

Advanced Sampled-Data Control (038798)

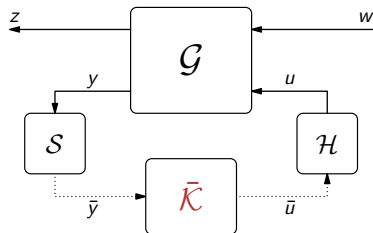
lecture no. 3

Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT



Setup



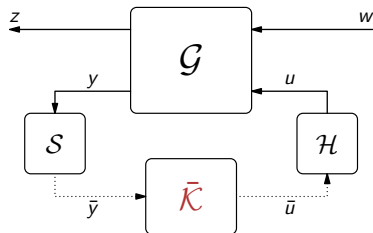
Given:

- LTI generalized plant $\mathcal{G}$
- periodic sampling with $h > 0$
- shift-invariant sampler $\mathcal{S}$ (say, $\mathcal{S}_{\text{idl}}$)
- shift-invariant hold $\mathcal{H}$ (say, $\mathcal{H}_{\text{ZOH}}$)

Design:

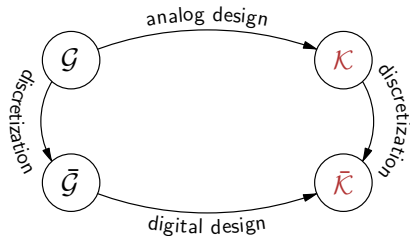
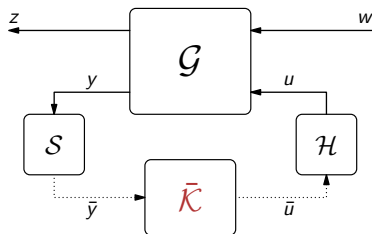
- discrete controller $\bar{\mathcal{K}}$

Difficulties



1. hybrid, analog / discrete, dynamics
 - aliasing at $\mathcal{S}$
 - limited bandwidth of $\mathcal{H}$
2. time invariance is lost

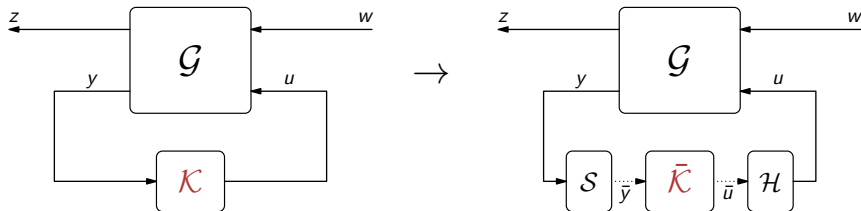
Two approaches



1. digital redesign of analog controllers
2. discrete-time design

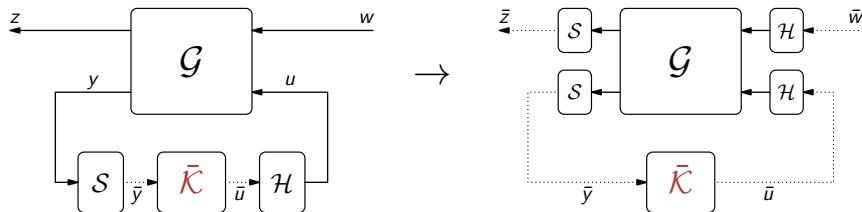
clockwise from $\mathcal{G}$ to $\bar{\mathcal{K}}$
counterclockwise from $\mathcal{G}$ to $\bar{\mathcal{K}}$

Digital redesign: methodology



1. design a “good” analog $\mathcal{K}$
2. find $\bar{\mathcal{K}}$ such that $\mathcal{K} \approx \mathcal{H}\bar{\mathcal{K}}\mathcal{S}$ for given $\mathcal{S}$ and $\mathcal{H}$

Discrete-time design: methodology



1. approximate the problem by a discrete standard problem
2. design a “good” discrete $\bar{\mathcal{K}}$

Content

Digital redesign of analog controllers: frequency-domain background

Digital redesign of analog controllers

Discrete-time design: discretization background

Discrete-time design

Content

Digital redesign of analog controllers: frequency-domain background

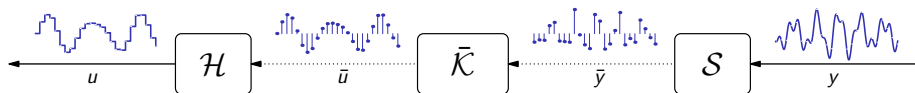
Digital redesign of analog controllers

Discrete-time design: discretization background

Discrete-time design

Sampled-data controller

Consider

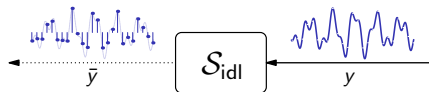


assuming that

- sampling is uniform, with sampling period $h > 0$
- $\mathcal{S}$ is shift invariant, with $\bar{y}[j] = \int_{\mathbb{R}} \psi(jh - s)y(s)ds$
- $\bar{\mathcal{K}}$ is shift invariant, with $\bar{u}[i] = \sum_{j \in \mathbb{Z}} \bar{k}[i - j]\bar{y}[j]$
- $\mathcal{H}$ is shift invariant, with $u(t) = \sum_{i \in \mathbb{Z}} \phi(t - ih)\bar{u}[i]$

Let us now take a glimpse of the relation between $Y(j\omega)$ and $U(j\omega) \dots$

What do we loose by sampling analog signals ?



