

Advanced Sampled-Data Control (038798)

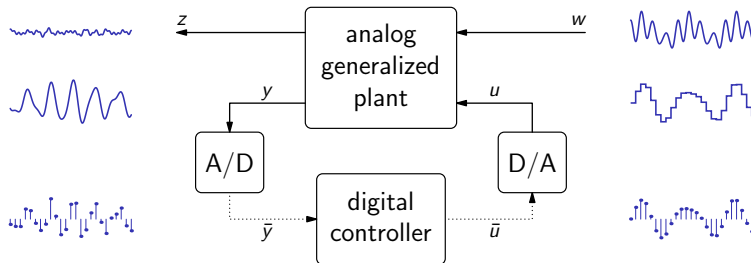
lecture no. 4

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Sampled-data standard problem



Next question:

- how to combine continuous- and discrete-time behavior?

Content

Lifting

History

Lifting analog systems

Lifting A/D and D/A converters

Content

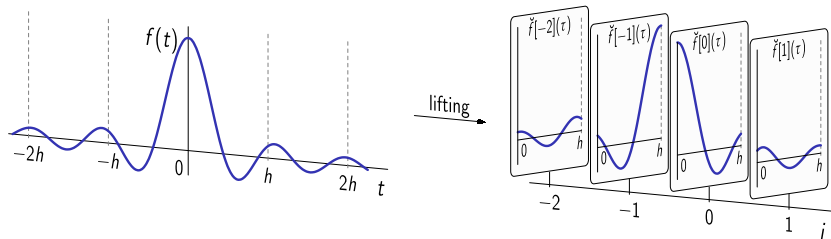
Lifting

History

Lifting analog systems

Lifting A/D and D/A converters

The lifting transformation

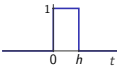
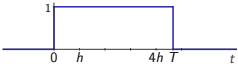
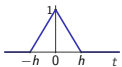


Given an $h > 0$, any analog signal $f : \mathbb{R} \rightarrow \mathbb{F}^n$ can be equivalently cast as a sequence of functions $\{\check{f}[i]\}_{i \in \mathbb{Z}}$ such that $\check{f}[i] : [0, h) \rightarrow \mathbb{F}^n$ is defined as

$$\check{f}[i](\tau) = f(ih + \tau), \quad \forall i \in \mathbb{Z} \text{ and } \tau \in [0, h).$$

The **lifting operator** is defined as the (bijective) mapping $\mathcal{L}_h : f \mapsto \check{f}$ and the discrete signal $\check{f}$ is called the lifting of the analog signal f .

Some lifting pairs

$f(t)$	$\check{f}[i]$
 $= \mathbb{1}_{[0,h]}(t)$	$\begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$
 $= \mathbb{1}_{[0,T]}(t)$	$\begin{cases} 1 & \text{if } i \in \mathbb{Z}_{0..\lfloor \frac{T-h}{h} \rfloor} \\ \mathbb{1}_{[0, T \bmod h]}(\tau) & \text{if } i = \lfloor \frac{T}{h} \rfloor \\ 0 & \text{otherwise} \end{cases}$
 $= (1 - \frac{ t }{h}) \mathbb{1}_{[-h,h]}(t)$	$\begin{cases} \frac{\tau}{h} & \text{if } i = -1 \\ 1 - \frac{\tau}{h} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$
$(1 + \frac{t}{h}) \mathbb{1}_{[0,h]}(t) + (1 - \frac{t}{h}) \mathbb{1}_{[h,2h]}(t)$	$\begin{cases} 1 + \frac{\tau}{h} & \text{if } i = 0 \\ -\frac{\tau}{h} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$

Signal spaces

Let $\mathbb{L}^n := L_2^n[0, h)$ (just $\mathbb{L}$, when the dimension is irrelevant). If $f \in L_2(\mathbb{R})$,

$$\begin{aligned}\|f\|_2^2 &= \int_{-\infty}^{\infty} \|f(t)\|^2 dt = \sum_{i \in \mathbb{Z}} \int_{ih}^{(i+1)h} \|f(t)\|^2 dt \\ &= \sum_{i \in \mathbb{Z}} \int_0^h \|f(ih + \tau)\|^2 d\tau = \sum_{i \in \mathbb{Z}} \int_0^h \|\check{f}[i](\tau)\|^2 d\tau \\ &= \sum_{i \in \mathbb{Z}} \|\check{f}[i]\|_2^2\end{aligned}$$

Thus, $\mathcal{L}_h$ is an *isometry* $L_2(\mathbb{R}) \rightarrow \ell_2^{\mathbb{L}}(\mathbb{Z})$, where

$$\ell_2^{\mathbb{L}}(\mathbb{Z}) := \left\{ \check{f} : \mathbb{Z} \rightarrow \mathbb{L} \mid \|\check{f}\|_2^2 := \sum_{i \in \mathbb{Z}} \|\check{f}[i]\|_2^2 < \infty \right\}.$$

The space $\ell_{2,\text{loc}}^{\mathbb{L}}(\mathbb{Z})$ is defined correspondingly.

Lifted Fourier transform

The Fourier transform of a lifted signal $\check{f}$:

$$\mathfrak{F}\{\check{f}\} = \check{F}(e^{j\theta}) := \sum_{i \in \mathbb{Z}} \check{f}[i] e^{-j\theta i},$$

where $\theta \in [-\pi, \pi]$ called the (angular) frequency and measured in rad/step.
At each frequency θ

- lifted Fourier transform is still a function of intersample time $\tau \in [0, h)$.

When this dependence is important we use the notation $\check{F}(e^{j\theta}; \tau)$.

