

Advanced Sampled-Data Control (038798)

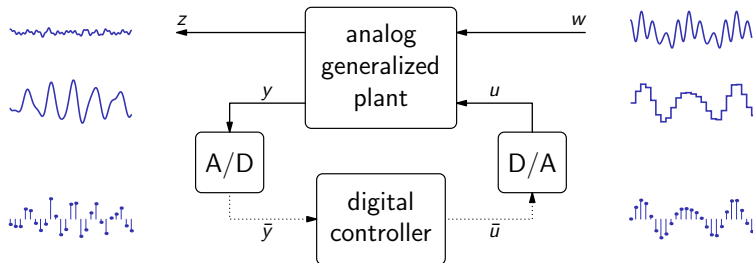
lecture no. 2

Leonid Mirkin

Faculty of Mechanical Engineering
Technion — IIT



Sampled-data standard problem



Next question:

- how analog and digital components talk to each other?

Content

A/D conversion

D/A conversion

Duality

Single-rate sampled-data controllers

Example

Content

A/D conversion

D/A conversion

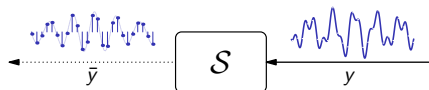
Duality

Single-rate sampled-data controllers

Example

Sampling

Analog-to-digital (A/D) converters, aka **samplers**, are devices transforming continuous-time signals into discrete sequences:



We associate $\mathcal{S}$ with a *strictly increasing* sequence of real numbers $\{t_i\}_{i \in \mathbb{Z}}$, called *sampling instances*, which set up a correspondence between the input and output time axes, $\mathbb{R}$ and $\mathbb{Z}$:

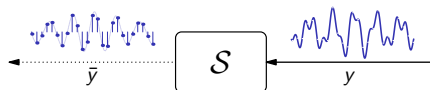
$$\bar{y}[i] \triangleq y(t_i).$$

It may be handy to assume that $t_0 = 0$. The quantities $h_i \triangleq t_{i+1} - t_i > 0$ are called *sampling intervals*. Note that

— the sequence of $y(t_i)$ is *not* necessarily periodic

Sampling

Analog-to-digital (A/D) converters, aka **samplers**, are devices transforming continuous-time signals into discrete sequences:



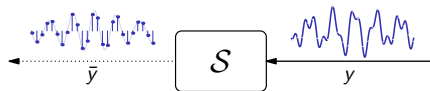
We associate S with a *strictly increasing* sequence of real numbers $\{t_i\}_{i \in \mathbb{Z}}$, called **sampling instances**, which set up a correspondence between the input and output time axes, $\mathbb{R}$ and $\mathbb{Z}$:

$$\bar{y}[i] \hat{=} y(t_i).$$

It may be handy to assume that $t_0 = 0$. The quantities $h_i := t_{i+1} - t_i > 0$ are called **sampling intervals**. Note that

- dimensions of $y(t)$ and $\bar{y}[i]$ need not coincide

Sampling: more terminology



Sampling instances are said to be

uniform if $t_i = ih$ for some $h > 0$, called the sampling period

$\mathcal{S}$ is said to be

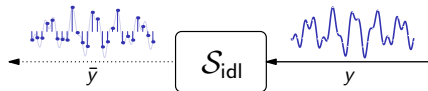
causal if $\bar{y}[i]$ may only depend on $y(t)$ for $t \leq t_i$

static (or memoryless), if $\bar{y}[i]$ may only depend on $y(t)$ for $t \in [t_{i-1}, t_i]$

stable if it is a bounded operator $\mathfrak{D}(\mathcal{S}) \cap L_2(\mathbb{R}) \rightarrow \ell_2(\mathbb{Z})$

$\mathfrak{D}(\mathcal{S})$ stands for the domain of $\mathcal{S}$

The ideal sampler



Acts as

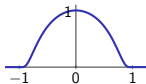
$$\bar{y}[i] = y(t_i)$$

and requires the *continuity* of $y(t)$ at sampling instances. $\mathcal{S}_{idl}$ is

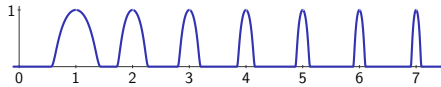
- causal
- static
- **unstable**, strangely enough

The ideal sampler: stability

Bump function:

$$f_{\text{bump}}(t) = \begin{cases} e^{t^2/(t^2-1)} & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases} = \text{graph}$$


(smooth). Let now

$$y(t) = \sum_{i \in \mathbb{N}} f_{\text{bump}}\left(\frac{t-i}{(i+1)^{1.1}}\right) = \text{graph}$$


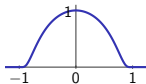
for which

$$\begin{aligned} \|y\|_2^2 &= \sum_{i \in \mathbb{N}} \int_{\mathbb{R}} \left[f\left(\frac{t-i}{(i+1)^{1.1}}\right) \right]^2 dt = \sum_{i \in \mathbb{N}} \frac{1}{(i+1)^{1.1}} \int_{\mathbb{R}} [f(t-i)]^2 dt \\ &= \sum_{i \in \mathbb{N}} \frac{\|f_{\text{bump}}\|_2^2}{(i+1)^{1.1}} \approx 9.42516 < \infty, \end{aligned}$$

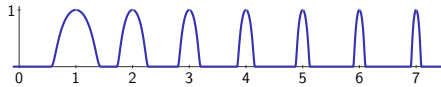
so $y \in L_2(\mathbb{R})$. But $S_M y = \sqrt{T} \notin L_2(\mathbb{Z})$, which means that S_M is unstable.