From the time-domain relation

$$\bar{y}[i] = y(ih)$$

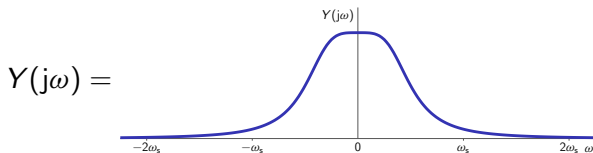
we know that all **intersample information** about $y(t)$ is **lost**. But

- to what extend is it important (if at all) ?

To answer this kind of questions, frequency-domain analysis is indispensable.

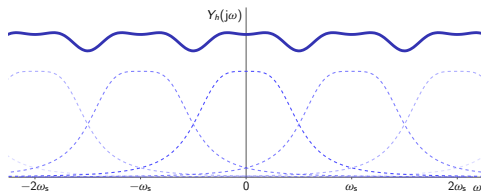
A weird function

Consider an analog signal $y(t)$ with the spectrum $Y(j\omega)$, e.g.



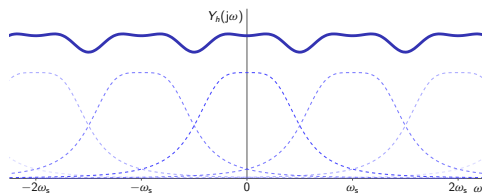
and define, for some $h > 0$, the function

$$Y_h(j\omega) := \frac{1}{h} \sum_{i \in \mathbb{Z}} Y(j(\omega + \omega_s i)) =$$



where $\omega_s := \frac{2\pi}{h}$.

A weird function: Fourier series



As $Y_h(j\omega)$ is ω_s -periodic, we can bring in its Fourier series expansion

$$Y_h(j\omega) = \sum_{i \in \mathbb{Z}} c_i e^{j \frac{2\pi}{\omega_s} i \omega} = \sum_{i \in \mathbb{Z}} c_i e^{j \omega h i},$$

where Fourier coefficients are calculated as

$$c_i = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} Y_h(j\omega) e^{-j \omega h i} d\omega.$$

A weird function: calculating Fourier coefficients

With some extra efforts:

$$\begin{aligned} c_i &= \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} \frac{1}{h} \sum_{j \in \mathbb{Z}} Y(j(\omega + \omega_s j)) e^{-j\omega h i} d\omega \\ &= \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} \int_{-\omega_s/2}^{\omega_s/2} Y(j(\omega + \omega_s j)) e^{-j(\omega + \omega_s j) h i} d\omega \\ &= \frac{1}{2\pi} \sum_{j \in \mathbb{Z}} \int_{-\omega_s/2 + \omega_s j}^{\omega_s/2 + \omega_s j} Y(j\omega) e^{-j\omega h i} d\omega \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} Y(j\omega) e^{-j\omega h i} d\omega \quad \left(\text{remember, } y(t) = \frac{1}{2\pi} \int_{\mathbb{R}} Y(j\omega) e^{j\omega t} d\omega \right) \\ &= y(-ih). \end{aligned}$$

A weird function: Fourier series (contd)

Thus, we end up with

$$Y_h(j\omega) = \sum_{i \in \mathbb{Z}} y(-ih) e^{j\omega hi} = \sum_{i \in \mathbb{Z}} y(ih) e^{-j\omega hi}.$$

Let us compare it with the discrete-time Fourier transform of $\tilde{y} = S_{1/h}y$,

$$\tilde{Y}(e^{j\theta}) = \sum_{i \in \mathbb{Z}} \tilde{y}[i] e^{-j\theta i} = \sum_{i \in \mathbb{Z}} y(ih) e^{-j\theta i}.$$

We can see that

$$Y_h(j\omega) = \frac{1}{h} \sum_{i \in \mathbb{Z}} \tilde{Y}(e^{j(\omega + \omega_s i)}), \quad \text{where } \omega_s = \frac{2\pi}{h} \text{ (the sampling frequency)}$$

is the DTFT of the sampled signal $\tilde{y}$ modulo scaling, $\theta = \omega h$, i.e.

$$\tilde{Y}(e^{j\omega h}) = Y_h(j\omega h).$$

A weird function: Fourier series (contd)

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We can see that

$$Y_h(j\omega) = \frac{1}{h} \sum_{i \in \mathbb{Z}} Y(j(\omega + \omega_s/h)), \quad \text{where } \omega_s = \frac{2\pi}{h} \text{ (the sampling frequency)}$$

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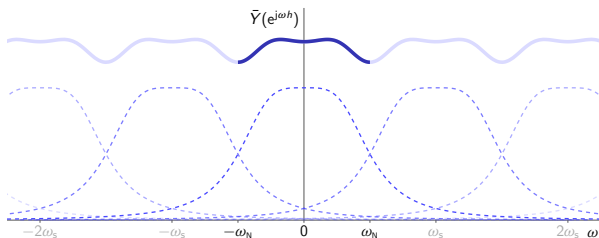
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Spectrum of sampled signal

In other words, spectrum of sampled signals is $\bar{Y}(e^{j\omega h})$ in $\omega \in [-\frac{\omega_s}{2}, \frac{\omega_s}{2}]$:

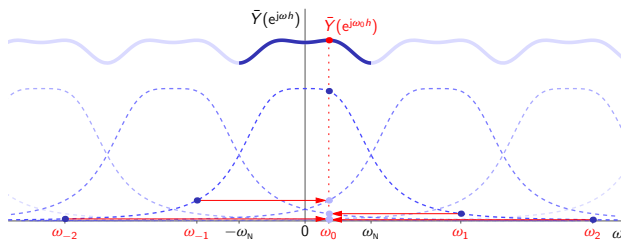


Frequency

$$\omega_N := \frac{\omega_s}{2} = \frac{\pi}{h}$$

is called the **Nyquist frequency** (measured in rad/sec).

