

Advanced Sampled-Data Control (038798)

lecture no. 1

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Course info

- Credit points: 3
- Prerequisite: Control Theory (035188)
or any **reasonable advanced linear control** course discussing state space
- Grading policy: homework **100%** (5 best out of 6 assignments)
homework solutions must be submitted **electronically** to ***c038798@technion.ac.il***
- Course site: **<http://leo.technion.ac.il/Courses/ASDC/>**

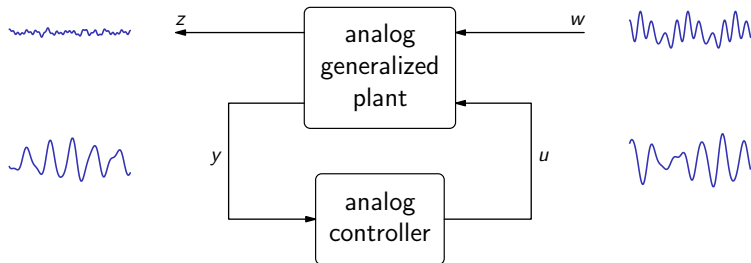
Literature:

1. My slides.
2. K. J. Åström and B. Wittenmark, *Computer-Controlled Systems: Theory and Design*. Englewood Cliffs, NJ: Prentice-Hall, 1997, 3rd ed.
3. T. Chen and B. A. Francis, *Optimal Sampled-Data Control Systems*. London: Springer-Verlag, 1995.
4. E. Garcia, P. J. Antsaklis, and L. A. Montestruque, *Model-Based Control of Networked Systems*. Boston: Birkhäuser, 2014.

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Continuous-time standard problem



w : exogenous input

how the outside world affects us; contains disturbances, references, noises, ...

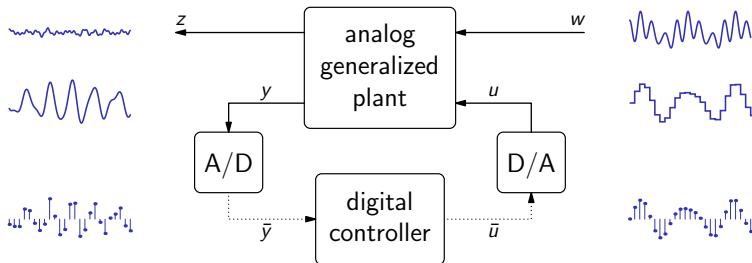
z : regulated output

all we wanna keep small, like tracking errors, control signals, ...

y : measured output (everything that can be used by the controller)

u : control input (our means to affect controlled variables)

Sampled-data standard problem



Just because

- controllers are implemented mostly digitally nowadays, this is cheaper, more reliable, more flexible, ...

Notation

$\mathbb{R}$	the set of real numbers, $\mathbb{R} = (-\infty, \infty)$
$\mathbb{R}_+$	the set of nonnegative real numbers, $\mathbb{R}_+ = [0, \infty)$
$\mathbb{R}_-$	the set of nonpositive real numbers, $\mathbb{R}_- = (-\infty, 0]$
$j\mathbb{R}$	the set of pure imaginary numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}_\alpha$	the half plane to the right of $\alpha \in \mathbb{R}$, i.e. $\mathbb{C}_\alpha: \{z \in \mathbb{C} : \operatorname{Re} z > \alpha\}$
$\bar{\mathbb{C}}_\alpha$	the closure of $\mathbb{C}_\alpha$, i.e. $\bar{\mathbb{C}}_\alpha: \{z \in \mathbb{C} : \operatorname{Re} z \geq \alpha\}$
$\mathbb{T}$	the unit circle, $\mathbb{T} := \{z \in \mathbb{C} : z = 1\}$
$\mathbb{D}_\alpha$	the open α -disk, $\mathbb{D} := \{z \in \mathbb{C} : z < \alpha\}$
$\bar{\mathbb{D}}_\alpha$	the closed α -disk, $\bar{\mathbb{D}} := \{z \in \mathbb{C} : z \leq \alpha\} = \mathbb{D}_\alpha \cup (\alpha\mathbb{T})$
$\mathbb{F}$	alias of either $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of positive integers (natural numbers)
$\mathbb{Z}_+$	the set of nonnegative integers
$\mathbb{Z}_-$	the set of nonpositive integers, $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}$
$\mathbb{Z}_{i_1..i_2}$	the interval $\{i_1, i_1 + 1, \dots, i_2\}$

Notation (contd)

$\mathbb{1}_{\mathbb{I}}(t)$ (continuous-time) characteristic function of a set $\mathbb{I} \subset \mathbb{R}$

$$\mathbb{1}_{\mathbb{I}}(t) = \begin{cases} 1 & \text{if } t \in \mathbb{I} \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{1}(t)$ the unit step, $\mathbb{1}(t) := \mathbb{1}_{\mathbb{R}_+}(t)$

$\delta(t)$ the Dirac delta function

$\bar{\mathbb{1}}_{\mathbb{I}}[i]$ (discrete-time) characteristic function of a set $\mathbb{I} \subset \mathbb{Z}$

$$\bar{\mathbb{1}}_{\mathbb{I}}[i] = \begin{cases} 1 & \text{if } i \in \mathbb{I} \\ 0 & \text{otherwise} \end{cases}$$

$\bar{\mathbb{1}}[i]$ the unit step, $\bar{\mathbb{1}}[i] := \mathbb{1}_{\mathbb{Z}_+}[i]$

$\bar{\delta}[i]$ the unit pulse at $i = 0$, $\bar{\delta}[i] = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$

I_n the $n \times n$ identity matrix (or simply I)

M' (complex-conjugate) transpose of a matrix $M \in \mathbb{F}^{p \times m}$

$\|M\|$ spectral norm of a matrix $M \in \mathbb{F}^{p \times m}$ (Euclidean norm for vectors)

$\|M\|_{\text{F}}$ Frobenius norm of a matrix $M \in \mathbb{F}^{p \times m}$, $\|M\|_{\text{F}} := \text{tr}(M'M)$

Content

Signals

Discrete-time systems

Continuous-time systems

LTI systems in transformed domain

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Signals

Signals represent

- conveyed information about (potentially changing) real-life phenomena

In mathematical language, signals are

- function of independent variables

Independent variables of interest are associated with time:

- continuous, and then they are variables from $\mathbb{R}$
- discrete, and then they are variables from $\mathbb{Z}$

Analog signals

n -dimensional continuous-time signals $f(t)$ are described as

$$f : \mathbb{R} \rightarrow \mathbb{F}^n \quad \text{or} \quad f : \mathbb{I} \rightarrow \mathbb{F}^n \quad \text{for some } \mathbb{I} \subset \mathbb{R}$$