The ideal sampler: stability

Bump function:

$$f_{\text{bump}}(t) = \begin{cases} e^{t^2/(t^2-1)} & \text{if } |t| < 1 \\ 0 & \text{otherwise} \end{cases} = \text{graph}$$


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for which

$$\begin{aligned} \|y\|_2^2 &= \sum_{i \in \mathbb{N}} \int_{\mathbb{R}} \left[f\left(\frac{t-i}{(i+1)^{1.1}}\right) \right]^2 dt = \sum_{i \in \mathbb{N}} \frac{1}{(i+1)^{1.1}} \int_{\mathbb{R}} [f(t-i)]^2 dt \\ &= \sum_{i \in \mathbb{N}} \frac{\|f_{\text{bump}}\|_2^2}{(i+1)^{1.1}} \approx 9.42516 < \infty, \end{aligned}$$

so $y \in L_2(\mathbb{R})$. But $\mathcal{S}_{\text{idl}} y = \bar{u}\bar{1} \notin \ell_2(\mathbb{Z})$, which means that $\mathcal{S}_{\text{idl}}$ is **unstable**.

The averaging sampler



Acts as

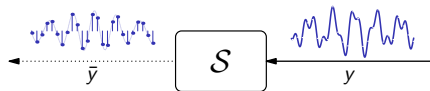
$$\bar{y}[i] = \frac{1}{\Delta} \int_{t_i - \Delta}^{t_i} y(s) ds$$

and does not require the continuity of $y(t)$ at sampling instances. $\mathcal{S}_{\text{avg}}$ is

- causal
- static, iff $\Delta \leq \inf_i h_i$
- stable (not obvious in this stage)

$\mathcal{S}_{\text{avg}}$ approaches $\mathcal{S}_{\text{idl}}$ as $\Delta \rightarrow 0$.

Generalized sampler



Acts as

$$\bar{y}[i] = \int_{\mathbb{R}} \psi(t_i, s) y(s) ds,$$

for some¹ $\psi(t, s)$ called the sampling function (so ψ). $\psi(t_i, s)$ shows the response of $\mathcal{S}$, at the time instance i , to the δ -impulse applied at the time instance s . This sampler is *linear* (as $\mathcal{S}(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 \mathcal{S}y_1 + \alpha_2 \mathcal{S}y_2$).

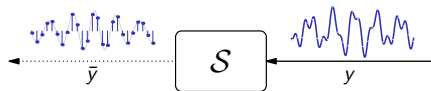
Special cases:

$$\mathcal{S}_{\text{Dir}} \psi(t, s) = \delta(t - s)$$

$$\mathcal{S}_{\text{avg}} \psi(t, s) = \frac{1}{\Delta} (1(t - s) - 1(t - s - \Delta))$$

¹Although $\psi(t, s)$ must not be defined at every t , we hereafter assume that it is.

Generalized sampler



Acts as

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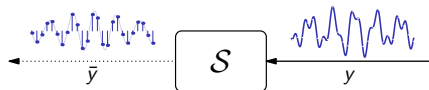
Special cases:

$$\mathcal{S}_{\text{idl}}: \psi(t_i, s) = \delta(t_i - s)$$

$$\mathcal{S}_{\text{avg}}: \psi(t_i, s) = \frac{1}{\Delta} (\mathbb{1}(t_i - s) - \mathbb{1}(t_i - s - \Delta))$$

¹Although $\psi(t, s)$ must not be defined at every t , we hereafter assume that it is.

Generalized sampler: generation via $\mathcal{S}_{idl}$



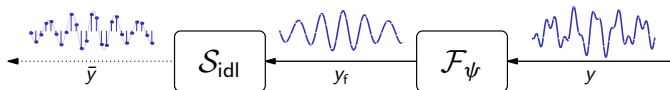
Let

$$y_f(t) = \int_{\mathbb{R}} \psi(t, s) y(s) ds \quad \implies \quad \bar{y}[i] = y_f(t_i)$$

so that any generalized sampler can be viewed as

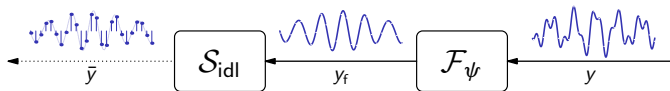
- the cascade of an analog filter and the ideal sampler,

i.e. as



where $\mathcal{F}_\psi$ is the linear analog filter having the impulse response $\psi(t, s)$.

Generalized sampler: shift invariance



We say that $\mathcal{S}$ is shift-invariant if

1. sampling is uniform (with some sampling period $h > 0$)
2. $\mathcal{F}_\psi$ is time invariant

In this case,

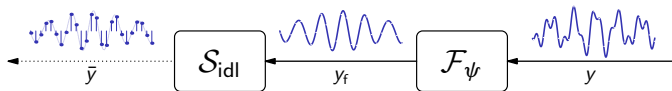
$$\bar{y}[i] = \int_{\mathbb{R}} \psi(ih - s)y(s)ds =: (\psi \star y)[i],$$

which may be called the “A/D convolution integral.” Special cases:

$$\Delta_{\text{ID}}: \psi(t) = \delta(t)$$

$$\Delta_{\text{ZOH}}: \psi(t) = \frac{1}{\Delta} 1_{[0, \Delta]}(t), \text{ so that } F_\psi(s) = \frac{1 - e^{-\Delta s}}{\Delta s}$$