Example: pulses

Let $f(t) = \mathbb{1}_{[0, \nu h]}(t)$ for some $\nu \in \mathbb{N}$. Its lifted Fourier transform

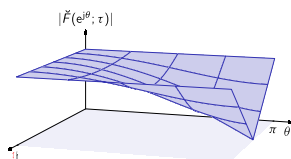
$$\check{F}(e^{j\theta}; \tau) = \sum_{i=0}^{\nu-1} e^{-j\theta i} = \begin{cases} \frac{1}{\nu} |\check{F}(e^{j\theta}; \tau)| & \text{if } \nu = 1 \\ \frac{1}{\nu} |\check{F}(e^{j\theta}; \tau)| & \text{if } \nu = 3 \\ \frac{1}{\nu} |\check{F}(e^{j\theta}; \tau)| & \text{if } \nu = 8 \end{cases}$$

(it happens to be independent of τ , will depend for $\mathbb{1}_{[0, T]}(t)$ with $T > 0$).

Example: tent (the hold function of the FOH)

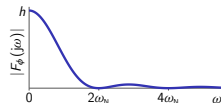
Let $f(t) = (1 - \frac{|t|}{h}) \mathbb{1}_{[-h,h]}(t)$. Its lifted Fourier transform

$$\breve{F}(e^{j\theta}; \tau) = \frac{\tau}{h} e^{j\theta} + \left(1 - \frac{\tau}{h}\right) =$$



As a matter of fact, it conveys the same information as

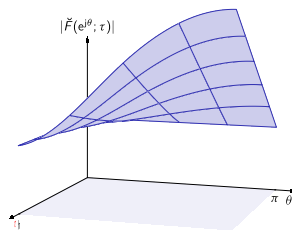
$$\mathfrak{F}\{f\} = F(j\omega) = \frac{2(1 - \cos(\omega h))}{\omega^2 h} =$$



Example: the hold function of the PFOH

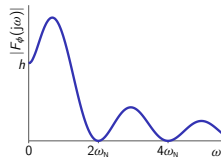
Let $f(t) = (1 + \frac{t}{h})\mathbb{1}_{[0,h]}(t) + (1 - \frac{t}{h})\mathbb{1}_{[h,2h]}(t)$. Its lifted Fourier transform

$$\check{F}(e^{j\theta}; \tau) = \left(1 + \frac{\tau}{h}\right) - \frac{\tau}{h} e^{-j\theta} =$$



It also conveys the same information as

$$\mathfrak{F}\{f\} = F(j\omega) = \frac{4(1 + j\omega h) \sin^2 \frac{\omega h}{2} e^{-j\omega h}}{\omega^2 h} =$$



Lifted z-transform

The two-sided (bilateral) z-transform of a lifted signal $\check{f}$:

$$\mathcal{Z}\{\check{f}\} = \check{F}(z) := \sum_{i \in \mathbb{Z}} \check{f}[i] z^{-i}$$

for all $z \in \mathbb{C}$ for which the sum converges (RoC). Again, at each such s

- lifted z-transform is still a function of intersample time $\tau \in [0, h)$.

When this dependence is important we use the notation $\check{F}(z; \tau)$.

Key Lifting Formula

Consider $f(t)$ and let $s_0 \in \mathbb{C}$ be such that $f(t)e^{-s_0 t} \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$. Then

$$\begin{aligned}
 F(s_0) &= \int_{\mathbb{R}} f(t)e^{-s_0 t} dt = \sum_{i \in \mathbb{Z}} \int_0^h f(\tau + ih)e^{-s_0(\tau + ih)} d\tau \\
 &= \int_0^h \sum_{i \in \mathbb{Z}} \check{f}[i](\tau) e^{s_0 ih} e^{-s_0 \tau} d\tau = \int_0^h \sum_{i \in \mathbb{Z}} \check{f}[i](\tau) (e^{s_0 h})^{-i} e^{-s_0 \tau} d\tau \\
 &= \int_0^h \check{F}(e^{s_0 h}; \tau) e^{-s_0 \tau} d\tau.
 \end{aligned}$$

Define now $s_i := s_0 + j\frac{2\pi}{h}i$, so that (mind that $e^{s_0 h} = e^{s_0 h}$ for all $i \in \mathbb{Z}$)

$$\frac{1}{h} F(s_i) = \frac{1}{h} \int_0^h \check{F}(e^{s_0 h}; \tau) e^{-s_i \tau} d\tau = \frac{1}{h} \int_0^h (\check{F}(e^{s_0 h}; \tau) e^{-s_0 \tau}) e^{-j\frac{2\pi}{h}i\tau} d\tau$$

is the i th Fourier series coefficient of $\check{F}(e^{s_0 h}; \tau) e^{-s_0 \tau}$.

Key Lifting Formula

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Define now $s_i := s_0 + j\frac{2\pi}{h}i$, so that (mind that $e^{s_i h} = e^{s_0 h}$ for all $i \in \mathbb{Z}$)

$$\frac{1}{h} F(s_i) = \frac{1}{h} \int_0^h \check{F}(e^{s_i h}; \tau) e^{-s_i \tau} d\tau = \frac{1}{h} \int_0^h (\check{F}(e^{s_0 h}; \tau) e^{-s_0 \tau}) e^{-j\frac{2\pi}{h}i\tau} d\tau$$

is the i th Fourier series coefficient of $\check{f}(e^{s_0 h}; \tau) e^{-s_0 \tau}$.

Key Lifting Formula (contd)

Thus,

$$\check{f}(e^{s_0 h}; \tau) e^{-s_0 \tau} = \sum_{i \in \mathbb{Z}} \frac{1}{h} F(s_i) e^{j \frac{2\pi}{h} i \tau}$$

and we end up with the following formula:

$$\check{f}(e^{s_0 h}; \tau) = \frac{1}{h} \sum_{i \in \mathbb{Z}} F(s_i) e^{s_i \tau},$$

which is referred to as the Key Lifting Formula (KLF). If $f(t)e^{-s_0 t} \in L_2(\mathbb{R})$, the KLF holds in L_2 sense (by Plancherel) and thus holds pointwise almost everywhere.

