{j\theta})\dot{\Phi}(e^{j\theta}) = e^{-j\theta} \int_0^h \dot{\Psi}(e^{j\theta}; h - \sigma) \dot{\Phi}(e^{j\theta}; \sigma) d\sigma \\ &= e^{-j\theta} \int_0^h \check{\psi}(e^{j\theta}; h - \sigma) \check{\phi}(e^{j\theta}; \sigma) d\sigma.\end{aligned}$$

By the Key Lifting Formula (with $\omega_i := \frac{\theta}{h} + \frac{2\pi}{h}i$)

$$\check{\psi}(e^{j\theta}; \sigma) = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Psi(j\omega_i) e^{j\omega_i \sigma} \quad \text{and} \quad \check{\phi}(e^{j\theta}; \sigma) = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Phi(j\omega_i) e^{j\omega_i \sigma},$$

so that

$$\begin{aligned}\check{\psi}(e^{j\theta}; h - \sigma) \check{\phi}(e^{j\theta}; \sigma) &= \frac{1}{h^2} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \Psi(j\omega_i) \Phi(j\omega_j) e^{j\omega_i h} e^{j(\omega_i - \omega_j) \sigma} \\ &= \frac{1}{h^2} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \Psi(j\omega_i) \Phi(j\omega_j) e^{j\theta} e^{j2\pi(i-j)\sigma/h}\end{aligned}$$

Frequency response of $\bar{\mathcal{G}}_I = \mathcal{SH}$ (contd)

Hence,

$$\begin{aligned}\bar{G}_I(e^{j\theta}) &= e^{-j\theta} \int_0^h \frac{1}{h^2} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \Psi(j\omega_i) \Phi(j\omega_j) e^{j\theta} e^{j2\pi(i-j)\sigma/h} d\sigma \\ &= \frac{1}{h^2} \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \Psi(j\omega_i) \Phi(j\omega_j) \int_0^h e^{j2\pi(i-j)\sigma/h} d\sigma = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Psi(j\omega_i) \Phi(j\omega_i)\end{aligned}$$

(by $\int_0^h e^{j2\pi i\sigma/h} d\sigma = h \bar{\delta}[i]$). We thus end up with two alternative forms of the frequency response of $\mathcal{SH}$:

$$\bar{G}_I(e^{j\theta}) = e^{-j\theta} \int_0^h \check{\psi}(e^{j\theta}; h - \sigma) \check{\phi}(e^{j\theta}; \sigma) d\sigma = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Psi(j\omega_i) \Phi(j\omega_i)$$

Each one of them may be useful...

Frequency response of $\bar{\mathcal{G}}_I = \mathcal{SH}$: special cases

If $\bar{\mathcal{G}}_I = \mathcal{H}^* \mathcal{H}$,

$$\bar{G}_I(e^{j\theta}) = \int_0^h [\check{\phi}(e^{j\theta}; \tau)]' \check{\phi}(e^{j\theta}; \tau) d\tau = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Phi'(-j\omega_i) \Phi(j\omega_i)$$

If $\bar{\mathcal{G}}_I = \mathcal{SS}^*$,

$$\bar{G}_I(e^{j\theta}) = \int_0^h \check{\psi}(e^{j\theta}; \sigma) [\check{\psi}(e^{j\theta}; \sigma)]' d\sigma = \frac{1}{h} \sum_{i \in \mathbb{Z}} \Psi(j\omega_i) \Psi'(-j\omega_i)$$

Frequency response and stability

The space $L_{\infty}^{\mathbb{L} \rightarrow \mathbb{L}}(\mathbb{T})$ is defined as

$$L_{\infty}^{\mathbb{L} \rightarrow \mathbb{L}}(\mathbb{T}) := \left\{ \check{G} : \mathbb{T} \rightarrow \{\mathbb{L} \rightarrow \mathbb{L}\} \mid \|\check{G}\|_{\infty} := \operatorname{ess\,sup}_{\theta \in [-\pi, \pi]} \|\check{G}(e^{j\theta})\|_{\mathbb{L} \rightarrow \mathbb{L}} < \infty \right\}$$

and the spaces $L_{\infty}^{\mathbb{L} \rightarrow \mathbb{F}}(\mathbb{T})$ and $L_{\infty}^{\mathbb{F} \rightarrow \mathbb{L}}(\mathbb{T})$ of operators $\acute{G}$ and $\grave{G}$, respectively, are defined accordingly. We say that systems $\check{G} \in L_{\infty}$ / $\acute{G} \in L_{\infty}$ / $\grave{G} \in L_{\infty}$ if their frequency responses belong to the corresponding $L_{\infty}(\mathbb{T})$ and

$$\|\check{G}\|_{\infty} := \|\check{G}\|_{\infty}, \quad \|\acute{G}\|_{\infty} := \|\acute{G}\|_{\infty}, \quad \text{and} \quad \|\grave{G}\|_{\infty} := \|\grave{G}\|_{\infty}.$$

It can be shown that $\|\check{G}\|_{\infty}$ equals $\ell_2^{\mathbb{L}}(\mathbb{Z})$ -induced norm of $\check{G}$, i.e.

$$\|\check{G}\|_{\infty} = \sup_{\|\check{u}\|_2=1} \|\check{G}\check{u}\|_2.$$

and the same is about $\|\acute{G}\|_{\infty}$ and $\|\grave{G}\|_{\infty}$. Thus,

- L_{∞} are the spaces of stable systems.