Generalized sampler: shift invariance



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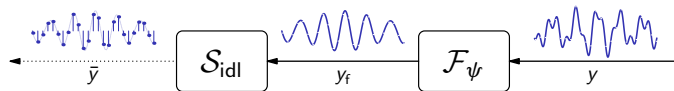
$$\bar{y}[i] = \int_{\mathbb{R}} \psi(ih - s)y(s)ds =: (\psi \ast y)[i],$$

which may be called the “A/D convolution integral.” Special cases:

$\mathcal{S}_{idl}$: $\psi(t) = \delta(t)$

$\mathcal{S}_{avg}$: $\psi(t) = \frac{1}{\Delta} \mathbb{1}_{[0, \Delta]}(t)$, so that $F_\psi(s) = \frac{1 - e^{-\Delta s}}{\Delta s}$

Generalized sampler: shift invariance (contd)



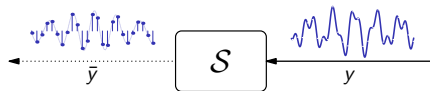
Shift invariance implies that

$$\begin{aligned} \int_{\mathbb{R}} \psi(ih - s)y(s - h)ds &= \int_{\mathbb{R}} \psi(ih - (s + h))y(s)ds \\ &= \int_{\mathbb{R}} \psi((i - 1)h - s)y(s)ds, \end{aligned}$$

i.e. that

- h -shift of the analog input $\implies$ unit shift of the discrete output
or, equivalently, $\mathcal{S}\mathcal{D}_h = \bar{\mathcal{U}}\mathcal{S}$ (and then $\mathcal{S}\mathcal{D}_{\nu h} = \bar{\mathcal{U}}^\nu\mathcal{S}$ for all $\nu \in \mathbb{Z}$).

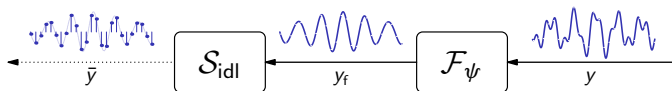
Generalized sampler: causality



As

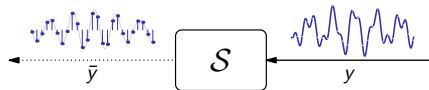
$$\bar{y}[i] = \int_{-\infty}^{t_i} \psi(t_i, s) y(s) ds + \int_{t_i}^{\infty} \psi(t_i, s) y(s) ds,$$

the causality of $\mathcal{S}$ is equivalent to $\psi(t_i, s) = 0$ whenever $s > t_i$, $\forall i \in \mathbb{Z}$. In



the causality of $\mathcal{S}$ is *guaranteed* by that of $\mathcal{F}_\psi$ (in which case $\psi(t, s) = 0$ whenever $s > t$, for all $t \in \mathbb{R}$, not only at the sampling instances). If $\mathcal{F}_\psi$ is time invariant, $\mathcal{S}$ is causal iff $\mathcal{F}_\psi$ is causal, as $\psi(t_i, s) = \psi(t_i + \kappa, s + \kappa)$.

Static generalized samplers



As

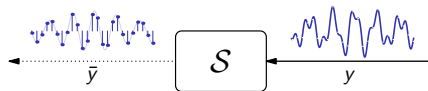
$$\bar{y}[i] = \int_{-\infty}^{t_{i-1}} \psi(t_i, s) y(s) ds + \int_{t_{i-1}}^{t_i} \psi(t_i, s) y(s) ds + \int_{t_i}^{\infty} \psi(t_i, s) y(s) ds,$$

a sampler is static iff

$$\psi(t_i, s) = 0 \text{ whenever } s \notin [t_{i-1}, t_i],$$

which normally corresponds to a causal FIR (finite impulse response) $\mathcal{F}_\psi$.

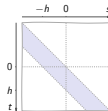
Shift invariant generalized samplers



If $\mathcal{S}$ is causal, its A/D convolution reads

$$\bar{y}[i] = \int_{-\infty}^{ih} \psi(ih - s)y(s)ds$$

If $\mathcal{S}$ is static, $\psi(t - s)$:



and its A/D convolution reads

$$\bar{y}[i] = \int_{(i-1)h}^{ih} \psi(ih - s)y(s)ds$$

Content

A/D conversion

D/A conversion

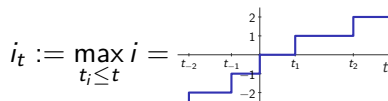
Duality

Single-rate sampled-data controllers

Example

Preliminary: generalized “floor” function

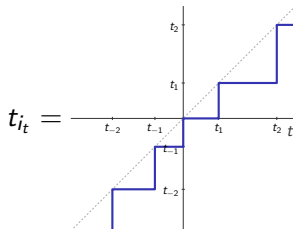
Let $\{t_i\}_{i \in \mathbb{Z}}$ be sampling instances. Define the $\mathbb{R} \rightarrow \mathbb{Z}$ function



which shows the last sampling instance preceding t . If sampling is uniform,

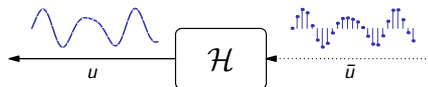
$$i_t = \left\lfloor \frac{t}{h} \right\rfloor.$$

Then



Holds

Digital-to-analog (D/A) converters, aka **holds** or interpolators, are devices transforming discrete sequences into continuous-time signals:

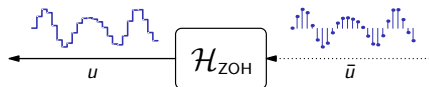


$\mathcal{H}$ is also associated with a sampling sequence $\{t_i\}_{i \in \mathbb{Z}}$ and is said to be **causal** if $u(t)$ may only depend on $\bar{u}[i]$ for $i \leq i_t$

static (or memoryless), if $u(t)$ may only depend on $\bar{u}[i]$ for $i = i_t$

stable if it is a bounded operator $\ell_2(\mathbb{Z}) \rightarrow L_2(\mathbb{R})$