A particular case (in the frequency domain):

$$\check{f}(e^{j\theta}; \tau) = \frac{1}{h} \sum_{i \in \mathbb{Z}} F(\omega_i) e^{j\omega_i \tau}, \quad \text{where } \omega_i := \frac{\theta}{h} + \frac{2\pi}{h} i$$

Key Lifting Formula (contd)

Thus,

$$\check{f}(e^{s_0 h}; \tau) e^{-s_0 \tau} = \sum_{i \in \mathbb{Z}} \frac{1}{h} F(s_i) e^{j \frac{2\pi}{h} i \tau}$$

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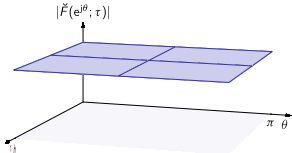
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Example: sinc's

Let $f(t) = \text{sinc}_h(t)$, for which $F(j\omega) = h\mathbb{1}_{[-\frac{\pi}{h}, \frac{\pi}{h}]}(\omega)$. Applying the KLF:

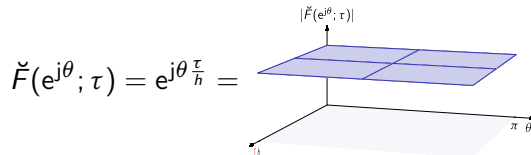
$$\check{F}(e^{j\theta}; \tau) = e^{j\theta \frac{\tau}{h}} =$$


Let $f(t) = \text{sinc}_h^2(t)$, for which $F(j\omega) = h(1 - \frac{h|\omega|}{2\pi})\mathbb{1}_{[-\frac{\pi}{h}, \frac{\pi}{h}]}(\omega)$ and then

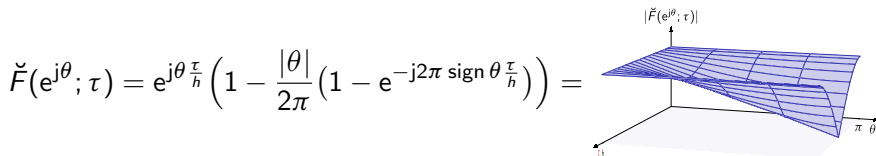
$$\check{F}(e^{j\theta}; \tau) = e^{j\theta \frac{\tau}{h}} \left(1 - \frac{|\theta|}{2\pi} (1 - e^{-j2\pi \sin \theta \frac{\tau}{h}}) \right) =$$

Example: sinc's

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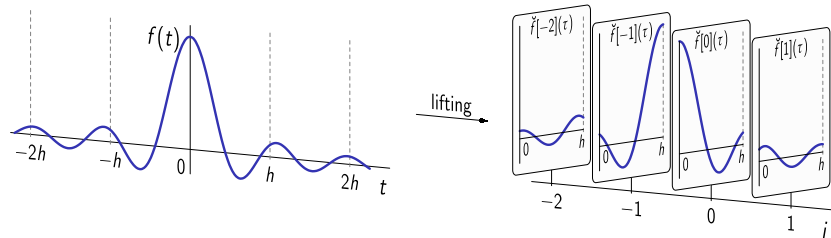


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The lifting transformation (contd)

Analog signals may be lifted over a non-uniform split of the time axis.



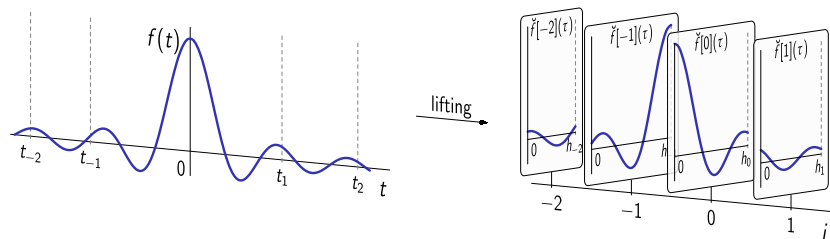
Here, for a given $\{t_i\}_{i \in \mathbb{Z}}$ with $h_i := t_{i+1} - t_i > 0$, a function $f : \mathbb{R} \rightarrow \mathbb{F}^n$ is transformed to a sequence $\{\check{f}[i]\}_{i \in \mathbb{Z}}$ such that $\check{f}[i] : [0, h_i) \rightarrow \mathbb{F}^n$ verifies

$$\check{f}[i](\tau) = f(t_i + \tau), \quad \forall i \in \mathbb{Z} \text{ and } \tau \in [0, h_i).$$

This again defines a bijective mapping $\mathcal{L}_{\{t_i\}} : f \mapsto \check{f}$.

The lifting transformation (contd)

Analog signals may be lifted over a non-uniform split of the time axis:



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'50s: modified (advanced) z-transform

For a given $\mu \in [0, 1]$, the modified z-transform of a continuous-time signal f is

$$\mathfrak{Z}_\mu\{f\} = \bar{F}(z, \mu) := \sum_{i \in \mathbb{Z}} f((i + \mu)h)z^{-i}$$

for all $z \in \mathbb{C}$ for which the sum converges (RoC). By an appropriate choice of μ , we can look into different intersample points.

- introduced by E. I. Jury (1956) to analyze intersample behavior
- limited use, at least lately (as normally used with a fixed μ)

'60s: discrete lifting

Any sequence, like

$$\{\dots, \bar{f}[-3], \bar{f}[-2], \bar{f}[-1], \bar{f}[0], \bar{f}[1], \bar{f}[2], \bar{f}[3], \bar{f}[4], \bar{f}[5], \dots\},$$

can be equivalently presented as

$$\left\{ \dots, \begin{bmatrix} \bar{f}[-2] \\ \bar{f}[-1] \end{bmatrix}, \begin{bmatrix} \bar{f}[0] \\ \bar{f}[1] \end{bmatrix}, \begin{bmatrix} \bar{f}[2] \\ \bar{f}[3] \end{bmatrix}, \begin{bmatrix} \bar{f}[4] \\ \bar{f}[5] \end{bmatrix}, \dots \right\},$$

or

$$\left\{ \dots, \begin{bmatrix} \bar{f}[-3] \\ \bar{f}[-2] \\ \bar{f}[-1] \end{bmatrix}, \begin{bmatrix} \bar{f}[0] \\ \bar{f}[1] \\ \bar{f}[2] \end{bmatrix}, \begin{bmatrix} \bar{f}[3] \\ \bar{f}[4] \\ \bar{f}[5] \end{bmatrix}, \dots \right\},$$

etcetera. This transformation was

- introduced by B. Friedland (1960) to handle periodic systems

'60s: Zak transform

The Zak transform of a continuous-time signal f may be defined as

$$\mathbf{Z}\{f\} = \tilde{F}(t, \omega) := \sum_{i \in \mathbb{Z}} f(t + ih) e^{-j\omega hi}$$

and is a function of two variables, time t (can be viewed as the intersample time) and frequency ω .