Admissibility $\rightarrow$ signal spaces:

$$L_2^n(\mathbb{I}) := \left\{ f : \mathbb{I} \rightarrow \mathbb{F}^n \mid \|f\|_2^2 := \int_{\mathbb{I}} \|f(t)\|^2 dt < \infty \right\}$$

$$L_{2,\text{loc}}^n(\mathbb{R}) := \left\{ f : \mathbb{R} \rightarrow \mathbb{F}^n \mid f \mathbb{1}_{[t_1, t_2]} \in L_2^n(\mathbb{R}) \text{ for all } -\infty < t_1 < t_2 < \infty \right\}$$

$$L_{2+}^n(\mathbb{R}) := \left\{ f : \mathbb{R} \rightarrow \mathbb{F}^n \mid f \in L_2^n(\mathbb{R}) \text{ and } f(t) = 0 \text{ whenever } t < 0 \right\}$$

$$L_{2-}^n(\mathbb{R}) := \left\{ f : \mathbb{R} \rightarrow \mathbb{F}^n \mid f \in L_2^n(\mathbb{R}) \text{ and } f(t) = 0 \text{ whenever } t > 0 \right\}$$

Discrete signals

n -dimensional discrete-time signals $\bar{f}[i]$ are described as

$$\bar{f} : \mathbb{Z} \rightarrow \mathbb{F}^n \quad \text{or} \quad \bar{f} : \mathbb{I} \rightarrow \mathbb{F}^n \quad \text{for some } \mathbb{I} \subset \mathbb{Z},$$

Admissibility $\rightarrow$ signal spaces:

$$\ell_2^n(\mathbb{I}) := \left\{ \bar{f} : \mathbb{I} \rightarrow \mathbb{F}^n \mid \|\bar{f}\|_2^2 := \sum_{i \in \mathbb{I}} \|\bar{f}[i]\|^2 < \infty \right\}$$

$$\ell_{2,\text{loc}}^n(\mathbb{Z}) := \left\{ \bar{f} : \mathbb{Z} \rightarrow \mathbb{F}^n \mid \bar{f} \mathbb{1}_{\mathbb{Z}_{i_1 \dots i_2}} \in \ell_2^n(\mathbb{Z}) \text{ for all } -\infty < i_1 < i_2 < \infty \right\}$$

$$\ell_{2+}^n(\mathbb{Z}) := \left\{ \bar{f} : \mathbb{Z} \rightarrow \mathbb{F}^n \mid \bar{f} \in \ell_2^n(\mathbb{Z}) \text{ and } \bar{f}[i] = 0 \text{ whenever } i < 0 \right\}$$

$$\ell_{2-}^n(\mathbb{Z}) := \left\{ \bar{f} : \mathbb{Z} \rightarrow \mathbb{F}^n \mid \bar{f} \in \ell_2^n(\mathbb{Z}) \text{ and } \bar{f}[i] = 0 \text{ whenever } i \geq 0 \right\}$$

Continuous-time Fourier transform

The Fourier transform of a continuous-time signal f :

$$\mathfrak{F}\{f\} = F(j\omega) := \int_{\mathbb{R}} f(t)e^{-j\omega t} dt,$$

where $\omega \in \mathbb{R}$ called the (angular) frequency and measured in rad/sec. The signal $\mathfrak{F}\{f\}$ is called the *frequency-domain* representation of f .

$$\mathfrak{F}^{-1}\{F\} = f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} F(j\omega)e^{j\omega t} d\omega,$$

i.e. $f(t)$ is a superposition, weighted by $F(j\omega)$, of the harmonic signals $e^{j\omega t}$ with frequencies ω .

Continuous-time Fourier transform (contd)

The Fourier transform is

- well defined (absolute convergence) on $L_1(\mathbb{R})$
- extendible (weaker convergence) to $L_2(\mathbb{R})$ by Plancherel Theorem

Then the Fourier transform preserves sizes

$$\underbrace{\int_{\mathbb{R}} \|f(t)\|^2 dt}_{\|f\|_2^2} = \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} \|F(j\omega)\|^2 d\omega}_{\|F\|_2^2}$$

and angles

$$\underbrace{\int_{\mathbb{R}} [f_2(t)]' f_1(t) dt}_{\langle f_1, f_2 \rangle_2} = \underbrace{\frac{1}{2\pi} \int_{\mathbb{R}} [F_2(j\omega)]' F_1(j\omega) d\omega}_{\langle F_1, F_2 \rangle_2}$$

as mapping $L_2(\mathbb{R}) \rightarrow L_2(j\mathbb{R})$ (Parseval's theorem, in engineering literature).

Discrete-time Fourier transform

The Fourier transform of a discrete-time signal $\bar{f}$:

$$\mathfrak{F}\{\bar{f}\} = \bar{F}(e^{j\theta}) := \sum_{i \in \mathbb{Z}} \bar{f}[i] e^{-j\theta i},$$

where $\theta \in [-\pi, \pi]$ called the (angular) frequency and measured in rad/step. The periodic signal $\mathfrak{F}\{\bar{f}\}$ is the *frequency-domain* representation of $\bar{f}$, as

$$\mathfrak{F}^{-1}\{\bar{F}\} = \bar{f}[i] := \frac{1}{2\pi} \int_{-\pi}^{\pi} \bar{F}(e^{j\theta}) e^{j\theta i} d\theta,$$

i.e. $\bar{f}[i]$ is a superposition, weighted by $\bar{F}(e^{j\theta})$, of the harmonic signals $e^{j\theta i}$ with frequencies θ .

Discrete-time Fourier transform (contd)

The Fourier transform is

- well defined (absolute convergence) on $\ell_1(\mathbb{Z})$
- extendible (weaker convergence) to $\ell_2(\mathbb{Z})$ by Plancherel Theorem

Then the discrete-time Fourier transform preserves sizes

$$\underbrace{\sum_{i \in \mathbb{Z}} \|\bar{f}[i]\|^2}_{\|\bar{f}\|_2^2} = \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} \|\bar{F}(e^{j\theta})\|^2 d\theta}_{\|\bar{F}\|_2^2}$$

and angles

$$\underbrace{\sum_{i \in \mathbb{Z}} (\bar{f}_2[i])' \bar{f}_1[i]}_{\langle \bar{f}_1, \bar{f}_2 \rangle_2} = \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} [\bar{F}_2(e^{j\theta})]' \bar{F}_1(e^{j\theta}) d\theta}_{\langle \bar{F}_1, \bar{F}_2 \rangle_2}$$

as mapping $\ell_2(\mathbb{Z}) \rightarrow L_2(\mathbb{T})$ (Parseval's theorem, in engineering literature).