The zero-order hold



Acts as

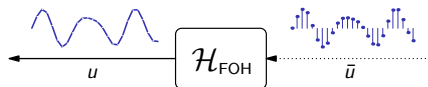
$$u(t) = \bar{u}[i_t] \quad \text{or} \quad u(t) = \bar{u}[i], \quad \forall t \in [t_i, t_{i+1})$$

It is

- causal
- static
- stable (whenever sampling intervals h_i are uniformly bounded):

$$\begin{aligned} \|u\|_2^2 &= \int_{\mathbb{R}} \|u(t)\|^2 dt = \sum_{i \in \mathbb{R}} \int_{t_i}^{t_{i+1}} \|u(t)\|^2 dt = \sum_{i \in \mathbb{R}} h_i \|\bar{u}[i]\|^2 \\ &\leq (\sup_i h_i) \|\bar{u}\|_2^2. \end{aligned}$$

The first-order hold



Acts as

$$u(t) = \frac{t_{i_t+1} - t}{h_{i_t}} \bar{u}[i_t] + \frac{t - t_{i_t}}{h_{i_t}} \bar{u}[i_t + 1]$$

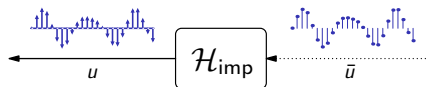
(linear interpolation between $\bar{u}[i]$ and $\bar{u}[i + 1]$ at each $i \in \mathbb{Z}$). It is

- **non**-causal

(delayed first-order hold, $u(t) = \frac{t_{i_t+1} - t}{h_{i_t}} \bar{u}[i_t - 1] + \frac{t - t_{i_t}}{h_{i_t}} \bar{u}[i_t]$, is causal)

- dynamic
- stable (whenever sampling intervals h_i are uniformly bounded)

The impulse hold



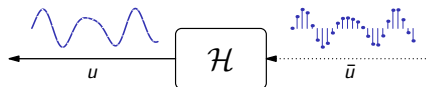
Acts as

$$u(t) = \delta(t - t_{i_t}) \bar{u}[i_t]$$

It is

- causal
- static
- unstable
- not quite practical by itself...

Generalized hold



Acts as

$$u(t) = \sum_{i \in \mathbb{Z}} \phi(t, t_i) \bar{u}[i]$$

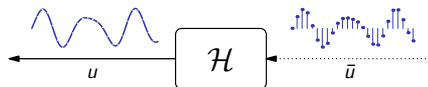
for some $\phi(t, s)$ called the **hold function** (so ϕ). $\phi(t, t_i)$ is the response of $\mathcal{H}$, at the time instance t , to the unit pulse applied at the time instance i . This hold is *linear* (as $\mathcal{H}(\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2) = \alpha_1 \mathcal{H} \bar{u}_1 + \alpha_2 \mathcal{H} \bar{u}_2$). *Special cases*

Imp $\phi(t, t_i) = \delta(t - t_i)$

ZOH $\phi(t, t_i) = \mathbb{1}(t - t_i) - \mathbb{1}(t - t_{i+1})$

FOH $\phi(t, t_i) =$

Generalized hold



Acts as

$$u(t) = \sum_{i \in \mathbb{Z}} \phi(t, t_i) \bar{u}[i]$$

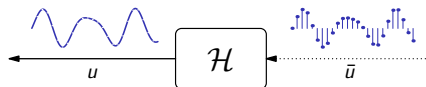
for some $\phi(t, s)$ called the **hold function** (so phi). $\phi(t, t_i)$ is the response of $\mathcal{H}$, at the time instance t , to the unit pulse applied at the time instance i . This hold is *linear* (as $\mathcal{H}(\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2) = \alpha_1 \mathcal{H} \bar{u}_1 + \alpha_2 \mathcal{H} \bar{u}_2$). Special cases:

$\mathcal{H}_{\text{imp}}$: $\phi(t, t_i) = \delta(t - t_i)$

ZOH : $\phi(t, t_i) = \begin{cases} 1 & t_i \leq t < t_{i+1} \\ 0 & \text{otherwise} \end{cases} = \mathbb{1}(t - t_i) - \mathbb{1}(t - t_{i+1})$

FOH : $\phi(t, t_i) = \begin{cases} \frac{t - t_{i-1}}{t_i - t_{i-1}} & t_{i-1} \leq t < t_i \\ \frac{t_{i+1} - t}{t_{i+1} - t_i} & t_i \leq t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$

Generalized hold: generation via $\mathcal{H}_{\text{imp}}$



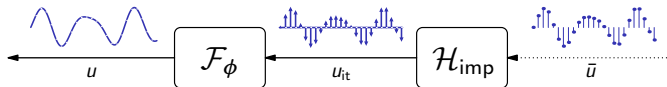
Let

$$u_{\text{it}}(s) = \sum_{i \in \mathbb{Z}} \delta(s - t_i) \bar{u}[i] \quad \Longrightarrow \quad u(t) = \int_{\mathbb{R}} \phi(t, s) u_{\text{it}}(s) ds$$

so that any generalized hold can be viewed as

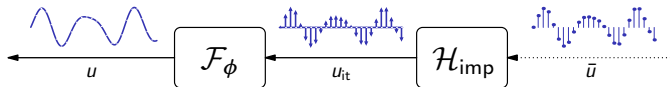
- the cascade of the impulse hold and an analog filter,

i.e. as



where $\mathcal{F}_\phi$ is the analog filter having the impulse response $\phi(t, s)$.