- introduced and developed by a solid-state physicist J. Zak (1967), but can be traced back at least to A. Weil (1951) and I. M. Gel'fand (1950)
- used in physics, signal processing, but not in control

'90s: continuous-time lifting

- introduced by Y. Yamamoto (1990, submitted c. Mar 1990; 1994)
- also done by H. Toivonen (1992, submitted Jun 1990)
- also done by G. Tadmor (1991; 1992, submitted Sep 1990)
- also done by B. A. Bamieh & J. B. Pearson (1992, submitted Nov 1990)

Il n'y a de nouveau que ce qui est oublié

—Rose Bertin

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Analog kernel representation revised

Consider an analog system $\mathcal{G}$ described by

$$y(t) = \int_{\mathbb{R}} g(t, s) u(s) ds$$

Then, for any increasing sequence $\{t_i\}$, we have that $\forall i \in \mathbb{Z}, \tau \in [0, h_i)$:

$$\begin{aligned} y(t_i + \tau) &= \sum_{j \in \mathbb{Z}} \int_{t_j}^{t_{j+1}} g(t_i + \tau, s) u(s) ds \\ &= \sum_{j \in \mathbb{Z}} \int_0^{h_j} g(t_i + \tau, t_j + \sigma) u(t_j + \sigma) d\sigma. \end{aligned}$$

This can be rewritten as the following relation between the lifted signals $\check{u}$ and $\check{y}$:

$$\check{y}[i](\tau) = \sum_{j \in \mathbb{Z}} \int_0^{h_j} g(t_i + \tau, t_j + \sigma) \check{u}[j](\sigma) d\sigma.$$

Lifting analog systems

Define integral operators $\{[0, h_j) \rightarrow \mathbb{F}^{n_u}\} \rightarrow \{[0, h_i) \rightarrow \mathbb{F}^{n_y}\}$:

$$\check{g}[i, j] : \check{\zeta}_j \mapsto \int_0^{h_j} g(t_i + \tau, t_j + \sigma) \check{\zeta}_j(\sigma) d\sigma$$

Thus any analog system $\mathcal{G} : u \mapsto y$ can be equivalently viewed as a discrete system $\check{u} \mapsto \check{y}$ described by

$$\check{y}[i] = \sum_{j \in \mathbb{Z}} \check{g}[i, j] \check{u}[j].$$

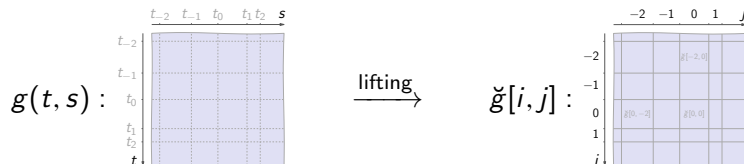
This system is called the **lifting** of $\mathcal{G}$ and is denoted $\check{\mathcal{G}}$. Note that at each (i, j) the impulse response $\check{g}[i, j]$ of $\check{\mathcal{G}}$ is an integral operator,

$$\check{g}[i, j] : \{[0, h_j) \rightarrow \mathbb{F}^{n_u}\} \rightarrow \{[0, h_i) \rightarrow \mathbb{F}^{n_y}\}, \quad \forall i, j \in \mathbb{Z}$$

but otherwise $\check{\mathcal{G}}$ is a standard discrete system.

Lifting analog systems (contd)

In terms of kernels, lifting of systems may be viewed as slicing:



nothing else changes.

Remark: In the operator terminology $\check{\mathcal{G}} = \mathcal{L}_{\{t_i\}} \mathcal{G} \mathcal{L}_{\{t_i\}}^{-1}$.

Lifted system matrix

Alternative form,

$$\begin{bmatrix} \vdots \\ \check{y}[i-1] \\ \check{y}[i] \\ \check{y}[i+1] \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \cdots & \check{g}[i-1, i-1] & \check{g}[i-1, i] & \check{g}[i-1, i+1] & \cdots \\ \cdots & \check{g}[i, i-1] & \check{g}[i, i] & \check{g}[i, i+1] & \cdots \\ \cdots & \check{g}[i+1, i-1] & \check{g}[i+1, i] & \check{g}[i+1, i+1] & \cdots \\ \ddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\text{lifted system matrix of } \check{\mathcal{G}}, [\check{\mathcal{G}}]} \begin{bmatrix} \vdots \\ \check{u}[i-1] \\ \check{u}[i] \\ \check{u}[i+1] \\ \vdots \end{bmatrix}$$

It is effectively the matrix representation of $\mathcal{G}$ for the basis $\{\mathbb{1}_{[t_i, t_{i+1})}\}_{i \in \mathbb{Z}}$ of the analog time axis $\mathbb{R}$, cf.

