Introduction to Control (034040)

tutorial 11

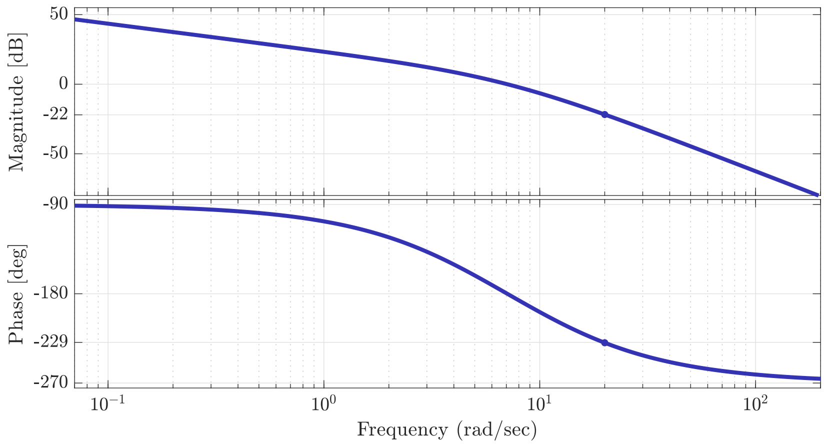


Fig. 1: Bode plot of $P(s)$ in Question 1

Question 1. Fig. 1 presents the Bode plot of

$$P(s) = \frac{750}{s(s+5)(s+10)}.$$

Design a stabilizing controller $C(s)$ in the unity-feedback configuration so that

1. closed-loop bandwidth $\omega_b \geq 20$ [rad/sec],
2. phase margin $\mu_{ph} \geq 45°$,
3. steady-state error for a step reference signal does not exceed 1% of the step height,
4. steady-state error for a step load disturbance does not exceed 1% of the step height.

Repeat it with the phase margin bound $\mu_{ph} \geq 35°$, explain differences.

Solution. First of all, we need to translate the closed-loop bandwidth requirement to a requirement to the crossover frequency. This is because the latter is readily incorporable into the loop-shaping design, whereas the former is not. Such a translation may start with the heuristic relation $\omega_b \approx 1.2 \div 1.5\omega_c$ and then go through trial and error procedure: pick ω_c, design a $C(s)$ satisfying all specs, check the resulted ω_b. Here ω_c should be increased to increase ω_b and vise versa. With such a procedure, it was found that the required $\omega_c \approx 11.04$ [rad/sec]. In what follows, consider the design with

$$\omega_c = 11 \text{ [rad/sec]},$$

which the closest integer.

We also need to translate the last two specifications into requirements on the static gain of the controller. Because the plant has an integrator, the third requirement is satisfied by any stabilizing controller. The fourth specification effectively requires that

$$|T_d(0)| = \left|\frac{1}{1/P(0) + C(0)}\right| = \frac{1}{|C(0)|} \leq 0.01 \iff |C(0)| \geq 100.$$

The design steps are then as follows:

- To set the required crossover, use the gain

$$k = 1/|P(\mathrm{j}11)| \approx 2.63 \approx 8.41\,[\mathrm{dB}].$$

  This result in the loop presented in Figs. 2(a) (Bode) and 2(b) (polar).

- Because k above is smaller than 100, we shall need to increase the static gain of the controller by a lag element. At the same time, it can be seen, from Fig. 2(a), that the loop $P(s)k$ has a phase of $\approx -203°$ at the crossover frequency. Hence, a phase lead is necessary to stabilize the system (this is clear from the polar plot in Fig. 2(b)). It is normally convenient to start with shaping the phase, to be able to calculate the required low-frequency gain in the last stage (the lead alters the static gain). The required phase lead is (here we use a more accurate, calculated, value of $\arg P(\mathrm{j}11) = -203.3°$)

$$\phi = -180° + 203.3° + \mu_{\mathrm{ph}} + 5.7° = 74°$$

  (5.7° is required to counteract any future phase lag due to a lag element). This phase lead necessitates the use of a second-order lead element, like

$$C_{\mathrm{lead}}(s) = \left(\frac{2s+11}{s+22}\right)^2, \qquad (♠)$$

  calculated with $\alpha = 4$ for the maximum phase lead of $2\arcsin\frac{\alpha-1}{\alpha+1} = 74°$. The resulted loop is presented in Figs. 2(c) (Bode) and 2(d) (polar).

- The static gain of the controller that we have by now, $C(0) = k/\alpha \approx 0.6586$, is far from 100 that we need. Thus we need to add a lag controller with

$$\beta = \frac{100\,\alpha}{k} \approx 154.83.$$

  The overall controller then is

$$C(s) = kC_{\mathrm{lead}}(s)C_{\mathrm{lag}}(s) = 2.63\left(\frac{2s+11}{s+22}\right)^2 \frac{10s+11}{10s+0.07245}$$

  and the resulted loop can be viewed in Figs. 2(e) (Bode) and 2(f) (polar). We can see that the phase margin is indeed $\mu_{\mathrm{ph}} \approx 45°$.