Generalized hold: shift invariance



We say that $\mathcal{H}$ is shift-invariant if

1. sampling is uniform (with some sampling period $h > 0$)
2. $\mathcal{F}_\phi$ is time invariant

In this case,

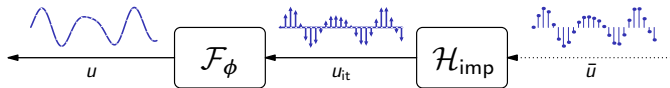
$$u(t) = \sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i] =: (\phi \ast \bar{u})(t)$$

which may be called the “**D/A convolution** sum.” Special cases:

ZOH: $\phi(t) = 1_{[0,h)}(t)$, so $F_\phi(s) = \frac{1 - e^{-sh}}{s}$

FQZ: $\phi(t) = \frac{1}{h}(1_{[-h,0]} \ast 1_{[0,h)})(t)$, so $F_\phi(s) = \frac{e^{sh} + e^{-sh} - 2}{s^2 h}$

Generalized hold: shift invariance



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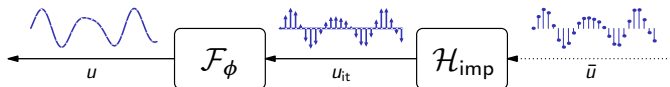
$$u(t) = \sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i] =: (\phi * \bar{u})(t)$$

which may be called the “D/A convolution sum.” Special cases:

ZOH: $\phi(t) = \begin{array}{c} 1 \\ \text{---} \\ 0 \quad h \quad t \end{array} = \mathbb{1}_{[0,h]}(t)$, so $F_\phi(s) = \frac{1 - e^{-sh}}{s}$

FOH: $\phi(t) = \begin{array}{c} 1 \\ \text{---} \\ -h \quad 0 \quad h \quad t \end{array} = \frac{1}{h}(\mathbb{1}_{[-h,0]} * \mathbb{1}_{[0,h]})(t)$, so $F_\phi(s) = \frac{e^{sh} + e^{-sh} - 2}{s^2 h}$

Generalized hold: shift invariance (contd)



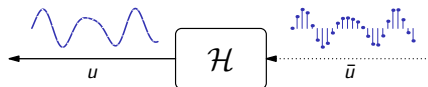
Shift invariance implies that

$$\sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i - 1] = \sum_{i \in \mathbb{Z}} \phi(t - (i + 1)h) \bar{u}[i] = \sum_{i \in \mathbb{Z}} \phi((t - h) - ih) \bar{u}[i]$$

i.e. that

- unit shift of the discrete input $\implies h$ -shift of the analog output
or, equivalently, $\mathcal{H}\bar{u} = \mathcal{D}_h \mathcal{H}$ (and then $\mathcal{H}\bar{u}^v = \mathcal{D}_{vh} \mathcal{H}$ for all $v \in \mathbb{Z}$).

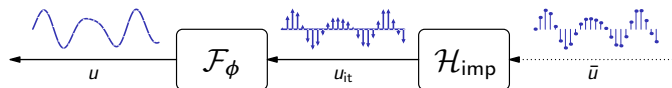
Generalized hold: causality



As

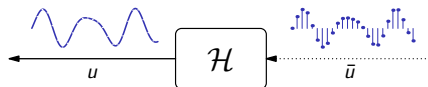
$$u(t) = \sum_{i=-\infty}^{i_t} \phi(t, t_i) \bar{u}[i] + \sum_{i=i_t+1}^{\infty} \phi(t, t_i) \bar{u}[i],$$

the causality of $\mathcal{H}$ is equivalent to $\phi(t, t_i) = 0$ whenever $i > i_t, \forall t \in \mathbb{R}$. In



the causality of $\mathcal{H}$ is *guaranteed* by that of $\mathcal{F}_\phi$ (in which case $\phi(t, s) = 0$ whenever $s > t$, for all $t \in \mathbb{R}$, not only at the sampling instances). If $\mathcal{F}_\phi$ is time invariant, $\mathcal{H}$ is causal iff $\mathcal{F}_\phi$ is causal, as $\phi(t, t_i) = \phi(t + \kappa, t_i + \kappa)$.

Static generalized holds



As

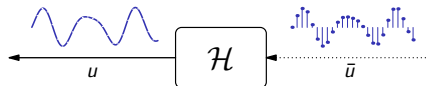
$$u(t) = \sum_{i=-\infty}^{i_t-1} \phi(t, t_i) \bar{u}[i] + \phi(t, t_{i_t}) \bar{u}[i_t] + \sum_{i=i_t+1}^{\infty} \phi(t, t_i) \bar{u}[i],$$

a hold is static iff

$$\phi(t, t_i) = 0 \text{ whenever } t \notin [t_{i_t}, t_{i_t+1}].$$

which normally corresponds to a causal FIR (finite impulse response) $\mathcal{F}_\phi$.