$$\begin{bmatrix} \vdots \\ \check{y}[i-1] \\ \check{y}[i] \\ \check{y}[i+1] \\ \vdots \end{bmatrix} \cong \begin{bmatrix} \vdots \\ y\mathbb{1}_{[t_{i-1}, t_i]} \\ y\mathbb{1}_{[t_i, t_{i+1}]} \\ y\mathbb{1}_{[t_{i+1}, t_{i+2}]} \\ \vdots \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \vdots \\ \check{u}[i-1] \\ \check{u}[i] \\ \check{u}[i+1] \\ \vdots \end{bmatrix} \cong \begin{bmatrix} \vdots \\ u\mathbb{1}_{[t_{i-1}, t_i]} \\ u\mathbb{1}_{[t_i, t_{i+1}]} \\ u\mathbb{1}_{[t_{i+1}, t_{i+2}]} \\ \vdots \end{bmatrix}.$$

Example: lifting a non-causal integrator

Let $\mathcal{G}$ be an LTI system with the impulse response $g(t) = 1$. Then

$$\check{y}[i](\tau) = \int_{\mathbb{R}} u(s) ds = \sum_{j \in \mathbb{Z}} \int_{jh}^{(j+1)h} u(s) ds = \sum_{j \in \mathbb{Z}} \int_0^h \check{u}[j](\sigma) d\sigma,$$

i.e.

$$\check{y}[i] = \sum_{j \in \mathbb{Z}} \check{g}[i, j] \check{u}[j] \quad \text{with } \check{g}[i, j] = \check{g}_{\text{int}},$$

where the integral operator $\check{g}_{\text{int}} : \{[0, h) \rightarrow \mathbb{F}\} \rightarrow \{[0, h) \rightarrow \mathbb{F}\}$ acts as

$$\check{g}_{\text{int}} : \check{\zeta} \mapsto \int_0^h \zeta(\sigma) d\sigma.$$

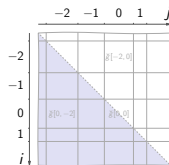
Causality of lifted systems

Causality would better be preserved under lifting. Accordingly, $\tilde{\mathcal{G}} : \tilde{u} \mapsto \tilde{y}$ is said to be **causal** if

- the output $\tilde{y}[i](\tau)$ at each i and τ depends only on the current and the past inputs, i.e. on $\tilde{u}[j]$ for $j < i$ and on $\tilde{u}[i](\sigma)$ for $\sigma \leq \tau$.

Causality is equivalent to

$$\begin{cases} \tilde{g}[i, j] = 0 \text{ whenever } j > i \text{ and} \\ \tilde{g}[i, i] : L_2[0, h_i) \rightarrow L_2[0, h_i) \text{ is causal} \end{cases} \quad \text{i.e.}$$



System matrices of causal lifted systems must thus

- be lower triangular
- have causal feedthrough terms

We say that $\tilde{\mathcal{G}}$ is strictly causal if $\tilde{g}[i, j] = 0$ whenever $j \geq i$ (a faux notion).

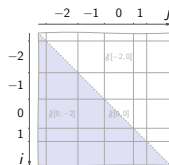
Causality of lifted systems

Causality would better be preserved under lifting. Accordingly, $\check{\mathcal{G}} : \check{u} \mapsto \check{y}$ is said to be **causal** if

- the output $\check{y}[i](\tau)$ at each i and τ depends only on the current and the past inputs, i.e. on $\check{u}[j]$ for $j < i$ and on $\check{u}[i](\sigma)$ for $\sigma \leq \tau$.

Causality is equivalent to

$$\begin{cases} \check{g}[i, j] = 0 \text{ whenever } j > i \text{ and} \\ \check{g}[i, i] : L_2[0, h_i) \rightarrow L_2[0, h_i) \text{ is causal} \end{cases} \quad \text{i.e.}$$



System matrices of causal lifted systems must thus

- be lower triangular
- have causal feedthrough terms

We say that $\check{\mathcal{G}}$ is **strictly causal** if $\check{g}[i, j] = 0$ whenever $j \geq i$ (a faux notion).

Example: lifting a causal integrator

Let $\mathcal{G}$ be an LTI system with the impulse response $g(t) = \mathbb{1}(t)$. Then

$$\begin{aligned}\check{y}[i](\tau) &= \int_{-\infty}^{ih+\tau} u(s)ds = \int_{ih}^{ih+\tau} u(s)ds + \sum_{j=-\infty}^{i-1} \int_{jh}^{(j+1)h} u(s)ds \\ &= \int_0^{\tau} \check{u}[i](\sigma)d\sigma + \sum_{j=-\infty}^{i-1} \int_0^h \check{u}[j](\sigma)d\sigma,\end{aligned}$$

i.e.

$$\check{y}[i] = \sum_{j \in \mathbb{Z}} \check{g}[i, j] \check{u}[j] \quad \text{with} \quad \check{g}[i, j] = \begin{cases} 0 & \text{if } j > i \\ \check{g}_{\text{cint}} & \text{if } j = i \\ \check{g}_{\text{int}} & \text{if } j < i \end{cases}$$

where the integral operator $\check{g}_{\text{cint}} : \{[0, h) \rightarrow \mathbb{F}\} \rightarrow \{[0, h) \rightarrow \mathbb{F}\}$ acts as

$$\check{g}_{\text{cint}} : \check{\zeta} \mapsto \int_0^{\tau} \zeta(\sigma)d\sigma$$

and $\check{g}_{\text{int}}$ is as defined earlier. This is a causal system.

Lifting periodic systems

Let $\mathcal{G}$ be h -periodic, i.e. $g(t, s + h) = g(t - h, s)$, and $\check{\mathcal{G}}$ be its lifting under the uniform axis split with the interval h . Then

$$\begin{aligned}(\check{g}[i, j + 1]\check{\zeta})(\tau) &= \int_0^h g(ih + \tau, jh + h + \sigma)\check{\zeta}(\sigma)d\sigma \\&= \int_0^h g(ih - h + \tau, jh + \sigma)\check{\zeta}(\sigma)d\sigma \\&= (\check{g}[i - 1, j]\check{\zeta})(\tau)\end{aligned}$$

In other words, $\check{g}[i, j + 1] = \check{g}[i - 1, j]$, which implies that

- $\check{\mathcal{G}}$ is shift invariant.