Spectrum of sampled signal: aliasing



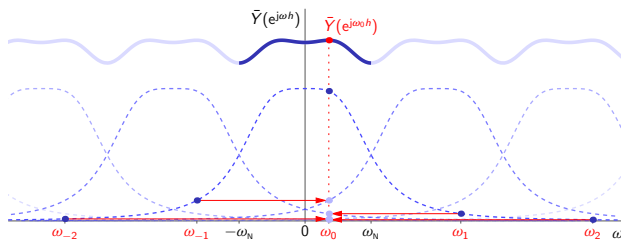
Thus, the spectrum of $\bar{y}$ at each discrete frequency $\theta_0 = \omega_0 h$ is a

- blend of analog frequency responses at $\omega_i := \omega_0 h + \omega_s i$, $\forall i \in \mathbb{Z}$.

Blending means **information lost**, we can no longer tell $Y(j\omega_i)$ from $Y(j\omega_j)$ in their effect on $\bar{Y}(e^{j\theta_0})$ (unless we know their dependencies). We say that

every discrete frequency $\theta_0 \in [-\pi, \pi]$ is an alias of all ω_i , $i \in \mathbb{Z}$, and call this phenomenon **aliasing**.

Spectrum of sampled signal: aliasing



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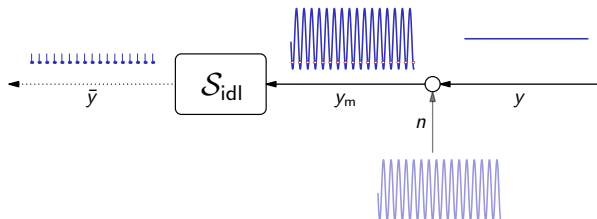
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Aliasing: example



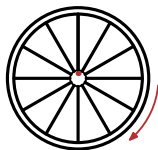
Let $y(t) = \text{const}$ be measured via a noisy sensor with $n(t) = \sin(2\omega_N t + \alpha)$ (their spectra are well separated). But the sampled measured signal

$$\bar{y}[i] = y(ih) + \sin(\alpha)$$

is offset, with no way to separate y from n .

Aliasing: non-control examples

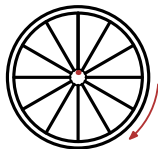
Wagon-wheel effect:



(shot with 12 FPS frame rate)

Aliasing: non-control examples

Wagon-wheel effect:



(shot with 12 FPS frame rate)

Moiré pattern:



downsampling →



Remark 1: instability of the ideal sampler (again)

Let $y(t)$ be a signal with

$$Y(j\omega) = \frac{1}{\sqrt{\omega^2 + 1}}.$$

By Parseval,

$$\|y\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{\omega^2 + 1} d\omega = \frac{1}{2},$$

so $y \in L_2(\mathbb{R})$. Frequency response of $\bar{y} = \mathcal{S}_{\text{idl}} y$ is

$$\bar{Y}(e^{j\theta}) = \frac{1}{h} \sum_{i \in \mathbb{Z}} \frac{1}{\sqrt{(\theta/h + 2\pi/h i)^2 + 1}} = \sum_{i \in \mathbb{Z}} \frac{1}{\sqrt{(\theta + 2\pi i)^2 + h^2}} = \infty$$

for every $\theta \in [-\pi, \pi]$. This means that $\bar{y}$ is unbounded, i.e. that

– the ideal sampler $\mathcal{S}_{\text{idl}}$ is unstable
(which we already know).

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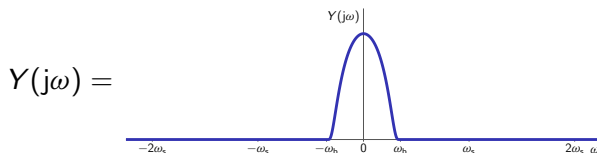
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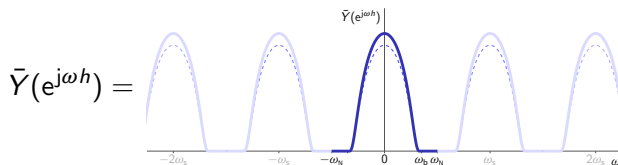
(which we already know).

Remark 2: Spectrum of sampled bandlimited signal

An analog signal $y(t)$ is said to be **bandlimited** if its spectrum, $Y(j\omega)$, has support in $[-\omega_b, \omega_b]$ for some $\omega_b > 0$ (bandwidth), like



If $\omega_b \leq \omega_N$, shifted $Y(j(\omega + \omega_s i))$ are mutually non-overlapping, so that



and there is no frequency blending in $\bar{Y}(e^{j\omega h})$, i.e. **no information is lost**. In fact, $\bar{Y}(e^{j\omega h}) = \frac{1}{h} Y(j\omega)$ for every $\omega \in [-\omega_N, \omega_N]$ and we may expect that y is reconstructable from $\bar{y}$.