Content

L_2 non-causal model-matching

Lifted systems in frequency domain

L_2 norm of sampled-data systems

Time-domain deterministic approach: naïve extension

Let $\mathcal{G}$ be an h -periodic system with the impulse response $g(t, s)$, define

$$\|\mathcal{G}\|_{2,\text{naïve}} := \left(\int_{-\infty}^{\infty} \|g(t, 0)\|_{\mathbb{F}}^2 dt \right)^{1/2},$$

i.e. the energy of its response to the input $u(t) = \delta(t)$. This definition was used

- implicitly in the sampled-data LQR (early '70s)
- explicitly in early sampled-data H_2 (early '90s)

The problem is that

- $\|\mathcal{G}\|_{2,\text{naïve}}$ is not a norm, it's a semi-norm
(i.e. $\|\mathcal{G}\|_{2,\text{naïve}} = 0 \not\Rightarrow \mathcal{G} = 0$; for example, the (unstable) periodic $\mathcal{G}$ with

$$g(t, s) = \begin{cases} e^{t-s} \mathbf{1}(t-s) & \text{if } s \bmod h \in (0.5h, h) \\ 0 & \text{otherwise} \end{cases}$$

has $\|\mathcal{G}\|_{2,\text{naïve}} = 0$), which might be inconvenient.

Time-domain deterministic approach: naïve extension

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has $\|\mathcal{G}\|_{2,\text{naïve}} = 0$), which might be inconvenient.

Time-domain deterministic approach

It may appear natural to average energies of the responses to every impulse applied in $[0, h]$, like

$$\|\mathcal{G}\|_2 := \left(\frac{1}{h} \int_0^h \int_{-\infty}^{\infty} \|g(t, s)\|_{\mathbb{F}}^2 dt ds \right)^{1/2},$$

which is

- indeed a norm
- reduces to the standard definition of $\mathcal{G}$ is time invariant

With this interpretation in mind, consider defining it in the lifted domain...

Preliminary: Hilbert-Schmidt operators

Let $\mathbb{H}_i$ and $\mathbb{H}_o$ be Hilbert spaces, $\{e_i\}$ be an orthonormal basis on $\mathbb{H}_i$, and $\|\cdot\|_{\mathbb{H}_o}$ be the norm on $\mathbb{H}_o$ and consider a bounded operator $O : \mathbb{H}_i \rightarrow \mathbb{H}_o$. The Hilbert-Schmidt norm of O is defined as

$$\|O\|_{\text{HS}} := \left(\sum_i \|Oe_i\|_{\mathbb{H}_o}^2 \right)^{1/2}$$

(is independent of the choice of $\{e_i\}$). We say that O is a Hilbert-Schmidt operator if $\|O\|_{\text{HS}} < \infty$. It can be shown that $\|O\|_{\text{HS}} = \|O^*\|_{\text{HS}}$.

The inner product of two Hilbert-Schmidt operators $\mathbb{H}_i \rightarrow \mathbb{H}_o$ is defined as

$$\langle O_1, O_2 \rangle_{\text{HS}} := \sum_i \langle O_1 e_i, O_2 e_i \rangle_{\mathbb{H}_o} = \sum_i \langle O_2^* O_1 e_i, e_i \rangle_{\mathbb{H}_i} =: \text{tr}(O_2^* O_1)$$

so that $\|O\|_{\text{HS}}^2 = \langle O, O \rangle_{\text{HS}}$.

Preliminary: Hilbert-Schmidt operators (contd)

Special cases:

- the Hilbert-Schmidt norm of $O : \mathbb{F}^m \rightarrow \mathbb{F}^p$ equals the Frobenius norm of its matrix representation M_O , $\|O\|_{\text{HS}} = \|M_O\|_{\text{F}}$.
- the Hilbert-Schmidt norm of an integral operator $O : \mathbb{L} \rightarrow \mathbb{L}$ with the kernel $k_o(t, s)$ is

$$\|O\|_{\text{HS}}^2 = \int_0^h \int_0^h \|k_o(t, s)\|_{\text{F}}^2 ds dt$$

(unbounded whenever $k_o(t, s)$ contains Dirac deltas) and then

$$\langle O_1, O_2 \rangle_{\text{HS}} = \int_0^h \int_0^h \text{tr}([k_{o2}(t, s)]' k_{o1}(t, s)) ds dt$$

Remark 1: The identity operator I on any infinite-dimensional Hilbert space is *not* a Hilbert-Schmidt operator, $\|I\|_{\text{HS}}^2 = \sum_i \|e_i\|_{\mathbb{H}_0}^2 = \sum_i 1 = \infty$.

Remark 2: A bounded *finite-rank* O is always a Hilbert-Schmidt operator.