Thus h -periodic analog systems become shift invariant in the lifted domain, which is a key property of the lifting transformation.

This property may be explained by the fact that lifting effectively

forbids any time shift that is not a multiple of h .

Lifting periodic systems

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- forbids any time shift that is not a multiple of h .

Shift invariant lifted systems

If $\check{\mathcal{G}}$ is shift invariant, we may write its action as the lifted convolution

$$\check{y}[i] = \sum_{j \in \mathbb{Z}} \check{g}[i-j] \check{u}[j] =: (\check{g} * \check{u})[i]$$

(lifted convolution), where the impulse response $\check{g}[i]$ at each $i \in \mathbb{Z}$ is the integral operator $\{[0, h) \rightarrow \mathbb{F}^{n_u}\} \rightarrow \{[0, h) \rightarrow \mathbb{F}^{n_y}\}$ acting as

$$\check{g}[i] : \check{\zeta} \mapsto \int_0^h g(ih + \tau, \sigma) \check{\zeta}(\sigma) d\sigma.$$

The system matrix of a shift-invariant $\check{\mathcal{G}}$,

$$\llbracket \check{\mathcal{G}} \rrbracket = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \check{g}[0] & \check{g}[-1] & \check{g}[-2] & \cdots \\ \cdots & \check{g}[1] & \check{g}[0] & \check{g}[-1] & \cdots \\ \cdots & \check{g}[2] & \check{g}[1] & \check{g}[0] & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

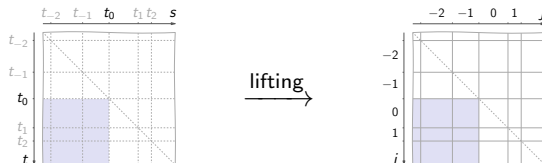
has then the Toeplitz structure.

State space in the lifted domain

For each t_i ,

$$\mathcal{L}_{\{t_i\}} u \mathbb{1}_{(-\infty, t_i)} = \tilde{u} \mathbb{1}_{\mathbb{Z}_{-\infty \dots i-1}} \quad \text{and} \quad \mathcal{L}_{\{t_i\}} y \mathbb{1}_{[t_i, \infty)} = \tilde{y} \mathbb{1}_{\mathbb{Z}_{i \dots \infty}},$$

i.e. the past (future) remains the past (future) under lifting. Therefore, the Hankel operator—which connects past inputs and future outputs—remains the same under lifting (as long as “the present” is synchronized with $\{t_i\}$):



Consequently,

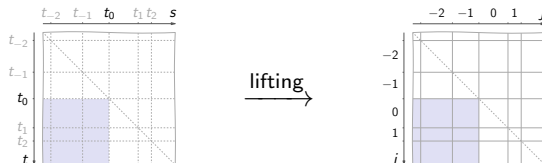
— lifting preserves the state dimension and the lifted state $\tilde{x}[i] = x(t_i)$, where $x(t)$ is the state vector of $\mathcal{G}$ (at least if $\mathcal{G}$ is LTI).

State space in the lifted domain

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$$\mathcal{L}_{\{t_i\}} u \mathbb{1}_{(-\infty, t_i)} = \tilde{u} \mathbb{1}_{\mathbb{Z}_{-\infty \dots i-1}} \quad \text{and} \quad \mathcal{L}_{\{t_i\}} y \mathbb{1}_{[t_i, \infty)} = \tilde{y} \mathbb{1}_{\mathbb{Z}_{i \dots \infty}},$$

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Consequently,

- lifting **preserves the state dimension** and the lifted state $\bar{x}[i] = x(t_i)$, where $x(t)$ is the state vector of $\mathcal{G}$ (at least if $\mathcal{G}$ is LTI).

Summary



Lifting

- converts analog systems to equivalent *pure discrete* systems
- preserves stability (check it)
- preserves causality
- turns periodic systems to shift invariant
- preserves state dimensions

Content

Lifting

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Lifting analog systems

Lifting A/D and D/A converters

Generalized sampler revised

A generalized sampler $\mathcal{S} : y \mapsto \bar{y}$ associated with $\{t_i\}$ can be expressed as

$$\begin{aligned}\bar{y}[i] &= \int_{\mathbb{R}} \psi(t_i, s) y(s) ds = \sum_{j \in \mathbb{Z}} \int_{t_j}^{t_{j+1}} \psi(t_i, s) y(s) ds \\ &= \sum_{j \in \mathbb{Z}} \int_0^{h_j} \psi(t_i, t_j + \sigma) y(t_j + \sigma) d\sigma\end{aligned}$$

This, in turn, can be rewritten as the following relation between the lifted signal $\check{y} = \mathcal{L}_{\{t_i\}} y$ and the standard discrete signal $\bar{y}$:

$$\bar{y}[i] = \sum_{j \in \mathbb{Z}} \int_0^{h_j} \psi(t_i, t_j + \sigma) \check{y}[j](\sigma) d\sigma.$$

Lifting samplers

Define operators $\{[0, h_j) \rightarrow \mathbb{F}^{n_y}\} \rightarrow \mathbb{F}^{n_{\bar{y}}}$:

$$\psi[i, j] : \check{\xi}_j \mapsto \int_0^{h_j} \psi(t_i, t_j + \sigma) \check{\xi}_j(\sigma) d\sigma$$

Then the sampler $\mathcal{S} : y \mapsto \bar{y}$ can also be viewed as a discrete system $\check{y} \mapsto \bar{y}$ described by

$$\bar{y}[i] = \sum_{j \in \mathbb{Z}} \psi[i, j] \check{y}[j].$$

This system is called the **lifting** of $\mathcal{S}$ and is denoted $\acute{\mathcal{S}}$. Note that at each (i, j) the impulse response $\psi[i, j]$ of $\acute{\mathcal{S}}$ is an operator,

$$\psi[i, j] : \{[0, h_j) \rightarrow \mathbb{F}^{n_y}\} \rightarrow \mathbb{F}^{n_{\bar{y}}}, \quad \forall i, j \in \mathbb{Z}$$

but otherwise $\acute{\mathcal{S}}$ is a standard discrete system.