Remark 2: how to reconstruct bandlimited signal

Let $y(t)$ be bandlimited, with $\omega_b \leq \omega_N$. Then $Y(j\omega) = h\bar{Y}(e^{j\omega h})$ and

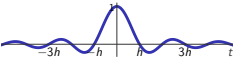
$$\begin{aligned}
 y(t) &= \frac{1}{2\pi} \int_{\mathbb{R}} Y(j\omega) e^{j\omega t} d\omega \\
 &= \frac{h}{2\pi} \int_{-\omega_N}^{\omega_N} \bar{Y}(e^{j\omega h}) e^{j\omega t} d\omega = \frac{h}{2\pi} \int_{-\omega_N}^{\omega_N} \sum_{i \in \mathbb{Z}} y(ih) e^{-j\omega h i} e^{j\omega t} d\omega \\
 &= \sum_{i \in \mathbb{Z}} y(ih) \frac{1}{2\omega_N} \int_{-\omega_N}^{\omega_N} e^{j\omega(t-ih)} d\omega = \sum_{i \in \mathbb{Z}} y(ih) \frac{1}{2\omega_N} \frac{e^{j\omega(t-ih)} \Big|_{-\omega_N}^{\omega_N}}{j(t-ih)} \\
 &= \sum_{i \in \mathbb{Z}} y(ih) \frac{e^{j\omega_N(t-ih)} - e^{-j\omega_N(t-ih)}}{2j\omega_N(t-ih)} = \sum_{i \in \mathbb{Z}} y(ih) \frac{\sin(\omega_N(t-ih))}{\omega_N(t-ih)} \\
 &= \sum_{i \in \mathbb{Z}} y(ih) \operatorname{sinc}_h(t-ih)
 \end{aligned}$$

where $\operatorname{sinc}_h(t) := \frac{\sin(\omega_N t)}{\omega_N t} =$ 

Remark 2: the Sampling Theorem

Theorem (Whittaker-Kotel'nikov-Shannon)

Let $y(t)$ be analog bandlimited signal with bandwidth ω_b . If $\omega_b \leq \omega_N$, $y(t)$ can be perfectly reconstructed from its sampled measurements $\bar{y}[i] = y(ih)$ by the generalized hold $\mathcal{H}_{\text{sinc}}$ with the hold function

$$\phi(t) = \text{sinc}_h(t) =$$


Remark 2: the Sampling Theorem

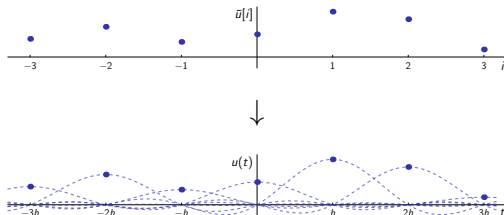
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Works as follows:

$$u(t) = \sum_{i \in \mathbb{Z}} \text{sinc}_h(t - ih) \bar{u}[i]$$



Remark 2: the Sampling Theorem

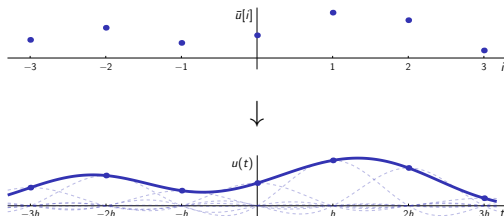
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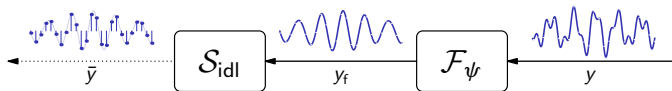
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Spectrum of signals sampled by shift-invariant samplers



Because

$$Y_f(j\omega) = F_\psi(j\omega) Y(j\omega)$$

we have that

$$\bar{Y}(e^{j\omega h}) = \frac{1}{h} \sum_{i \in \mathbb{Z}} F_\psi(j(\omega + \omega_s i)) Y(j(\omega + \omega_s i))$$

It makes sense to render $\mathcal{F}_\psi$ a low-pass filter, to eliminate high-frequency components of y *before* sampling. Such $\mathcal{F}_\psi$ are dubbed

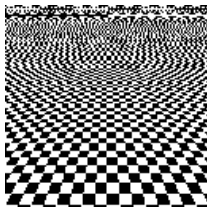
- antialiasing filters

(the best is the ideal low-pass, $\psi(t) = \frac{1}{h} \text{sinc}_h(t)$, but it is not practical).

Antialiasing filtering: non-control examples

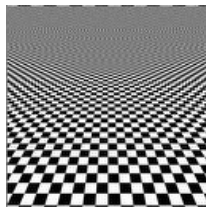
w/o antialiasing filter

התחזות

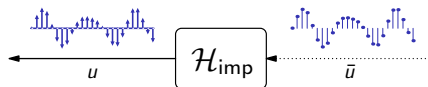


with antialiasing filter

התחזות



Spectrum of signals reconstructed by the impulse hold



Let

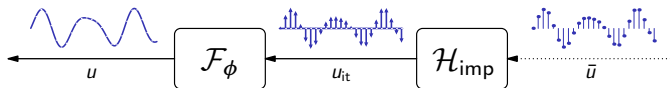
$$u(t) = \sum_{i \in \mathbb{Z}} \delta(t - ih) \bar{u}[i]$$

Then

$$U(j\omega) = \int_{\mathbb{R}} \sum_{i \in \mathbb{Z}} \delta(t - ih) \bar{u}[i] e^{-j\omega t} dt = \sum_{i \in \mathbb{Z}} \bar{u}[i] e^{-j\omega h i} = \bar{U}(e^{j\omega h})$$

is a periodic function of ω and therefore has infinite bandwidth (bad idea).