Laplace transform

The two-sided (bilateral) Laplace transform of a continuous-time signal f :

$$\mathfrak{L}\{f\} = F(s) := \int_{\mathbb{R}} f(t)e^{-st}dt$$

for all $s \in \mathbb{C}$ for which the integral converges (region of convergence, aka RoC). We need $f(t)e^{-st}$ to belong to $L_1(\mathbb{R})$ for absolute convergence, so

- if $\operatorname{Re} s > 0$, $f(t)$ should decay “sufficiently fast” as $t \rightarrow -\infty$ and grow “not too fast” as $t \rightarrow +\infty$
- if $\operatorname{Re} s < 0$, $f(t)$ should decay “sufficiently fast” as $t \rightarrow +\infty$ and grow “not too fast” as $t \rightarrow -\infty$

These conditions are draconian, so the transform normally used for signals

- supported in $\mathbb{R}_+$, then RoC is typically $\mathbb{C}_\alpha$ (may be $\bar{\mathbb{C}}_\alpha$)
- supported in $\mathbb{R}_-$, then RoC is typically $\mathbb{C} \setminus \bar{\mathbb{C}}_\alpha$ (may be $\mathbb{C} \setminus \mathbb{C}_\alpha$)

for some $\alpha \in \mathbb{R}$.

z-transform

The two-sided (bilateral) z-transform of a discrete-time sequence $\bar{f}$:

$$\mathcal{Z}\{\bar{f}\} = \bar{F}(z) := \sum_{i \in \mathbb{Z}} \bar{f}[i] z^{-i}$$

for all $z \in \mathbb{C}$ for which the sum converges (also RoC). We need $\bar{f}[i]z^{-i}$ to belong to $\ell_1(\mathbb{Z})$ for absolute convergence, so

- if $|z| > 1$, $\bar{f}[i]$ should decay “sufficiently fast” as $i \rightarrow -\infty$ and grow “not too fast” as $i \rightarrow +\infty$
- if $|z| < 1$, $\bar{f}[i]$ should decay “sufficiently fast” as $i \rightarrow +\infty$ and grow “not too fast” as $i \rightarrow -\infty$

These conditions are draconian, so the transform normally used for signals

- supported in $\mathbb{Z}_+$, then RoC is typically $\mathbb{C} \setminus \bar{\mathbb{D}}_\alpha$ (may be $\mathbb{C} \setminus \mathbb{D}_\alpha$)
- supported in $\mathbb{Z}_-$, then RoC is typically $\mathbb{D}_\alpha$ (may be $\bar{\mathbb{D}}_\alpha$)

for some $\alpha > 0$.

Content

Signals

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Systems

From the control theory viewpoint, they are

- collections of laws defining interactions between phenomena (signals) of interest

Systems are normally represented by their *models*, which *approximate* the real world with some degree of accuracy. With some abuse of terminology, we frequently say “system” meaning its model (common practice).

Throughout, the input / output paradigm is adopted. Accordingly, systems are viewed as

- mappings of a set of input signals to a set of output signals.

This viewpoint assumes an action / effect relationship between signals and might not always be adequate¹. Nonetheless, it appears to be sufficiently general in the context of sampled-data systems.

¹The Willems's *behavioral* paradigm to modeling offers a less restrictive framework.

Linear systems

Linearity means that superposition and scaling properties hold, i.e.

$$\mathcal{G}(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \mathcal{G} u_1 + \alpha_2 \mathcal{G} u_2$$

$$\bar{\mathcal{G}}(\alpha_1 \bar{u}_1 + \alpha_2 \bar{u}_2) = \alpha_1 \bar{\mathcal{G}} \bar{u}_1 + \alpha_2 \bar{\mathcal{G}} \bar{u}_2$$

for all admissible inputs u_1 and u_2 (or $\bar{u}_1$ and $\bar{u}_2$) and all scalars α_1 and α_2 . This is a favorable property, which is

- often **assumed** in analysis, but seldom (if ever) holds in reality.

Nevertheless, linear models may approximate nonlinear reality sufficiently well in many control applications, especially because feedback is capable of weakening effects of modeling inaccuracies on controlled behaviors.

Kernel representation

Let $\bar{\mathcal{G}} : \ell_{2,\text{loc}}^m(\mathbb{Z}) \rightarrow \ell_{2,\text{loc}}^p(\mathbb{Z})$ be linear and write (appears awkward)

$$\bar{u}[i] = \sum_{j \in \mathbb{Z}} \bar{\delta}[i - j] \bar{u}[j].$$

By **linearity**, and passing over convergence issues, we have then that

$$\begin{aligned} \bar{y}[i] &= (\bar{\mathcal{G}}\bar{u})[i] = \left(\bar{\mathcal{G}} \sum_{j \in \mathbb{Z}} \bar{\delta}[\cdot - j] \bar{u}[j] \right)[i] = \sum_{j \in \mathbb{Z}} (\bar{\mathcal{G}}\bar{\delta}[\cdot - j])[i] \bar{u}[j] \\ &= \sum_{j \in \mathbb{Z}} \bar{g}[i, j] \bar{u}[j], \end{aligned}$$

where $p \times m$ matrix $\bar{g}[i, j] := (\bar{\mathcal{G}}\bar{\delta}[\cdot - j])[i]$ is the response of $\bar{\mathcal{G}}$, at the time instance i , to the unit pulse applied at the time instance j (the **impulse response** of $\bar{\mathcal{G}}$). This representation is called the kernel representation of $\bar{\mathcal{G}}$.

Kernel representation

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System matrix

Alternative form,

$$\begin{bmatrix} \vdots \\ \bar{y}[i-1] \\ \bar{y}[i] \\ \bar{y}[i+1] \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \cdots & \bar{g}[i-1, i-1] & \bar{g}[i-1, i] & \bar{g}[i-1, i+1] & \cdots \\ \cdots & \bar{g}[i, i-1] & \bar{g}[i, i] & \bar{g}[i, i+1] & \cdots \\ \cdots & \bar{g}[i+1, i-1] & \bar{g}[i+1, i] & \bar{g}[i+1, i+1] & \cdots \\ \ddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\text{system matrix of } \bar{\mathcal{G}}, [\bar{\mathcal{G}}]} \begin{bmatrix} \vdots \\ \bar{u}[i-1] \\ \bar{u}[i] \\ \bar{u}[i+1] \\ \vdots \end{bmatrix}$$

Causality

A system $\bar{\mathcal{G}} : \bar{u} \mapsto \bar{y}$ is said to be **causal** (**strictly causal**) if

- the output $\bar{y}[i]$ at each time instance i depends only on the current and the past (only on the past) inputs, i.e. on $\bar{u}[j]$ for $j \leq i$ ($j < i$).

Causality (strict causality) is equivalent to

$$\bar{g}[i, j] = 0 \text{ whenever } j > i \text{ } (j \geq i).$$

System matrices of causal (strictly causal) systems are thus lower triangular (lower triangular with zero diagonal blocks).

The diagonal elements of the system matrix, called the **feedthrough** terms of $\bar{\mathcal{G}}$, shape the dependence of $\bar{y}[i]$ on the current $\bar{u}[i]$. The system built on feedthrough part of $\bar{\mathcal{G}}$ is called its **static part**. It satisfies

$$\bar{y}[i] = \bar{g}[i, i] \bar{u}[i].$$

Shift operator

Backward shift operator $\bar{\mathcal{U}} : \ell_2(\mathbb{Z}) \rightarrow \ell_2(\mathbb{Z})$

$$(\bar{\mathcal{U}}\bar{u})[i] = \bar{u}[i-1], \quad \text{so that} \quad \llbracket \bar{\mathcal{U}} \rrbracket = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & 0 & 0 & 0 & \dots \\ \dots & / & 0 & 0 & \dots \\ \dots & 0 & / & 0 & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is invertible, with

$$(\bar{\mathcal{U}}^{-1}\bar{u})[i] = \bar{u}[i+1], \quad \text{so that} \quad \llbracket \bar{\mathcal{U}}^{-1} \rrbracket = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \dots & 0 & / & 0 & \dots \\ \dots & 0 & 0 & / & \dots \\ \dots & 0 & 0 & 0 & \dots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Shift invariance