Remark: In the operator terminology $\acute{\mathcal{S}} = \mathcal{S} \mathcal{L}_{\{t_i\}}^{-1}$.

Example: ideal sampler

Consider $\mathcal{S}_{\text{idl}}$ having the sampling function

$$\psi(t_i, s) = \delta(t_i - s).$$

Its lifting has the impulse response

$$\psi'[i, j] : \check{\xi}_j \mapsto \int_0^{h_j} \delta(t_i - t_j - \sigma) \check{\xi}_j(\sigma) d\sigma = \begin{cases} \check{\xi}_j(0) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

This corresponds to the static $\mathcal{S}'_{\text{idl}}$, acting according to

$$\bar{y}[i] = \psi'[i, i] \check{y}[i] = \check{y}[i](0).$$

Lifting shift-invariant samplers

Remember, we say that $\mathcal{S}$ is said to be shift invariant if sampling is uniform and $\psi(t, s + \kappa) = \psi(t - \kappa, s)$ for all $\kappa \in \mathbb{R}$. In this case,

$$\begin{aligned}\psi[i, j+1]\check{\xi} &= \int_0^h \psi(ih, jh + h + \sigma)\check{\xi}(\sigma)d\sigma = \int_0^h \psi(ih - h, jh + \sigma)\check{\xi}(\sigma)d\sigma \\ &= \psi[i-1, j]\check{\xi}\end{aligned}$$

In other words, $\psi[i, j+1] = \psi[i-1, j]$, which implies that

- $\mathcal{S}$ is a shift-invariant discrete system

and lifted shift-invariant sampler can be described by the convolution

$$\bar{y}[i] = \sum_{j \in \mathbb{Z}} \psi[i-j]\check{y}[j] =: (\psi * \check{y})[i]$$

where $\psi[i] : \{[0, h) \rightarrow \mathbb{F}^{n_y}\} \rightarrow \mathbb{F}^{n_{\bar{y}}}$ acts as

$$\psi[i] : \check{\xi} \mapsto \int_0^h \psi(ih - \sigma)\check{\xi}(\sigma)d\sigma = \int_0^h \check{\psi}[i-1](h - \sigma)\check{\xi}(\sigma)d\sigma.$$

Example: averaging sampler

Let sampling be uniform and consider $\mathcal{S}_{\text{avg}}$ having the sampling function

$$\psi(s) = \frac{1}{h} \mathbb{1}_{[0,h]}(s)$$

Its lifting has the impulse response

$$\psi[i] : \check{\xi} \mapsto \frac{1}{h} \int_0^h \mathbb{1}_{[0,h]}(ih - \sigma) \check{\xi}(\sigma) d\sigma = \begin{cases} \frac{1}{h} \int_0^h \check{\xi}(\sigma) d\sigma & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

This corresponds to the FIR $\mathcal{S}_{\text{avg}}$, acting according to

$$\bar{y}[i] = \psi[1] \check{y}[i-1] = \frac{1}{h} \int_0^h \check{y}[i-1](\sigma) d\sigma.$$

Causality of lifted samplers

A sampler is causal iff $\psi(t_i, s) = 0$ whenever $s > t_i$. This condition reads

$$\psi[i, j] = 0, \quad \text{whenever } j \geq i$$

(unless $\psi(t_i, s)$ contains $\delta(t_i - s)$, in which case $\psi[i, i]$ is also nonzero). In other words,

- the causality of $\mathcal{S}$ (normally) implies the **strict causality** of $\hat{\mathcal{S}}$.

Technically, we can assume that the ideal sampling component is $\delta(t_i^- - s)$ (may be done wlog; the continuity of its input at $t \in \{t_i\}$ must be assumed anyway), which renders the ideal sampler strictly causal as well:

$$\hat{y}[i] = y[i-1](h^-).$$

We then may ignore the “normally” part above.

Causality of lifted samplers

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Technically, we can assume that the ideal sampling component is $\delta(t_i^- - s)$ (may be done wlog, the continuity of its input at $t \in \{t_i\}$ must be assumed anyway), which renders the ideal sampler strictly causal as well:

$$\bar{y}[i] = \breve{y}[i - 1](h^-).$$

We then may ignore the “normally” part above.

Lifting static samplers

A sampler is causal iff $\psi(t_i, s) = 0$ whenever $s \notin [t_{i-1}, t_i]$. This condition reads

$$\psi[i, j] = 0, \quad \text{whenever } j \neq i - 1.$$

In other words,

- static $\mathcal{S}$ (normally) has **one step delayed static** $\acute{\mathcal{S}}$

and the “normally” part above may again be ignored under an appropriate modification of the definition of the ideal sampler.

Generalized hold revised

A generalized hold $\mathcal{H} : \bar{u} \mapsto u$ associated with $\{t_i\}$ can be expressed as

$$u(t_i + \tau) = \sum_{j \in \mathbb{Z}} \phi(t_i + \tau, t_j) \bar{u}[j]$$

or, equivalently, as the relation between the standard discrete signal $\bar{u}$ and the lifted signal $\check{u} = \mathcal{L}_{\{t_i\}} u$ of the form

$$\check{u}[i](\tau) = \sum_{j \in \mathbb{Z}} \phi(t_i + \tau, t_j) \bar{u}[j].$$

Lifting holds

Define multiplication operators $\mathbb{F}^{n_{\bar{u}}} \rightarrow \{[0, h_j) \rightarrow \mathbb{F}^{n_u}\}$:

$$\phi[i, j] : \bar{\xi}_j \mapsto \phi(t_i + \tau, t_j) \bar{\xi}_j$$

Then the hold $\mathcal{H} : \bar{u} \mapsto u$ can also be viewed as a discrete system $\bar{u} \mapsto \check{u}$ described by

$$\check{u}[i] = \sum_{j \in \mathbb{Z}} \phi[i, j] \bar{u}[j].$$

This system is called the **lifting** of $\mathcal{H}$ and is denoted $\hat{\mathcal{H}}$. Note that at each (i, j) the impulse response $\phi[i, j]$ of $\hat{\mathcal{H}}$ is an operator,

$$\phi[i, j] : \mathbb{F}^{n_{\bar{u}}} \rightarrow \{[0, h_j) \rightarrow \mathbb{F}^{n_u}\}, \quad \forall i, j \in \mathbb{Z}$$

but otherwise $\hat{\mathcal{H}}$ is a standard discrete system.