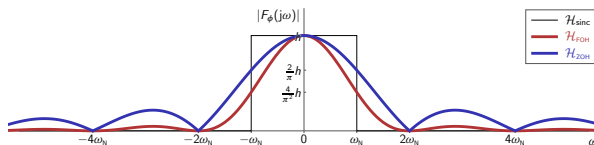
Spectrum of signals reconstructed by shift-invariant holds



Now,

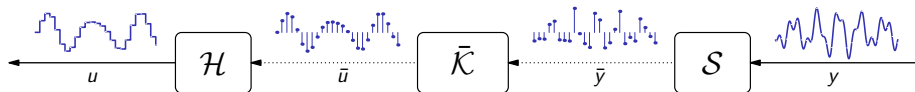
$$U(j\omega) = F_\phi(j\omega)U_{\text{it}}(j\omega) = F_\phi(j\omega)\bar{U}(e^{j\omega h})$$

and popular D/A converters use $\mathcal{F}_\phi$ to filter out high-frequency content of $\bar{U}(e^{j\omega h})$:



although none of them can outperform (non-implementable) $\mathcal{H}_{\text{sinc}}$.

Sampled-data controller (contd)



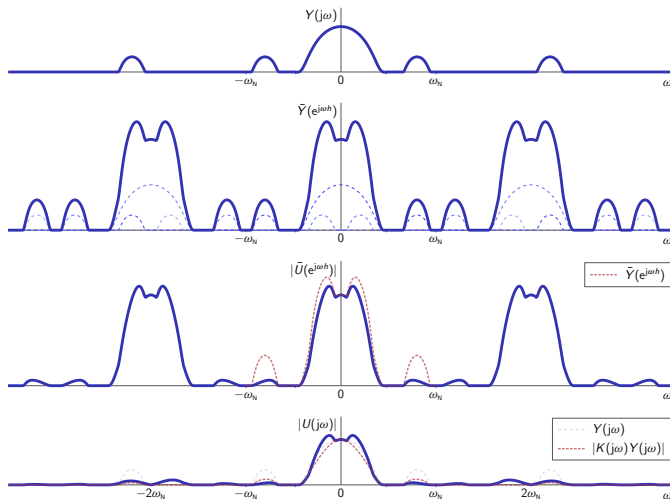
Combining all components:

$$U(j\omega) = \underbrace{\frac{1}{h} F_{\phi}(j\omega)}_{\text{low-pass}} \underbrace{\bar{K}(e^{j\omega h}) \sum_{i \in \mathbb{Z}} F_{\psi}(j(\omega + \frac{2\pi}{h}i)) Y(j(\omega + \frac{2\pi}{h}i))}_{\omega_s\text{-periodic, normally high-pass}}$$

Problems:

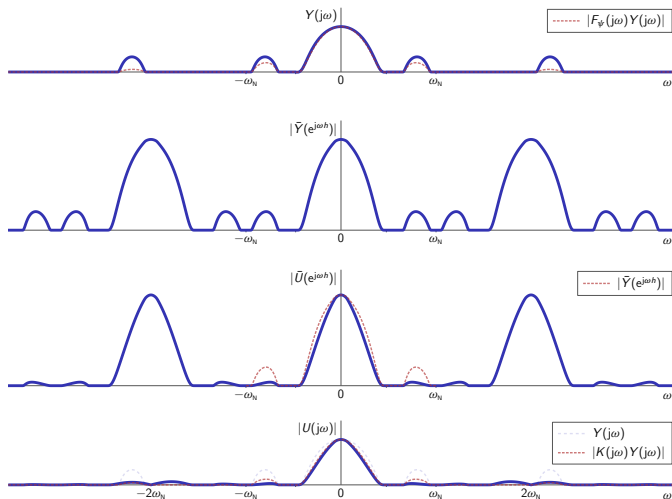
- aliasing at $\mathcal{S}$
high-frequency components of y pass through disguised as low-frequency harmonics
- insufficient low-pass prowess of $\mathcal{H}$
low-frequency components of $\bar{u}$ appear at higher frequencies

Example 1: no antialiasing, $F_{\psi}(s) = 1$



if $K(s) = \frac{2}{s+2}$ is discretized with $\omega_N = 2\pi$, $\bar{K}(z) = \frac{z+1}{3z-1}$, and $\mathcal{H} = \mathcal{H}_{\text{ZOH}}$.

Example 2: antialiasing with $F_\psi(s) = \left(\frac{2\pi}{s+2\pi}\right)^2$



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Content

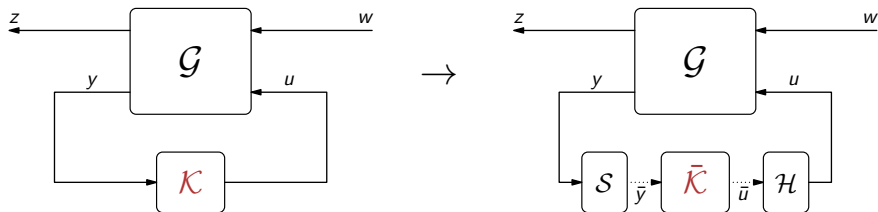
Digital redesign of analog controllers: frequency-domain background

Digital redesign of analog controllers

Discrete-time design: discretization background

Discrete-time design

Digital redesign: methodology

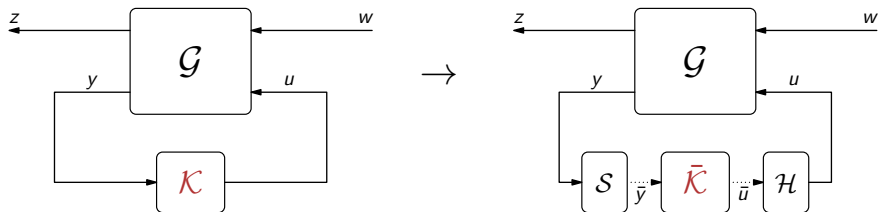


1. design a “good” analog $\mathcal{K}$
2. find $\bar{\mathcal{K}}$ such that $\mathcal{K} \approx \mathcal{H}\bar{\mathcal{K}}\mathcal{S}$ for given $\mathcal{S}$ and $\mathcal{H}$

Bad news: there's hardly any theory to support it in classical control, just plenty of rules of thumb...