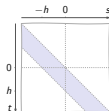
Shift invariant generalized holds



If $\mathcal{H}$ is causal, its D/A convolution reads

$$u(t) = \sum_{i=-\infty}^{\lfloor t/h \rfloor} \phi(t - ih) \bar{u}[i]$$

If $\mathcal{H}$ is static, $\phi(t - s)$:



and its D/A convolution reads

$$u(t) = \phi(t - \lfloor \frac{t}{h} \rfloor h) \bar{u}[\lfloor \frac{t}{h} \rfloor]$$

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Preliminary: adjoint operators

Let $\mathbb{H}_i$ and $\mathbb{H}_o$ be Hilbert spaces and $\langle \cdot, \cdot \rangle_{\mathbb{H}_i}$ and $\langle \cdot, \cdot \rangle_{\mathbb{H}_o}$ be corresponding inner products. Consider an operator

$$O : \mathbb{H}_i \rightarrow \mathbb{H}_o$$

There is (normally) an operator

$$O^* : \mathbb{H}_o \rightarrow \mathbb{H}_i$$

such that

$$\langle O\eta_i, \eta_o \rangle_{\mathbb{H}_o} = \langle \eta_i, O^*\eta_o \rangle_{\mathbb{H}_i}, \quad \forall \eta_i \in \mathbb{H}_i \text{ and } \eta_o \in \mathbb{H}_o.$$

The operator O^* is called the **adjoint** of O . Adjoints are extensively used in optimization methods. In particular (here induced norms are meant),

$$\|O\|^2 = \|OO^*\| = \bar{\sigma}(OO^*) = \|O^*O\| = \bar{\sigma}(O^*O).$$

Adjoint of ZOH

We know that $\mathcal{H}_{\text{ZOH}} : \ell_2(\mathbb{Z}) \rightarrow L_2(\mathbb{R})$. Then, assuming periodic sampling,

$$\begin{aligned}\langle \mathcal{H}_{\text{ZOH}} \bar{u}, y \rangle_{L_2} &= \int_{\mathbb{R}} [y(t)]' (\mathcal{H}_{\text{ZOH}} \bar{u})(t) dt = \sum_{i \in \mathbb{Z}} \int_{ih}^{(i+1)h} [y(t)]' \bar{u}[i] dt \\ &= \sum_{i \in \mathbb{Z}} \left(\int_{ih}^{(i+1)h} y(t) dt \right)' \bar{u}[i] \\ &= \langle \bar{u}, \mathcal{H}_{\text{ZOH}}^* y \rangle_{\ell_2},\end{aligned}$$

where $\mathcal{H}_{\text{ZOH}}^* : y \mapsto \bar{y}$ is actually a **sampler** and it acts as

$$\bar{y}[i] = \int_{ih}^{(i+1)h} y(t) dt$$

This is a predictive (i.e. non-causal) averaging sampler, modulo scaling.

Adjoint of ZOH (contd)

Now, $\mathcal{H}_{\text{ZOH}}^* \mathcal{H}_{\text{ZOH}} : \ell_2(\mathbb{Z}) \rightarrow \ell_2(\mathbb{Z})$ acts as

$$\mathcal{H}_{\text{ZOH}}^* \mathcal{H}_{\text{ZOH}} \bar{u} = \int_{ih}^{(i+1)h} (\mathcal{H}_{\text{ZOH}} \bar{u})(t) dt = \int_{ih}^{(i+1)h} \bar{u}[i] dt = h \bar{u}[i]$$

This is a static shift-invariant discrete system, and its H_∞ norm

$$\|\mathcal{H}_{\text{ZOH}}^* \mathcal{H}_{\text{ZOH}}\|_\infty = h.$$

Hence,

$$\|\mathcal{H}_{\text{ZOH}}\| = \sqrt{h},$$

which we already know.

Adjoint of generalized shift-invariant hold

Consider a shift-invariant generalized hold $\mathcal{H} : \bar{u} \mapsto u = \phi * \bar{u}$ and assume that $\mathcal{H} : \ell_2(\mathbb{Z}) \rightarrow L_2(\mathbb{R})$. Then

$$\begin{aligned}\langle \mathcal{H}\bar{u}, y \rangle_{L_2} &= \int_{\mathbb{R}} [y(t)]' \sum_{i \in \mathbb{Z}} \phi(t - ih) \bar{u}[i] dt = \sum_{i \in \mathbb{Z}} \int_{\mathbb{R}} [y(t)]' \phi(t - ih) \bar{u}[i] dt \\ &= \sum_{i \in \mathbb{Z}} \left(\int_{\mathbb{R}} \phi'(t - ih) y(t) dt \right)' \bar{u}[i] \\ &= \langle \bar{u}, \mathcal{H}^* y \rangle_{\ell_2}.\end{aligned}$$

Hence, $\mathcal{H}^* : L_2(\mathbb{R}) \rightarrow \ell_2(\mathbb{Z})$ is the generalized sampler having the sampling function

$$\psi(t) = \phi'(-t).$$

Note that if $\mathcal{H}$ is causal, the sampler $\mathcal{H}^*$ is anti-causal.

Adjoint of generalized shift-invariant sampler

Consider now a shift-invariant generalized sampler $\mathcal{S} : y \mapsto \bar{y} = \psi \star y$ and assume that $\mathcal{S} : L_2(\mathbb{R}) \rightarrow \ell_2(\mathbb{Z})$. Then

$$\begin{aligned} \langle \mathcal{S}y, \bar{y} \rangle_{\ell_2} &= \sum_{i \in \mathbb{Z}} (\bar{u}[i])' \int_{\mathbb{R}} \psi(ih - t) y(t) dt = \int_{\mathbb{R}} \sum_{i \in \mathbb{Z}} (\bar{u}[i])' \psi(ih - t) y(t) dt \\ &= \int_{\mathbb{R}} \left(\sum_{i \in \mathbb{Z}} \psi'(ih - t) \bar{u}[i] \right)' y(t) dt \\ &= \langle y, \mathcal{S}^* \bar{y} \rangle_{L_2}. \end{aligned}$$

Hence, $\mathcal{S}^* : \ell_2(\mathbb{Z}) \rightarrow L_2(\mathbb{R})$ is the generalized hold having the hold function

$$\phi(t) = \psi'(-t).$$

Note that if $\mathcal{S}$ is causal, the hold $\mathcal{S}^*$ is anti-causal, again.