A system $\bar{\mathcal{G}} : \bar{u} \mapsto \bar{y}$ is said to be **shift invariant** if

$$\bar{\mathcal{G}}\bar{u} = \bar{u}\bar{\mathcal{G}} \quad \text{or, equivalently,} \quad \bar{\mathcal{G}}\bar{u}^\nu = \bar{u}^\nu\bar{\mathcal{G}}, \quad \forall \nu \in \mathbb{Z}$$

Shift invariance means that

- a shift of the input $\implies$ the same shift of the output

In terms of the kernel representation:

$$\sum_{j \in \mathbb{Z}} \bar{g}[i, j] \bar{u}[j - 1] = \sum_{j \in \mathbb{Z}} \bar{g}[i, j + 1] \bar{u}[j] = \sum_{j \in \mathbb{Z}} \bar{g}[i - 1, j] \bar{u}[j].$$

Hence, $\bar{\mathcal{G}}$ is shift invariant iff

$$\bar{g}[i, j + 1] = \bar{g}[i - 1, j], \quad \forall i, j \in \mathbb{Z}$$

or iff $\bar{g}[i, j] = \bar{g}[i + \nu, j + \nu]$ for all $\nu \in \mathbb{Z}$.

Shift invariance (contd)

As shift-invariant $\bar{g}[i, j]$ does not change whenever $i - j$ is constant, it can be viewed as a function of one variable, $\bar{g}[i]$, which is the response to the pulse applied at $j = 0$. In this case the kernel representation reads

$$\bar{y}[i] = \sum_{j \in \mathbb{Z}} \bar{g}[i - j] \bar{u}[j] =: (\bar{g} * \bar{u})[i]$$

which is called the **convolution** sum. If $\bar{\mathcal{G}}$ is shift invariant, its system matrix

$$\llbracket \bar{\mathcal{G}} \rrbracket = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & \bar{g}[0] & \bar{g}[-1] & \bar{g}[-2] & \cdots \\ \cdots & \bar{g}[1] & \bar{g}[0] & \bar{g}[-1] & \cdots \\ \cdots & \bar{g}[2] & \bar{g}[1] & \bar{g}[0] & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

has **Toeplitz** structure and it is causal (strictly causal) iff $\bar{g}[i] = 0$ whenever $i < 0$ ($i \leq 0$). Matrices $\bar{g}[i]$ are also known as the **Markov parameters** of $\bar{\mathcal{G}}$.

Hankel operator

Let $\bar{\mathcal{G}}$ be shift invariant and causal. Then

$$\begin{bmatrix} \bar{y}[i] \\ \bar{y}[i+1] \\ \bar{y}[i+2] \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{g}[1] & \bar{g}[2] & \bar{g}[3] & \cdots \\ \bar{g}[2] & \bar{g}[3] & \bar{g}[4] & \cdots \\ \bar{g}[3] & \bar{g}[4] & \bar{g}[5] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\text{Hankel matrix, } \mathbf{H}_{\bar{\mathcal{G}}}} \begin{bmatrix} \bar{u}[i-1] \\ \bar{u}[i-2] \\ \bar{u}[i-3] \\ \vdots \end{bmatrix} \\
 + \underbrace{\begin{bmatrix} \bar{g}[0] & 0 & 0 & \cdots \\ \bar{g}[1] & \bar{g}[0] & 0 & \cdots \\ \bar{g}[2] & \bar{g}[1] & \bar{g}[0] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\text{Toeplitz matrix, } \mathbf{T}_{\bar{\mathcal{G}}}} \begin{bmatrix} \bar{u}[i] \\ \bar{u}[i+1] \\ \bar{u}[i+2] \\ \vdots \end{bmatrix}$$

Here $\mathbf{H}_{\bar{\mathcal{G}}}$ is a matrix representation of the **Hankel operator** associated with $\bar{\mathcal{G}}$, which connects the past with the present + future at each step i .

Example 1

Let $\mathcal{G}$ be described as

$$\bar{y}[i] = \sum_{j=-\infty}^i \bar{u}[j],$$

i.e. $\bar{g}[i] = \bar{1}[i]$. Then

$$\mathbf{H}_{\bar{\mathcal{G}}} = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots \end{bmatrix}$$

is rank-1 matrix...

Example 2

Let $\mathcal{G}$ be described as

$$\bar{y}[i] = \sum_{j=-\infty}^i \alpha^{i-j} \bar{u}[j]$$

for some $\alpha \in \mathbb{R}$, i.e. $\bar{g}[i] = \alpha^i \bar{1}[i]$. Then

$$\mathbf{H}_{\bar{g}} = \begin{bmatrix} \alpha^1 & \alpha^2 & \alpha^3 & \dots \\ \alpha^2 & \alpha^3 & \alpha^4 & \dots \\ \alpha^3 & \alpha^4 & \alpha^5 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} \alpha^1 \\ \alpha^2 \\ \alpha^3 \\ \vdots \end{bmatrix} \begin{bmatrix} 1 & \alpha^1 & \alpha^2 & \dots \end{bmatrix}$$

is still rank-1 matrix...

State vector

If the rank of (infinite-dimensional) $\mathbf{H}_{\bar{\mathcal{G}}}$ is finite, say n , $\bar{\mathcal{G}}$ is said to be **finite dimensional** and n is its dimension (order). We then can factorize

$$\mathbf{H}_{\bar{\mathcal{G}}} = \begin{bmatrix} O_1 \\ O_2 \\ O_3 \\ \vdots \end{bmatrix} \begin{bmatrix} R_1 & R_2 & R_3 & \cdots \end{bmatrix} =: \mathbf{O}_{\bar{\mathcal{G}}} \mathbf{R}_{\bar{\mathcal{G}}}$$

($R_i \in \mathbb{R}^{n \times m}$ and $O_i \in \mathbb{R}^{p \times n}$ for each $i \in \mathbb{N}$), where $\mathbf{O}_{\bar{\mathcal{G}}}$ and $\mathbf{R}_{\bar{\mathcal{G}}}$ are rank- n matrices having n columns and n rows, respectively. The n -dimensional

$$\bar{\mathbf{x}}[i] := \mathbf{R}_{\bar{\mathcal{G}}} \begin{bmatrix} \bar{u}[i-1] \\ \bar{u}[i-2] \\ \bar{u}[i-3] \\ \vdots \end{bmatrix} = \sum_{j=-\infty}^{i-1} R_{i-j} \bar{u}[j]$$

is then called the **state vector** of $\bar{\mathcal{G}}$ at i .

State vector as history accumulator

The state vector is actually the **history accumulator**, because

$$\begin{bmatrix} \bar{y}[i] \\ \bar{y}[i+1] \\ \bar{y}[i+2] \\ \vdots \end{bmatrix} = \begin{bmatrix} O_1 \\ O_2 \\ O_3 \\ \vdots \end{bmatrix} \bar{x}[i] + \begin{bmatrix} \bar{g}[0] & 0 & 0 & \cdots \\ \bar{g}[1] & \bar{g}[0] & 0 & \cdots \\ \bar{g}[2] & \bar{g}[1] & \bar{g}[0] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \bar{u}[i] \\ \bar{u}[i+1] \\ \bar{u}[i+2] \\ \vdots \end{bmatrix},$$

i.e. no knowledge of $\bar{u}[j]$ for $j < i$ is required. This suggests that

- causal systems can be regarded as operators $\ell_{2,\text{loc}}(\mathbb{Z}_+) \rightarrow \ell_{2,\text{loc}}(\mathbb{Z}_+)$, rather than $\ell_{2,\text{loc}}(\mathbb{Z})$, with the
- initial condition $\bar{x}[0]$ explaining the effect of prehistoric inputs.