Good news: they normally do the trick, if sampling rate is “fast enough”...

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Rules of thumb

Just a few of them:

- account for the need to add an antialiasing filter
- don't design analog loops with too high crossover(s)
it's considered safe to have $\omega_c < 0.05 \div 0.16 \omega_N$
- account for sampled-data implementation by adding loop delay $e^{-sh/2}$
perhaps motivated by the expansion $\frac{1-e^{-sh}}{sh} \approx 1 - \frac{1}{2}sh \approx e^{-sh/2}$
- approximating $\mathcal{K}$, use numerical integration rules to substitute s

$$\dot{x}(ih) \approx \frac{x(ih+h) - x(ih)}{h} \implies s \approx \frac{z-1}{h} \quad (\text{Euler's method})$$

$$\dot{x}(ih) \approx \frac{x(ih) - x(ih-h)}{h} \implies s \approx \frac{z-1}{hz} \quad (\text{backward difference})$$

$$\frac{\dot{x}(ih+h) + \dot{x}(ih)}{2} \approx \frac{x(ih+h) - x(ih)}{h} \implies s \approx \frac{2}{h} \frac{z-1}{z+1} \quad (\text{Tustin's approximat.})$$

— ...

More on Tustin's approximation

A general bilinear transformation

$$s \rightarrow \gamma \frac{z - 1}{z + 1}$$

transforms

$$\mathbb{C} \setminus \bar{\mathbb{C}}_0 \mapsto \mathbb{D}_1, \quad j\mathbb{R} \mapsto \mathbb{T}, \quad \{0\} \mapsto \{1\}, \quad \text{and} \quad \{j\infty\} \mapsto \{-1\}$$

i.e. it preserves stability, frequency response (although with warping), static gain, and high-frequency gain. Furthermore, frequencies warp as

$$\omega \mapsto 2 \arctan \frac{\omega}{\gamma},$$

so the choice (frequency prewarping) $\gamma = \omega_0 \cot \frac{\omega_0 h}{2}$ results in mapping

$$j\omega_0 \mapsto e^{j\omega_0 h},$$

which is useful under the choice $\omega_0 = \omega_c$ (yet another rule of thumb).

Content

Digital redesign of analog controllers: frequency-domain background

Digital redesign of analog controllers

Discrete-time design: discretization background

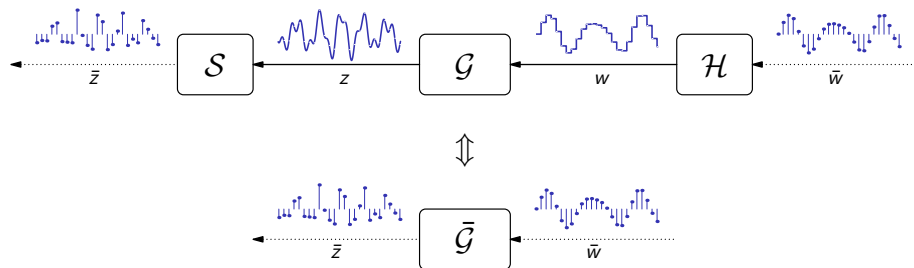
Discrete-time design

Discretizing analog systems

The discretization problem for an analog system $\mathcal{G}$ is to find (if it exists) a discrete system $\bar{\mathcal{G}}$, such that

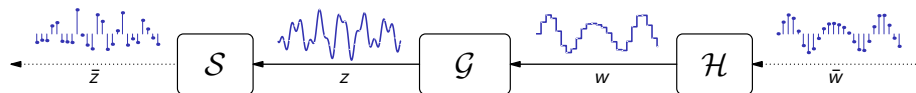
$$\bar{\mathcal{G}} = \mathcal{S}\mathcal{G}\mathcal{H}$$

for given hold and sampler, i.e.



Such $\bar{\mathcal{G}}$ is sometimes dubbed the **stroboscopic** model of $\mathcal{G}$.

Solvability



Let

$$\bar{z}[i] = \int_{\mathbb{R}} \psi(ih - t) z(t) dt, \quad z(t) = \int_{\mathbb{R}} g(t - s) w(s) ds,$$

$$\text{and } w(s) = \sum_{j \in \mathbb{Z}} \phi(s - jh) \bar{w}[j]$$

Then

$$\begin{aligned} \bar{z}[i] &= \int_{\mathbb{R}} \psi(ih - t) \int_{\mathbb{R}} g(t - s) \sum_{j \in \mathbb{Z}} \phi(s - jh) \bar{w}[j] ds dt \\ &= \sum_{j \in \mathbb{Z}} \underbrace{\int_{\mathbb{R}} \int_{\mathbb{R}} \psi(ih - t) g(t - s) \phi(s - jh) ds dt}_{\bar{g}[i,j]} \bar{w}[j] \end{aligned}$$

and we see that the stroboscopic model exists (which is hardly surprising).