$L_2(\mathbb{T})$ space of operators on $\mathbb{L}$

The space $L_2^{\mathbb{L} \rightarrow \mathbb{L}}(\mathbb{T})$ is defined as

$$L_2^{\mathbb{L} \rightarrow \mathbb{L}}(\mathbb{T}) := \left\{ \check{G} : \mathbb{T} \rightarrow \{\mathbb{L} \rightarrow \mathbb{L}\} \mid \|\check{G}\|_2 := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \|\check{G}(e^{j\theta})\|_{\text{HS}}^2 d\theta \right)^{1/2} < \infty \right\}$$

It is a Hilbert space, with the inner product

$$\langle \check{G}_1, \check{G}_2 \rangle_2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle \check{G}_1(e^{j\theta}), \check{G}_2(e^{j\theta}) \rangle_{\text{HS}} d\theta.$$

The spaces $L_2^{\mathbb{L} \rightarrow \mathbb{F}}(\mathbb{T})$ and $L_2^{\mathbb{F} \rightarrow \mathbb{L}}(\mathbb{T})$ of operators $\acute{G}$ and $\grave{G}$, respectively, are defined accordingly...

Remark: A nonzero constant $\acute{G}(e^{j\theta})$ belongs to $L_2(\mathbb{T})$ only if $\|\acute{G}\|_{\text{HS}} < \infty$. Nonzero constant $\acute{G}(e^{j\theta})$ and $\grave{G}(e^{j\theta})$ always belong to $L_2(\mathbb{T})$ because their rank is finite (boundedness is required, of course).

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L₂ space of lifted LTI system

Define the space of LTI systems whose frequency response, $\check{G}(e^{j\theta})$, belongs to $L_2(\mathbb{T})$ by L_2 . This is a Hilbert space with

$$\langle \check{G}_1, \check{G}_2 \rangle := \frac{1}{h} \langle \check{G}_1, \check{G}_2 \rangle_2 = \frac{1}{2\pi h} \int_{-\pi}^{\pi} \langle \check{G}_1(e^{j\theta}), \check{G}_2(e^{j\theta}) \rangle_{\text{HS}} d\theta$$

and

$$\|\check{G}\|_2 := \frac{1}{\sqrt{h}} \|\check{G}\|_2 = \left(\frac{1}{2\pi h} \int_{-\pi}^{\pi} \|\check{G}(e^{j\theta})\|_{\text{HS}}^2 d\theta \right)^{1/2}$$

(the factor $\frac{1}{h}$ is added to keep averaging interpretation in the time domain).

Again, we have that

- $\check{T}\check{G} \in L_2$ and $\check{G}\check{T} \in L_2$ whenever $\check{G} \in L_2$ and $\check{T} \in L_\infty$
- for all $\check{G}_1, \check{G}_2 \in L_2$ and $\check{T}_1, \check{T}_2 \in L_\infty$, $(\check{G}_1, \check{T}_1 \check{G}_2 \check{T}_2)_2 = (\check{T}_1^* \check{G}_1 \check{T}_2^*, \check{G}_2)_2$

L₂ space of lifted LTI system

Define the space of LTI systems whose frequency response, $\check{G}(e^{j\theta})$, belongs to $L_2(\mathbb{T})$ by L_2 . This is a Hilbert space with

$$\langle \check{G}_1, \check{G}_2 \rangle := \frac{1}{h} \langle \check{G}_1, \check{G}_2 \rangle_2 = \frac{1}{2\pi h} \int_{-\pi}^{\pi} \langle \check{G}_1(e^{j\theta}), \check{G}_2(e^{j\theta}) \rangle_{\text{HS}} d\theta$$

and

$$\|\check{G}\|_2 := \frac{1}{\sqrt{h}} \|\check{G}\|_2 = \left(\frac{1}{2\pi h} \int_{-\pi}^{\pi} \|\check{G}(e^{j\theta})\|_{\text{HS}}^2 d\theta \right)^{1/2}$$

(the factor $\frac{1}{h}$ is added to keep averaging interpretation in the time domain).

Again, we have that¹

- $\check{T}\check{G} \in L_2$ and $\check{G}\check{T} \in L_2$ whenever $\check{G} \in L_2$ and $\check{T} \in L_\infty$
- for all $\check{G}_1, \check{G}_2 \in L_2$ and $\check{T}_1, \check{T}_2 \in L_\infty$, $\langle \check{G}_1, \check{T}_1 \check{G}_2 \check{T}_2 \rangle_2 = \langle \check{T}_1^* \check{G}_1 \check{T}_2^*, \check{G}_2 \rangle_2$

¹We say that $\check{G} \in L_\infty$ if it's bounded $\ell_2^{\mathbb{L}}(\mathbb{Z}) \rightarrow \ell_2^{\mathbb{L}}(\mathbb{Z})$ (or, equivalently, is the lifting of $G : L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})$).

Bottom line

Lifted systems in the frequency domain are much like conventional discrete systems, just little different:

- integral operators instead of matrices
- Hilbert-Schmidt norm instead of Frobenius
- ...