Having a starting point may be intuitive in control applications, this is the time moment where the control law is applied.

Realizations

Rank- n and the Hankel structure of $\mathbf{H}_{\bar{g}}$ guarantee the existence of matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$ such that

$$\mathbf{O}_{\bar{g}} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \quad \text{and} \quad \mathbf{R}_{\bar{g}} = \begin{bmatrix} B & AB & A^2B & \dots \end{bmatrix}$$

(infinite observability and reachability matrices), so that $\bar{g}[i] = CA^{i-1}B$ for all $i \in \mathbb{N}$. First, this means that

$$\bar{x}[i+1] = \begin{bmatrix} B & AB & A^2B & \dots \end{bmatrix} \begin{bmatrix} \bar{u}[i] \\ \bar{u}[i-1] \\ \bar{u}[i-2] \\ \vdots \end{bmatrix} = A\bar{x}[i] + B\bar{u}[i],$$

so the state vector can be propagated **recursively**.

Realizations (contd)

Second, the output at each i is a linear function of current state and input:

$$\begin{aligned}\bar{y}[i] &= \sum_{j=-\infty}^i \bar{g}[i-j] \bar{u}[j] = C \sum_{j=-\infty}^{i-1} A^{i-j-1} B \bar{u}[j] + \bar{g}[0] \bar{u}[i] \\ &= C \bar{x}[i] + \bar{g}[0] \bar{u}[i].\end{aligned}$$

Thus, we end up with the following description (with $D := \bar{g}[0] \in \mathbb{R}^{p \times m}$):

$$\bar{\mathcal{G}} : \begin{cases} \bar{x}[i+1] = A\bar{x}[i] + B\bar{u}[i], & \bar{x}[0] = \bar{x}_0 \\ \bar{y}[i] = C\bar{x}[i] + D\bar{u}[i], \end{cases}$$

called the **state representation** of $\bar{\mathcal{G}}$ and the quadruple (A, B, C, D) is called the **state-space realization** of $\bar{\mathcal{G}}$. It's **minimal** iff the pair (A, B) is reachable and the pair (C, A) is observable.

Stability

We say that a system is

- **stable** if it is bounded as an operator $\ell_2(\mathbb{Z}) \rightarrow \ell_2(\mathbb{Z})$.

A system is

- **stable and causal** iff it is a bounded operator $\ell_{2+}(\mathbb{Z}) \rightarrow \ell_{2+}(\mathbb{Z})$.

In the finite-dimensional shift-invariant case the stability of a system can be verified via its *minimal* state-space realization: a system is

- stable iff its “ A ” matrix is *Schur*,

i.e. has all its eigenvalues in the open unit disk $\mathbb{D}_1$.

Content

Signals

Discrete-time systems

Continuous-time systems

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Kernel representation

A sufficiently general family of linear systems $\mathcal{G} : L_{2,\text{loc}}^m(\mathbb{R}) \rightarrow L_{2,\text{loc}}^p(\mathbb{R})$ can be described as

$$y(t) = \int_{\mathbb{R}} g(t, s) u(s) ds$$

for $p \times m$ impulse response (or kernel)

$$g(t, s) = \tilde{g}(t, s) + \sum_{i \in \mathbb{Z}} g_i(t) \delta(t - s - \kappa_i(t))$$

for a locally bounded (i.e. bounded on any bounded subset of $\mathbb{R}^2$) function $\tilde{g}(t, s)$, locally bounded $g_i(t)$, and an increasing, at each $t \in \mathbb{R}$, sequence $\{\kappa_i(t)\}_{i \in \mathbb{Z}}$ with $\kappa_0(t) \equiv 0$ and such that there is $\epsilon > 0$, independent of i and t , such that $\kappa_i(t) - \kappa_{i-1}(t) \geq \epsilon$ for all $i \in \mathbb{Z}$. Equivalently,

$$y(t) = \int_{\mathbb{R}} \tilde{g}(t, s) u(s) ds + \sum_{i \in \mathbb{Z}} g_i(t) u(t - \kappa_i(t)),$$

This representation is called the kernel representation of $\mathcal{G}$.

Kernel representation

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This representation is called the **kernel representation** of $\mathcal{G}$.

Causality

A system $\mathcal{G} : u \mapsto y$ is said to be **causal** if

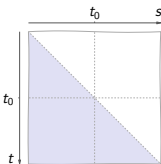
- at every $t_0 \in \mathbb{R}$, the output $y(t) = 0$ for all $t \leq t_0$ whenever $u(t) = 0$ for all $t \leq t_0$

In other words, a system is causal if

$$\int_{t_0}^{\infty} g(t, s) u(s) ds = 0, \quad \forall t < t_0$$

and all admissible u . This condition, in turn, is equivalent to

$$g(t, s) = 0 \text{ whenever } s > t \quad \text{i.e. } g(t, s) :$$



(requires that $g_i(t) \equiv 0$ for all $i < 0$; $g_0(t)$ is called the feedthrough term).

Time invariance

A system $\mathcal{G} : u \mapsto y$ is said to be **time invariant** if

$$\mathcal{G} \mathcal{D}_{\kappa} = \mathcal{D}_{\kappa} \mathcal{G}, \quad \forall \kappa \in \mathbb{R}$$

where $\mathcal{D}_{\kappa}$ is the κ -delay operator, $(\mathcal{D}_{\kappa}u)(t) = u(t - \kappa)$. Time invariance means that

- a shift of the input $\implies$ the same shift of the output

In terms of the kernel representation:

$$\int_{\mathbb{R}} g(t, s) u(s - \kappa) ds = \int_{\mathbb{R}} g(t, s + \kappa) u(s) ds = \int_{\mathbb{R}} g(t - \kappa, s) u(s) ds,$$

Hence, $\mathcal{G}$ is shift invariant iff

$$g(t, s + \kappa) = g(t - \kappa, s), \quad \forall t, s, \kappa \in \mathbb{R}.$$

or iff $g(t, s) = g(t + \kappa, s + \kappa)$ for all $\kappa \in \mathbb{R}$.

Time invariance (contd)

Since time-invariant $g(t, s)$ does not change whenever $t - s$ is constant, it can be viewed as a function of one variable, $g(t)$, which is the response to the pulse applied at $s = 0$. It is then of the form

$$g(t) = \tilde{g}(t) + \sum_{i \in \mathbb{Z}} g_i \delta(t - \tau_i)$$

The kernel representation of time-invariant systems reads

$$\begin{aligned} y(t) &= \int_{\mathbb{R}} g(t-s)u(s)ds =: (g * u)(t) \\ &= \int_{\mathbb{R}} \tilde{g}(t-s)u(s)ds + \sum_{i \in \mathbb{Z}} g_i u(t - \tau_i), \end{aligned}$$

which is called the **convolution** integral.