Solvability (contd)

Now

$$\begin{aligned}\bar{g}[i, j+1] &= \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(ih - t)g(t - s)\phi(s - h - jh)dsdt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(ih - t)g(t - h - s)\phi(s - jh)dsdt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(ih - h - t)g(t - s)\phi(s - jh)dsdt \\ &= \bar{g}[i-1, j],\end{aligned}$$

i.e. $\bar{g}$ is always shift invariant.

The special case

If $\mathcal{H} = \mathcal{H}_{\text{ZOH}}$ and $\mathcal{S} = \mathcal{S}_{\text{idl}}$ (by far the most common setting),

$$\begin{aligned}\bar{g}[i-j] &= \int_{\mathbb{R}} \int_{\mathbb{R}} \psi(ih-t)g(t-s)\phi(s-jh)dsdt \\ &= \int_{\mathbb{R}} g(ih-s)\phi(s-jh)ds = \int_{jh}^{(j+1)h} g(ih-s)ds \\ &= \int_0^h g((i-j)h-s)ds\end{aligned}$$

so that the **impulse response** of $\bar{\mathcal{G}}$ is

$$\bar{g}[i] = \int_0^h g(ih-t)dt.$$

If $\mathcal{G}$ is causal, then $\bar{g}[0] = \int_0^h g(-t)dt = 0$ (i.e. $\bar{\mathcal{G}}$ is strictly causal), unless $g(t)$ contains the term $g_0\delta(t)$ and then $\bar{g}[0] = g_0$.

The special case (contd)

If $\mathcal{G}$ is finite dimensional, then $g(t) = D\delta(t) + Ce^{At}B\mathbb{1}(t)$ as we have:

$$\begin{aligned}\bar{g}[i] &= \int_0^h g(ih - t)dt = \int_0^h (D\delta(ih - t) + Ce^{A(ih-t)}B\mathbb{1}(ih - t))dt \\ &= \begin{cases} 0 & \text{if } i < 0 \\ D & \text{if } i = 0 \\ Ce^{A(i-1)h} \int_0^h e^{At}dt B & \text{if } i > 0 \end{cases}\end{aligned}$$

In other words, $\bar{\mathcal{G}}$ is a discrete finite-dimensional system with the realization

$$(\bar{A}, \bar{B}, \bar{C}, \bar{D}) := \left(e^{Ah}, \int_0^h e^{At}dt B, C, D \right).$$

The finite dimensionality should not be taken for granted, after all

$\mathcal{G} = \mathcal{S}_{\text{id}} \mathcal{G}_{\Phi} \mathcal{H}_{\text{imp}}$ with infinite-dimensional $\mathcal{F}_{\Phi}$ having $F_{\Phi}(s) = \frac{1-s^{-2h}}{s}$

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Calculating $\bar{B}$

The integral

$$\int_0^h e^{At} dt B = \int_0^h e^{A(h-s)} B ds$$

may be thought of as the

- response (from zero i. c.) of $(sI - A)^{-1}B$ to $\mathbb{1}(t)$ at $t = h$

But $\mathbb{1}(t)$ is the impulse response of the integrator. Thus, the integral is the

- impulse response of $(sI - A)^{-1}B \frac{1}{s} = [I \ 0] (sI - \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix})^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}$ at $t = h$

Thus,

$$\int_0^h e^{At} dt B = [I \ 0] \exp \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} h \right) \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Moreover, $\bar{A}$ is the (1,1) sub-block of the same exponential, so that

$$\begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} = \begin{bmatrix} e^{Ah} & \int_0^h e^{A(h-s)} B ds \end{bmatrix} = [I \ 0] \exp \left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} h \right).$$

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Remark: Van Loan's formulae

Consider the matrix

$$M := \int_0^h e^{Et} G e^{Ft} ds.$$

It is readily seen that

$$e^{-Eh} M = e^{-Eh} \int_0^h e^{Et} G e^{Ft} ds = \int_0^h e^{-E(h-t)} G e^{Ft} ds$$

is the impulse response of the cascade

$$\left[\begin{array}{c|c} -E & G \\ \hline I & 0 \end{array} \right] \left[\begin{array}{c|c} F & I \\ \hline I & 0 \end{array} \right] = \left[\begin{array}{c|c} -E & G \\ \hline 0 & F \\ \hline I & 0 \end{array} \right]$$

Thus,

$$\int_0^h e^{Et} G e^{Ft} ds = \Gamma_{11}^{-1} \Gamma_{12}, \quad \text{where} \quad \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & \Gamma_{22} \end{bmatrix} := \exp \left(\begin{bmatrix} -E & G \\ 0 & F \end{bmatrix} h \right).$$

Content

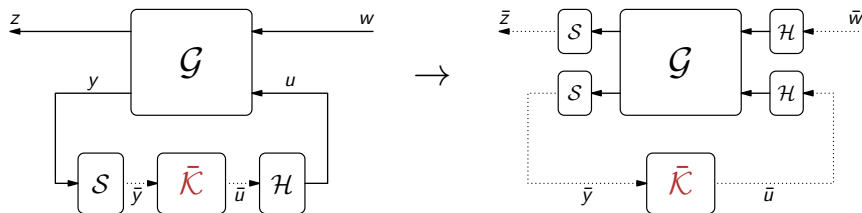
Digital redesign of analog controllers: frequency-domain background