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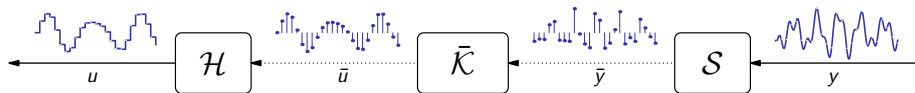
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General linear sampled-data controller

By a (single-rate) linear sampled-data controller $\mathcal{K} : y \mapsto u$ we understand the cascade

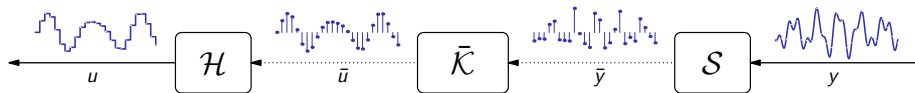


where

$$u(t) = \sum_{i \in \mathbb{Z}} \phi(t, t_i) \bar{u}[i], \quad \bar{u}[i] = \sum_{j \in \mathbb{Z}} \bar{k}[i, j] \bar{y}[j], \quad \bar{y}[j] = \int_{\mathbb{R}} \psi(t_j, s) y(s) ds$$

for some locally bounded ψ , $\bar{k}$, and ϕ having appropriate dimensions.

Kernel representation

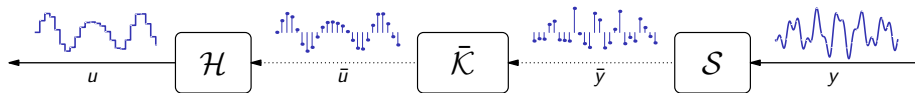


Passing again over convergence issues,

$$\begin{aligned}
 y(t) &= \sum_{i \in \mathbb{Z}} \phi(t, t_i) \sum_{j \in \mathbb{Z}} \bar{k}[i, j] \int_{\mathbb{R}} \psi(t_j, s) y(s) ds \\
 &= \int_{\mathbb{R}} \underbrace{\sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \phi(t, t_i) \bar{k}[i, j] \psi(t_j, s)}_{k(t, s)} y(s) ds,
 \end{aligned}$$

which is an analog linear system with the impulse response $k(t, s)$.

Sampled-data causality



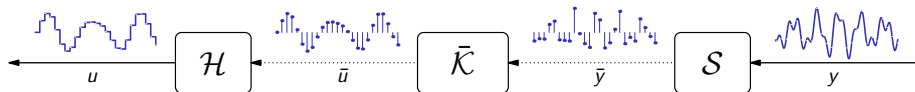
The causality of all components should imply the causality of $\mathcal{K}$. But not every causal analog controller is a sampled-data one. Indeed, we have that

- $\psi(t_j, s) = 0$ whenever $s > t_j$
- $\bar{k}[i, j] = 0$ whenever $j > i$
- $\phi(t, t_i) = 0$ whenever $i > i_t$ ($:= \max_{t_i \leq t} i$)

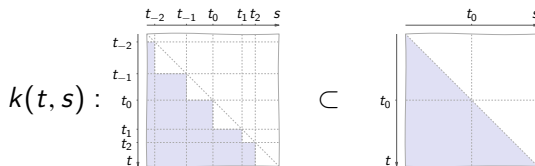
Hence,

- $\bar{k}[i, j]\psi(t_j, s) = 0$ whenever $s > t_i$
- $\phi(t, t_i)\bar{k}[i, j]\psi(t_j, s) = 0$ whenever $s > t_{i_t}$
- $k(t, s) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \phi(t, t_i)\bar{k}[i, j]\psi(t_j, s) = 0$ whenever $s > t_{i_t}$

Sampled-data causality (contd)



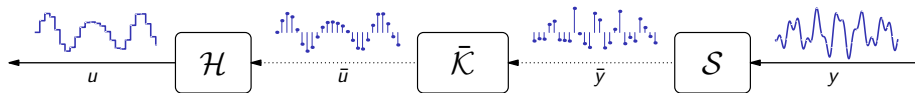
Condition “ $k(t, s) = 0$ whenever $s > t_{i_t}$ ” constrains



Thus,

- any sampled-data controller *must* have a “staircase” kernel although it’s not clear (yet?) whether this property is sufficient.

Sampled-data controller with shift-invariant components



- sampling is uniform, with sampling period $h > 0$
- $\mathcal{S}$ is shift invariant, with $\bar{y}[j] = \int_{\mathbb{R}} \psi(jh - s)y(s)ds$
- $\bar{\mathcal{K}}$ is shift invariant, with $\bar{u}[i] = \sum_{j \in \mathbb{Z}} \bar{k}[i - j]\bar{y}[j]$
- $\mathcal{H}$ is shift invariant, with $u(t) = \sum_{i \in \mathbb{Z}} \phi(t - ih)\bar{u}[i]$

Then

$$k(t, s) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \phi(t - ih) \bar{k}[i - j] \psi(jh - s).$$

Time invariance



In this case, $u(t) = y(\lfloor \frac{t}{h} \rfloor h)$. Let $h = 4$, then

$$\mathcal{K}\mathbb{1}_{[-1,1]} = \mathbb{1}_{[0,4]}$$

whereas

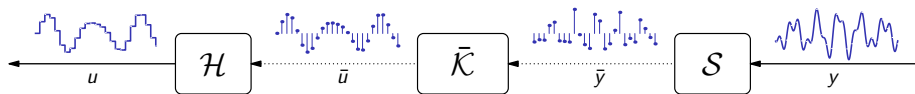
$$\mathcal{K}\mathcal{D}_2\mathbb{1}_{[-1,1]} = \mathcal{K}\mathbb{1}_{[1,3]} = 0 \neq \mathcal{D}_2\mathcal{K}\mathbb{1}_{[-1,1]} = \mathbb{1}_{[2,5]}$$

Thus, $\mathcal{K}\mathcal{D}_2 \neq \mathcal{D}_2\mathcal{K}$ and

- $\mathcal{K}$ is **not** time invariant

(at least, not always).