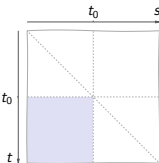
Hankel operator

Let $\mathcal{G}$ be time invariant and causal. Then at each $t \geq t_0$

$$y(t) = \int_{-\infty}^{t_0} g(t-s)u(s)ds + \int_{t_0}^t g(t-s)u(s)ds$$

The **Hankel operator** $\mathbf{\Gamma}_{\mathcal{G}} : L_{2,\text{loc}}(-\infty, t_0) \rightarrow L_{2,\text{loc}}(t_0, \infty)$, connects its past (with respect to any given t_0) inputs and future outputs as follows:

$$y(t) = \int_{-\infty}^{t_0} g(t-s)u(s)ds, \quad \forall t > t_0, \quad \text{i.e. } \gamma_g(t, s) : t_0$$



(γ_g stands for the kernel of $\mathbf{\Gamma}_{\mathcal{G}}$).

State vector

If the rank of $\mathbf{\Gamma}_{\mathcal{G}}$ is finite, say n , $\mathcal{G}$ is said to be **finite dimensional** and n is its dimension (order). Finite dimensionality necessitates that $g_i = 0, \forall i < 0$ (causality—that $g_i = 0, \forall i > 0$) and then guarantees the factorization

$$g(t-s) = \tilde{g}_o(t) \tilde{g}_R(s) + g_0 \delta(t-s), \quad t \geq s$$

for some locally bounded $\tilde{g}_o(t) \in \mathbb{R}^{p \times n}$ and $\tilde{g}_R(t) \in \mathbb{R}^{n \times m}$ such that

$$\int_{t_1}^{t_2} [\tilde{g}_o(t)]' \tilde{g}_o(t) dt > 0 \quad \text{and} \quad \int_{t_1}^{t_2} \tilde{g}_R(t) [\tilde{g}_R(t)]' dt > 0$$

whenever $t_2 > t_1 \geq 0$. The n -dimensional

$$\mathbf{x}(t) := \int_{-\infty}^t \tilde{g}_R(t-s) u(s) ds$$

is called the **state vector** of $\mathcal{G}$ at the time instance t .

State vector as history accumulator

The state vector is again the **history accumulator**, because at each $t \geq t_0$

$$\begin{aligned} y(t) &= \tilde{g}_0(t - t_0)x(t_0) + \int_{t_0}^t g(t - s)u(s)ds \\ &= \tilde{g}_0(t - t_0)x(t_0) + \tilde{g}_0(t - t_0) \int_{t_0}^t \tilde{g}_R(t_0 - s)u(s)ds + g_0u(t) \end{aligned}$$

i.e. no knowledge of $u(s)$ in $s < t_0$ is required. This suggests that

- causal systems can be regarded as operators $L_{2,\text{loc}}(\mathbb{R}_+) \rightarrow L_{2,\text{loc}}(\mathbb{R}_+)$, rather than $L_{2,\text{loc}}(\mathbb{R})$, with the
- initial condition $x(0)$ explaining the effect of prehistoric inputs.

Having a starting point may be intuitive in control applications, this is the time moment where the control law is applied.

Realizations

The rank- n property of the Hankel operator guarantees the existence of matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$ such that

$$\tilde{g}_0(t) = Ce^{At} \quad \text{and} \quad \tilde{g}_R(s) = e^{As}B.$$

so that $g(t) = Ce^{At}B + g_0\delta(t)$. First, this means (via Leibniz integral rule) that

$$\begin{aligned}\dot{x}(t) &= \frac{d}{dt} \int_{-\infty}^t e^{A(t-s)} Bu(s) ds \\ &= A \int_{-\infty}^t e^{A(t-s)} Bu(s) ds + e^{A(t-s)} Bu(s) \Big|_{s=t} \\ &= Ax(t) + Bu(t),\end{aligned}$$

so the state vector can be propagated by a **first-order** differential equation.

Realizations (contd)

Second, the output at each t is a linear function of current state and input:

$$\begin{aligned}y(t) &= \tilde{g}_0(t - t_0)x(t_0) + \tilde{g}_0(t - t_0) \int_{t_0}^t \tilde{g}_R(t_0 - s)u(s)ds + g_0u(t) \\&= \tilde{g}_0(0)x(t) + g_0u(t) \quad (\text{taking } t_0 = t) \\&= Cx(t) + g_0u(t),\end{aligned}$$

Thus, denoting $D := g_0 \in \mathbb{R}^{p \times m}$, we end up with the following description:

$$\mathcal{G} : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t) + Du(t). \end{cases}$$

called the **state representation** of $\mathcal{G}$ and the quadruple (A, B, C, D) is called the **state-space realization** of $\mathcal{G}$. It's **minimal** iff the pair (A, B) is reachable and the pair (C, A) is observable, again.

Stability

We say that a system is

- **stable** if it is bounded as an operator $L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})$.

A system is

- **stable and causal** iff it is a bounded operator $L_{2+}(\mathbb{R}) \rightarrow L_{2+}(\mathbb{R})$.

In the finite-dimensional shift-invariant case the stability of a system can be verified via its *minimal* state-space realization: a system is

- stable iff its “ A ” matrix is *Hurwitz*,

i.e. has all its eigenvalues in the open left half plane $\mathbb{C} \setminus \bar{\mathbb{C}}_0$.

Content

Signals

Discrete-time systems

Continuous-time systems

LTI systems in transformed domain

Transforming convolutions

Important properties:

- $\mathfrak{F}\{f_1 * f_2\} = \mathfrak{F}\{f_1\} \mathfrak{F}\{f_2\}$ continuous-time Fourier transform
- $\mathfrak{F}\{\bar{f}_1 * \bar{f}_2\} = \mathfrak{F}\{\bar{f}_1\} \mathfrak{F}\{\bar{f}_2\}$ discrete-time Fourier transform
- $\mathfrak{L}\{f_1 * f_2\} = \mathfrak{L}\{f_1\} \mathfrak{L}\{f_2\}$ (continuous-time) Laplace transform
- $\mathfrak{Z}\{\bar{f}_1 * \bar{f}_2\} = \mathfrak{Z}\{\bar{f}_1\} \mathfrak{Z}\{\bar{f}_2\}$ (discrete-time) z-transform

i.e. they all turn convolution into products.

LTI systems in frequency domain

Let $\mathcal{G} : u \mapsto y$ be LTI. Then the spectrum of the output

$$Y(j\omega) = \mathfrak{F}\{g * u\} = \mathfrak{F}\{g\} \mathfrak{F}\{u\} = G(j\omega)U(j\omega)$$

(provided $G(j\omega)$ and $U(j\omega)$ exist). The $p \times m$ function $G(j\omega)$, which is the continuous-time Fourier transform of the impulse response of $\mathcal{G}$, is called the **frequency response** of $\mathcal{G}$.

Likewise, let $\bar{\mathcal{G}} : \bar{u} \mapsto \bar{y}$ be LTI. Then the spectrum of the output

$$\bar{Y}(e^{j\theta}) = \mathfrak{F}\{\bar{g} * \bar{u}\} = \mathfrak{F}\{\bar{g}\} \mathfrak{F}\{\bar{u}\} = \bar{G}(e^{j\theta})\bar{U}(e^{j\theta})$$

(provided $\bar{G}(e^{j\theta})$ and $\bar{U}(e^{j\theta})$ exist). The $p \times m$ function $\bar{G}(e^{j\theta})$, which is the discrete-time Fourier transform of the impulse response of $\bar{\mathcal{G}}$, is called the **frequency response** of $\bar{\mathcal{G}}$.