Remark: In the operator terminology $\hat{\mathcal{H}} = \mathcal{L}_{\{t_i\}} \mathcal{H}$.

Example: zero-order hold

Consider $\mathcal{H}_{\text{ZOH}}$ having the hold function

$$\phi(t, t_i) = \mathbb{1}_{[t_i, t_{i+1})}(t)$$

Its lifting has the impulse response

$$\phi[i, j] : \bar{\xi}_j \mapsto \mathbb{1}_{[t_j, t_{j+1})}(t_i + \tau) \bar{\xi}_j = \begin{cases} \bar{\xi}_j & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

This corresponds to the static $\mathcal{H}_{\text{ZOH}}$, acting according to

$$\check{u}[i](\tau) = \phi[i, i] \bar{u}[i] = \bar{u}[i].$$

Lifting shift-invariant holds

Now, we say that $\mathcal{H}$ is said to be shift invariant if sampling is uniform and $\phi(t, s + \kappa) = \phi(t - \kappa, s)$ for all $\kappa \in \mathbb{R}$. In this case,

$$\phi[i, j + 1]\bar{\xi} = \phi(ih + \tau, jh + h)\bar{\xi} = \phi(ih - h + \tau, jh)\bar{\xi} = \phi[i - 1, j]\bar{\xi}$$

In other words, $\phi[i, j + 1] = \phi[i - 1, j]$, which implies that

- $\mathcal{H}$ is a shift-invariant discrete system

and lifted shift-invariant hold can be described by the convolution

$$\check{u}[i] = \sum_{j \in \mathbb{Z}} \phi[i - j]\bar{u}[j] =: (\phi * \bar{u})[i]$$

where $\phi[i] : \mathbb{F}^{n_{\bar{u}}} \rightarrow \{[0, h) \rightarrow \mathbb{F}^{n_u}\}$ acts as

$$\phi[i] : \bar{\xi} \mapsto \phi(ih + \tau)\bar{\xi} = \check{\phi}[i](\tau)\bar{\xi}$$

where $\check{\phi} = \mathcal{L}_h\phi$ is the lifting of the hold function.

Example: first-order hold

Let sampling be uniform and consider $\mathcal{H}_{\text{FOH}}$ having the hold function

$$\phi(t) = \left(1 - \frac{|t|}{h}\right) \mathbb{1}_{[-h, h]}(t)$$

Its lifting has the impulse response

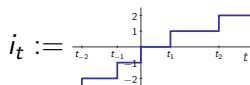
$$\phi[i] : \bar{\xi} \mapsto \left(1 - \frac{|ih + \tau|}{h}\right) \mathbb{1}_{[-h, h]}(ih + \tau) \bar{\xi} = \begin{cases} \frac{\tau}{h} \bar{\xi} & \text{if } i = -1 \\ \left(1 - \frac{\tau}{h}\right) \bar{\xi} & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

This corresponds to the FIR $\hat{\mathcal{H}}_{\text{FOH}}$, acting according to

$$\check{u}[i](\tau) = \phi[-1] \bar{u}[i+1] + \phi[0] \bar{u}[i] = \frac{\tau}{h} \bar{u}[i+1] + \left(1 - \frac{\tau}{h}\right) \bar{u}[i].$$

Causality of lifted holds

A hold is causal iff $\phi(t, t_i) = 0$ whenever $i > i_t$, where



With $\bar{\phi}[i, j] : \bar{\xi}_j \mapsto \phi(t_i + \tau, t_j) \bar{\xi}_j$, the causality condition reads

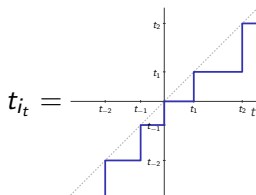
$$\bar{\phi}[i, j] = 0, \quad \text{whenever } j > i.$$

In other words,

- the causality of $\mathcal{H}$ implies the **causality** of $\bar{\mathcal{H}}$ (in the conventional sense).

Lifting static holds

A hold is static iff $\phi(t, t_i) = 0$ whenever $t \notin [t_{i_t}, t_{i_t+1}]$, where



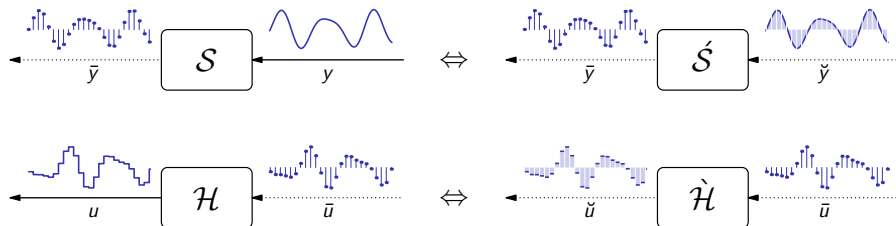
With $\phi[i, j] : \bar{\xi}_j \mapsto \phi(t_i + \tau, t_j) \bar{\xi}_j$, the static condition reads

$$\phi[i, j] = 0, \quad \text{whenever } j \neq i.$$

In other words,

- static $\mathcal{H}$ has **static** (memoryless) $\hat{\mathcal{H}}$.

Summary



Lifting

- converts A/D and D/A converters to equivalent *pure discrete* systems
- preserves stability (check it)
- preserves causality
- preserves shift invariance