Periodicity

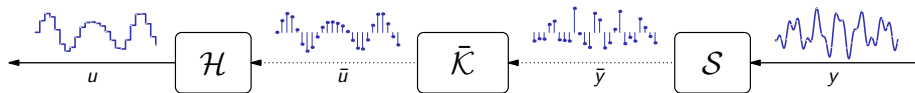


But it turns out that $\mathcal{K}\mathcal{D}_h = \mathcal{D}_h\mathcal{K}$. Indeed,

$$\begin{aligned}
 k(t, s + h) &= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \phi(t - ih) \bar{k}[i - j] \psi(jh - s - h) \\
 &= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \phi(t - ih) \bar{k}[i - j - 1] \psi(jh - s) \\
 &= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \phi(t - (i + 1)h) \bar{k}[i - j] \psi(jh - s) \\
 &= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \phi(t - h - ih) \bar{k}[i - j] \psi(jh - s) = k(t - h, s)
 \end{aligned}$$

Such systems are called *h-periodic*, which is a handy property (as we'll see).

Periodicity



But it turns out that $\mathcal{K}\mathcal{D}_h = \mathcal{D}_h\mathcal{K}$. Indeed,

$$\begin{aligned}
 k(t, s + h) &= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \phi(t - ih) \bar{k}[i - j] \psi(jh - s - h) \\
 &= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \phi(t - ih) \bar{k}[i - j - 1] \psi(jh - s) \\
 &= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \phi(t - (i + 1)h) \bar{k}[i - j] \psi(jh - s) \\
 &= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \phi(t - h - ih) \bar{k}[i - j] \psi(jh - s) = k(t - h, s)
 \end{aligned}$$

Such systems are called ***h-periodic***, which is a handy property (as we'll see).

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Approximating an analog controller

Assume we have an analog LTI controller $\mathcal{K}$ and we would like to design a sampler $\mathcal{S}$ that minimizes the norm of the error system, $\mathcal{G}_e = \mathcal{K} - \mathcal{H}_{\text{ZOH}}\mathcal{S}$. As $\|\mathcal{G}_e\|^2 = \|\mathcal{G}_e^* \mathcal{G}_e\|$, consider

$$\begin{aligned}\mathcal{G}_e^* \mathcal{G}_e &= (\mathcal{K} - \mathcal{H}_{\text{ZOH}}\bar{\mathcal{K}}\mathcal{S})^* (\mathcal{K} - \mathcal{H}_{\text{ZOH}}\bar{\mathcal{K}}\mathcal{S}) \\ &= \mathcal{K}^* \mathcal{K} - \mathcal{K}^* \mathcal{H}_{\text{ZOH}}\bar{\mathcal{K}}\mathcal{S} - \mathcal{S}^* \bar{\mathcal{K}}^* \mathcal{H}_{\text{ZOH}}^* \mathcal{K} + \mathcal{S}^* \bar{\mathcal{K}}^* \mathcal{H}_{\text{ZOH}}^* \mathcal{H}_{\text{ZOH}} \bar{\mathcal{K}}\mathcal{S} \\ &= \mathcal{K}^* \mathcal{K} - \mathcal{K}^* \mathcal{H}_{\text{ZOH}}\bar{\mathcal{K}}\mathcal{S} - \mathcal{S}^* \bar{\mathcal{K}}^* \mathcal{H}_{\text{ZOH}}^* \mathcal{K} + h \mathcal{S}^* \bar{\mathcal{K}}^* \bar{\mathcal{K}}\mathcal{S}\end{aligned}$$

(remember, $\mathcal{H}_{\text{ZOH}}^* \mathcal{H}_{\text{ZOH}} = h$). Denote $\mathcal{S}_0 := \frac{1}{\sqrt{h}} \mathcal{H}_{\text{ZOH}}^* \mathcal{K}$, then

$$\mathcal{G}_e^* \mathcal{G}_e = \underbrace{\mathcal{K}^* \mathcal{K} - \mathcal{S}_1^* \mathcal{S}_1}_{\text{independent of } \mathcal{S}} + \underbrace{(\mathcal{S}_1 - \sqrt{h} \bar{\mathcal{K}}\mathcal{S})^* (\mathcal{S}_1 - \sqrt{h} \bar{\mathcal{K}}\mathcal{S})}_{\geq 0}$$

and the obvious (although not unique) choice is $\mathcal{S}_{\text{opt}} = \frac{1}{\sqrt{h}} \mathcal{S}_1 = \frac{1}{h} \mathcal{H}_{\text{ZOH}}^* \mathcal{K}$. This is a non-causal shift-invariant generalized sampler with

$$\psi_{\text{opt}}(t) = \frac{1}{h} (\mathbb{1}_{[-h,0]} * k)(t) = \frac{1}{h} \int_t^{t+h} k(s) ds.$$