Frequency response interpretation

Let $u(t) = u_\omega e^{j\omega t}$, then (the *Frequency Response Theorem*)

$$\begin{aligned} y(t) &= \int_{\mathbb{R}} g(t-s) u_\omega e^{j\omega s} ds = \int_{\mathbb{R}} g(s) u_\omega e^{j\omega(t-s)} ds \\ &= \int_{\mathbb{R}} g(s) e^{-j\omega s} ds u_\omega e^{j\omega t} = [G(j\omega) u_\omega] e^{j\omega t}, \end{aligned}$$

so that the response to a harmonic input is also a harmonic signal with the same frequency, whose magnitude / direction are shaped by $G(j\omega)$. Usage:

- performance analysis of LTI systems
- system identification from harmonic inputs

Discrete version: $\bar{u}[i] = u_\theta e^{j\theta i} \implies \bar{y}[i] = [\bar{G}(e^{j\theta}) u_\theta] e^{j\theta i}$.

Frequency response and stability

$\mathcal{G}$ is stable iff its frequency response $G \in L_{\infty}^{p \times m}(\mathbb{j}\mathbb{R})$, where

$$L_{\infty}^{p \times m}(\mathbb{j}\mathbb{R}) := \left\{ G : \mathbb{j}\mathbb{R} \rightarrow \mathbb{C}^{p \times m} \mid \|G\|_{\infty} := \operatorname{ess\,sup}_{\omega \in \mathbb{R}} \|G(\mathbf{j}\omega)\| < \infty \right\}$$

Moreover, $\|G\|_{\infty}$ equals $L_2(\mathbb{R})$ -induced norm of $\mathcal{G}$, i.e.

$$\|G\|_{\infty} = \sup_{\|u\|_2=1} \|\mathcal{G}u\|_2$$

$\bar{\mathcal{G}}$ is stable iff its frequency response $\bar{G} \in L_{\infty}^{p \times m}(\mathbb{T})$, where

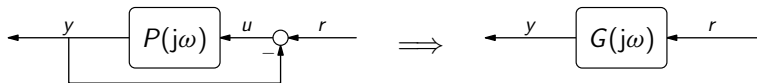
$$L_{\infty}^{p \times m}(\mathbb{T}) := \left\{ \bar{G} : \mathbb{T} \rightarrow \mathbb{C}^{p \times m} \mid \|\bar{G}\|_{\infty} := \operatorname{ess\,sup}_{\theta \in [-\pi, \pi]} \|\bar{G}(\mathbf{e}^{\mathbf{j}\theta})\| < \infty \right\}$$

Moreover, $\|\bar{G}\|_{\infty}$ equals $\ell_2(\mathbb{Z})$ -induced norm of $\bar{\mathcal{G}}$, i.e.

$$\|\bar{G}\|_{\infty} = \sup_{\|\bar{u}\|_2=1} \|\bar{\mathcal{G}}\bar{u}\|_2$$

Frequency domain and feedback: example

Consider



for

$$p(t) = -2e^{-t}\mathbb{1}(t) = \text{graph} \Rightarrow P(j\omega) = -\frac{2}{1+j\omega}$$

The graph shows a signal $p(t)$ that is zero for $t < 0$ and decays exponentially from -2 at $t = 0$ towards zero as t increases.

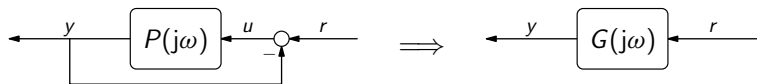
Signals in closed loop are $Y(j\omega) = P(j\omega)(R(j\omega) - Y(j\omega))$, hence

$$G(j\omega) = \frac{P(j\omega)}{1 + P(j\omega)} = \frac{2}{1 - j\omega} \Rightarrow g(t) = 2e^t\mathbb{1}(-t) =$$

This G is a stable anti-causal system, which makes no sense. The catch: Fourier transform cannot cope with exponentially growing signals, it's only good to analyze surely stable systems (e.g., in open-loop settings).

Frequency domain and feedback: example

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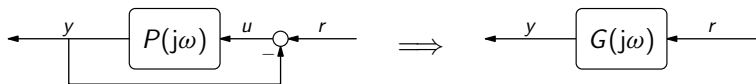
$$G(j\omega) = \frac{P(j\omega)}{1 + P(j\omega)} = \frac{2}{1 - j\omega} \implies g(t) = 2e^t\mathbb{1}(-t) = \text{graph of } 2e^t\mathbb{1}(-t)$$

This $\mathcal{G}$ is a **stable anti-causal** system, which **makes no sense**². The catch is that the Fourier transform cannot cope with exponentially growing signals, it's only good to analyze **surely stable** systems (e.g., in open-loop settings).

²No way $y(t)$ may depend on $r(t + \varkappa)$ here (especially as $\mathcal{P}$ has zero feedthrough).

Frequency domain and feedback: example

Consider



for

$$p(t) = -2e^{-t}\mathbb{1}(t) = \text{graph of } -2e^{-t}\mathbb{1}(t) \implies P(j\omega) = -\frac{2}{1+j\omega}$$

Signals in closed loop are $Y(j\omega) = P(j\omega)(R(j\omega) - Y(j\omega))$, hence

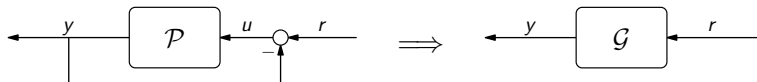
$$G(j\omega) = \frac{P(j\omega)}{1 + P(j\omega)} = \frac{2}{1 - j\omega} \implies g(t) = 2e^t\mathbb{1}(-t) = \text{graph of } 2e^t\mathbb{1}(-t)$$

This $\mathcal{G}$ is a **stable anti-causal** system, which **makes no sense**². The catch:

- Fourier transform cannot cope with exponentially growing signals, it's only good to analyze surely stable systems (e.g., in open-loop settings).

²No way $y(t)$ may depend on $r(t + \varkappa)$ here (especially as $\mathcal{P}$ has zero feedthrough).

Frequency domain and feedback: example (contd)



Time-domain analysis (state space) does show true closed-loop dynamics:

$$\mathcal{P} : \begin{cases} \dot{x}(t) = -x(t) - 2u(t), & x(0) = x_0 \\ y(t) = x(t) \end{cases}$$

and $u(t) = r(t) - y(t) = r(t) - x(t)$ result in

$$\mathcal{G} : \begin{cases} \dot{x}(t) = x(t) - 2r(t), & x(0) = x_0 \\ y(t) = x(t) \end{cases}$$

with the impulse response $g(t) = -2e^t \mathbb{1}(t)$ (is not Fourier transformable). This $\mathcal{G}$ is an **unstable causal** system, which does make sense.