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Discrete-time design

Discrete-time design: methodology



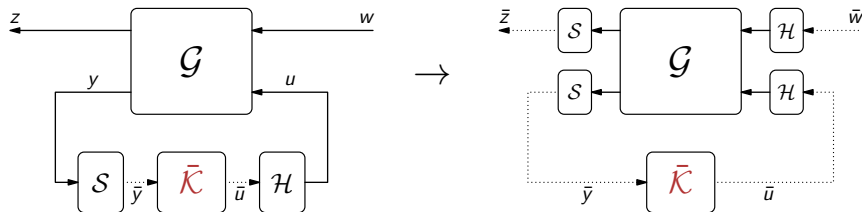
1. approximate the problem by a discrete standard problem
2. design a “good” discrete $\bar{\mathcal{K}}$

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Bad news: from the performance viewpoint, it's hardly justified in classical control

Good news: it frequently works, if sampling rate is “fast enough”

Discrete-time design: methodology



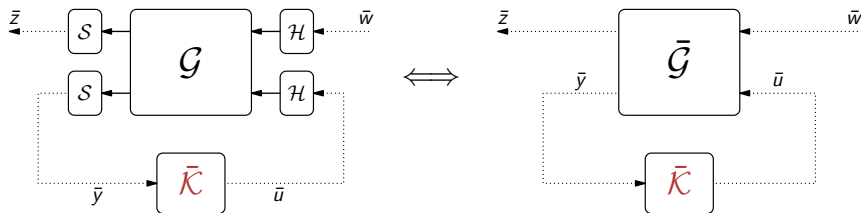
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Problem discretization: technical side

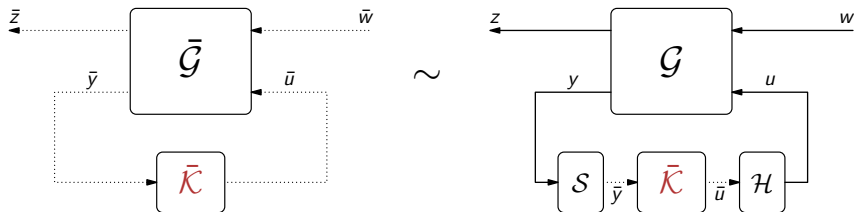


... and we end up with

- pure discrete
- LTI

problem, for which an extensive toolbox exists.

Problem discretization: stability

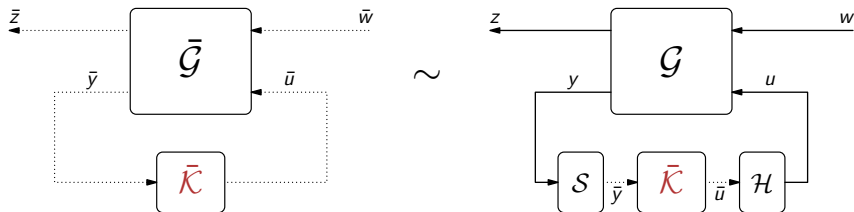


Assume that $\bar{\mathcal{K}}$ internally stabilizes $\bar{\mathcal{G}}$.

Q: Will $\bar{\mathcal{K}}$ internally stabilize the original hybrid system then?

A: Yes, almost always.

Problem discretization: stability



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Pathological sampling

Let $\mathcal{G}$ be a causal LTI system with a minimal realization as in

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right].$$

We say that sampling is **pathological** if there are at least two eigenvalues of A , say λ_i and λ_j , such that

$$\lambda_i - \lambda_j = j\omega_s l, \quad \text{for some } l \in \mathbb{Z} \setminus \{0\}$$

where $\omega_s := \frac{2\pi}{h}$. Otherwise, sampling is said to be non-pathological.

The minimal possible distance between two eigenvalues with the same real part is ω_s (i.e. under $l \equiv 1$). Thus,
→ sampling is never pathological if h is “small enough”

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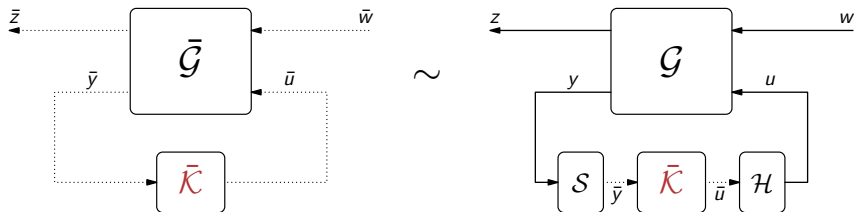
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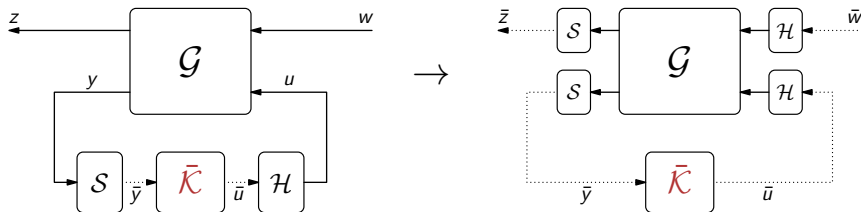
Problem discretization: stability (contd)



Lemma

Let $\bar{\mathcal{K}}$ stabilize $\bar{\mathcal{G}}$. Then $\mathcal{K} = \mathcal{H}_{\text{ZOH}} \bar{\mathcal{K}} \mathcal{S}_{\text{idl}}$ stabilizes $\mathcal{G}$ if h is non-pathological.

Problem discretization: performance



Two silently made assumptions:

1. w is close to $\mathcal{H}\bar{w}$
(e.g. w can be closely approximated by a piecewise-constant signal)
2. $\bar{z} = \mathcal{S}z$ reflects what happens with z
(e.g. behavior of $\bar{z}$ reflects what happens with z during the intersample)

But there are no rigorous justifications of them (especially, the second one) and we may only hope that they hold true if sampling is “fast enough”...