LTI systems in s - and z -domains

Let $\mathcal{G} : u \mapsto y$ be LTI. Then

$$Y(s) = \mathfrak{L}\{g * u\} = \mathfrak{L}\{g\} \mathfrak{L}\{u\} = G(s)U(s)$$

(in the intersection of RoC of $G(s)$ and $U(s)$). The $p \times m$ function $G(s)$, which is the Laplace transform of the impulse response of $\mathcal{G}$, is called the **transfer function** of $\mathcal{G}$.

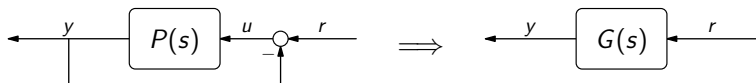
Likewise, let $\bar{\mathcal{G}} : \bar{u} \mapsto \bar{y}$ be LTI. Then

$$\bar{Y}(z) = \mathfrak{Z}\{\bar{g} * \bar{u}\} = \mathfrak{Z}\{\bar{g}\} \mathfrak{Z}\{\bar{u}\} = \bar{G}(z)\bar{U}(z)$$

(in the intersection of RoC of $\bar{G}(z)$ and $\bar{U}(z)$). The $p \times m$ function $\bar{G}(z)$, which is the z -transform of the impulse response of $\bar{\mathcal{G}}$, is called the **transfer function** of $\bar{\mathcal{G}}$.

s-domain and feedback: example (contd)

Return to



for

$$p(t) = -2e^{-t}\mathbb{1}(t) \quad \Rightarrow \quad P(s) = -\frac{2}{s+1}$$

whose RoC = $\mathbb{C}_{-1}$ (includes $j\mathbb{R}$). Then, via $Y(s) = P(s)(R(s) - Y(s))$,

$$G(s) = \frac{P(s)}{1 + P(s)} = \frac{2}{1-s} \quad \Rightarrow \quad g(t) = \begin{cases} 2e^t\mathbb{1}(-t) & \text{if RoC} = \mathbb{C} \setminus \bar{\mathbb{C}}_1 \\ -2e^t\mathbb{1}(t) & \text{if RoC} = \mathbb{C}_1 \end{cases}$$

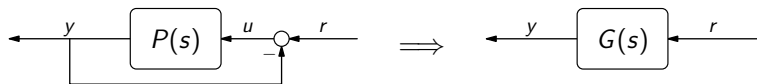
It is not unreasonable to assume that

— causality is preserved under this feedback $\Rightarrow$ RoC must remain a RHP

The correct impulse response can then be obtained immediately.

s-domain and feedback: example (contd)

Return to



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$$p(t) = -2e^{-t}\mathbb{1}(t) \quad \Rightarrow \quad P(s) = -\frac{2}{s+1}$$

whose RoC = $\mathbb{C}_{-1}$ (includes $j\mathbb{R}$). Then, via $Y(s) = P(s)(R(s) - Y(s))$,

$$G(s) = \frac{P(s)}{1 + P(s)} = \frac{2}{1-s} \quad \Rightarrow \quad g(t) = \begin{cases} 2e^t\mathbb{1}(-t) & \text{if RoC} = \mathbb{C} \setminus \bar{\mathbb{C}}_1 \\ -2e^t\mathbb{1}(t) & \text{if RoC} = \mathbb{C}_1 \end{cases}$$

It is not unreasonable to assume that

- causality is preserved under this feedback $\Rightarrow$ RoC must remain a RHP

The **correct impulse response** can then be obtained immediately.

Causality in s - and z -domains

Impulse responses of causal systems are in $\mathbb{R}_+$ ($\mathbb{Z}_+$), so

- RoC of transfer functions of causal systems are $\mathbb{C}_\alpha$ ($\mathbb{C} \setminus \bar{\mathbb{D}}_\alpha$),

which is what is silently assumed in control, based on the fact that well-posed interconnections of causal systems remain causal.

Also, an LTI system $\mathcal{G} : L_{2,\text{loc}}(\mathbb{R}) \rightarrow L_{2,\text{loc}}(\mathbb{R})$ and such that its output has a Laplace transform for every input in $L_{2,\text{loc}}(\mathbb{R})$ is

- causal iff its transfer functions $G(s)$ is proper,

i.e. bounded for all $s \in \mathbb{C}_\alpha$ for some $\alpha > 0$. Properness in the z -domain is defined as the boundedness of $G(z)$ for all $z \in \mathbb{C} \setminus \bar{\mathbb{D}}_\alpha$ for some $\alpha > 1$ and it also implies causality. Thus,

- systems with transfer functions $\frac{1}{s+1}$, $\frac{s}{s-1}$, e^{-s} , $\frac{z}{z-1}$, z^{-10} are causal
- systems with transfer functions e^s , $\frac{z^2+1}{z-1}$, z^{10} are non-causal

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Causality + stability in s -domain

An LTI $\mathcal{G}$ is causal and stable iff its transfer function $G \in H_\infty^{p \times m}$, where

$$H_\infty^{p \times m} := \left\{ G : \mathbb{C}_0 \rightarrow \mathbb{C}^{p \times m} \mid G(s) \text{ is analytic and bounded in } \mathbb{C}_0 \right\}$$

Every $G \in H_\infty$ has a unique boundary function $\tilde{G} \in L_\infty(j\mathbb{R})$ such that

$$\|\tilde{G}\|_\infty = \sup_{s \in \mathbb{C}_0} \|G(s)\| =: \|G\|_{H_\infty}.$$

It is customary to identify G with $\tilde{G}$ and regard

- H_∞ as a closed subspace of $L_\infty(j\mathbb{R})$.

In this case the H_∞ -norm of a function $G \in H_\infty$ is

$$\|G\|_\infty = \operatorname{ess\,sup}_{\omega \in \mathbb{R}} \|G(j\omega)\|,$$

which is easier to compute / interpret (in particular, it's the $L_2(\mathbb{R})$ -induced norm of $\mathcal{G}$).

Causality + stability in z -domain

An LTI $\bar{\mathcal{G}}$ is causal and stable iff its transfer function $\bar{G} \in H_\infty^{p \times m}$, where

$$H_\infty^{p \times m} := \left\{ \bar{G} : \mathbb{C} \setminus \bar{\mathbb{D}}_1 \rightarrow \mathbb{C}^{p \times m} \mid \bar{G}(z) \text{ is analytic and bounded in } \mathbb{C} \setminus \bar{\mathbb{D}}_1 \right\}$$

Every $\bar{G} \in H_\infty$ has a unique boundary function $\tilde{G} \in L_\infty(\mathbb{T})$ such that

$$\|\tilde{G}\|_\infty = \sup_{z \in \mathbb{C} \setminus \bar{\mathbb{D}}_1} \|\bar{G}(z)\| =: \|\bar{G}\|_{H_\infty}.$$

It is customary to identify $\bar{G}$ with $\tilde{G}$ and regard

- H_∞ as a closed subspace of $L_\infty(\mathbb{T})$.

In this case the H_∞ -norm of a function $\bar{G} \in H_\infty$ is

$$\|\bar{G}\|_\infty = \text{ess sup}_{\theta \in [-\pi, \pi]} \|G(e^{j\theta})\|,$$

which is easier to compute / interpret (in particular, it's the $\ell_2(\mathbb{Z})$ -induced norm of $\bar{\mathcal{G}}$).