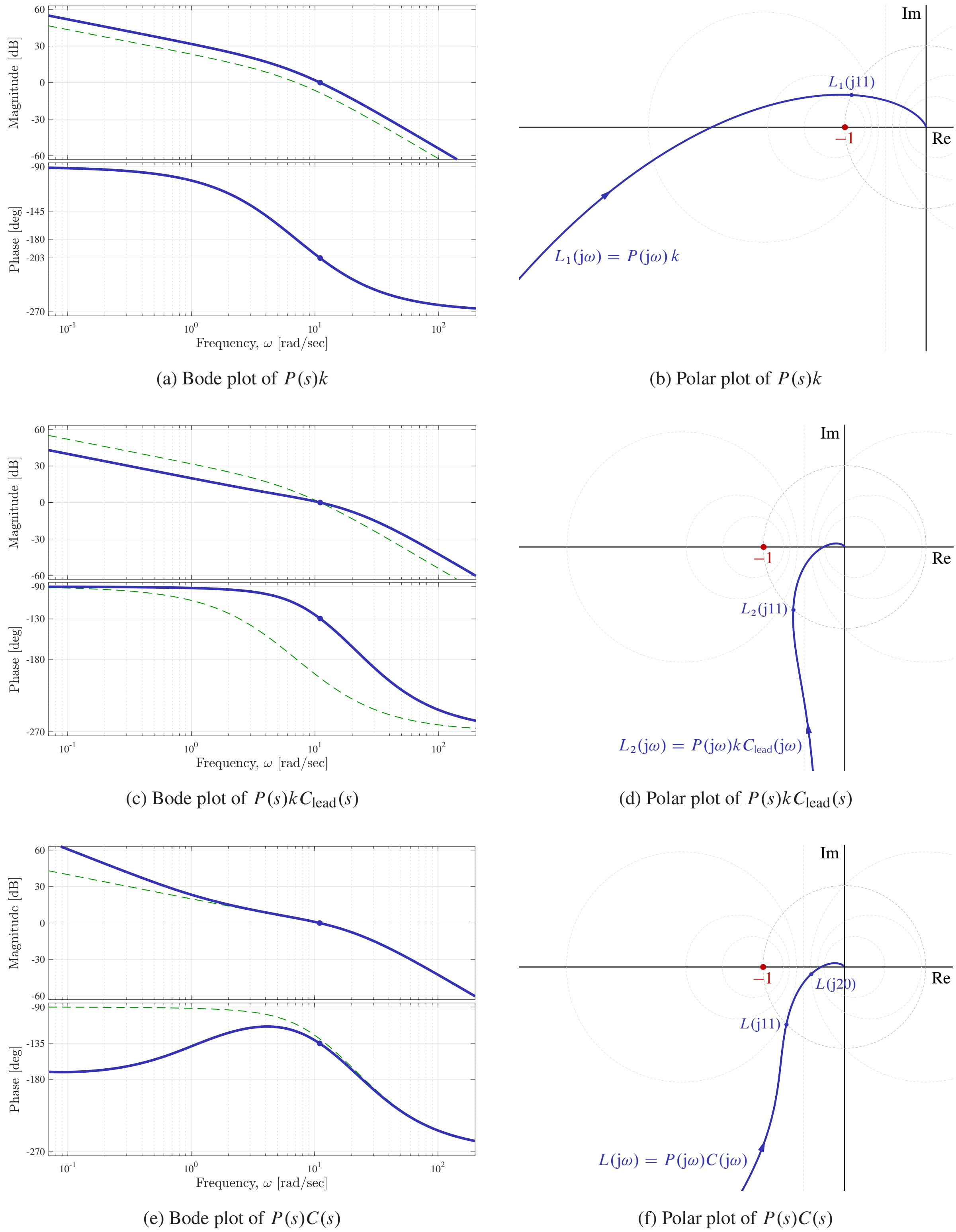
Note that all these steps can be easily automated, so it's a matter of button pressing to carry them out.

Having $C(s)$ designed, the next step is to analyze the resulting closed-loop system. This is done via the closed-loop frequency- and time-domain response plots presented in Fig. 3. Some points to notice:

- The complementary sensitivity has a resonant peak of 2.7 dB at $\omega \approx 13.4$ [rad/sec], which is slightly above the crossover, see Fig. 3(a). This can be expected in view of the fact that the polar plot in Fig. 2(f) around the crossover frequency a bit approaches the critical point as ω increases. The resonant peak in $|T(\mathrm{j}\omega)|$ causes the overshoot, about 28%, in the step response in Fig. 3(b).

- The disturbance sensitivity transfer function is below the 2.2 dB ≈ 1.29 level at all frequencies, see Fig. 3(c). Moreover, it is below the plant gain at all frequencies within the plant bandwidth, which is about 8.4 [rad/sec].

- The closed-loop bandwidth exceeds that of the plant by a factor of about 2.4, see Fig. 3(a). As a result, the control sensitivity grows in the high frequency range as can be seen in Fig. 3(e). This, in turn, gives rise to a relatively high spike in the control signal when the reference step is applied, see Fig. 3(f).



(a) Bode plot of $P(s)k$

(b) Polar plot of $P(s)k$

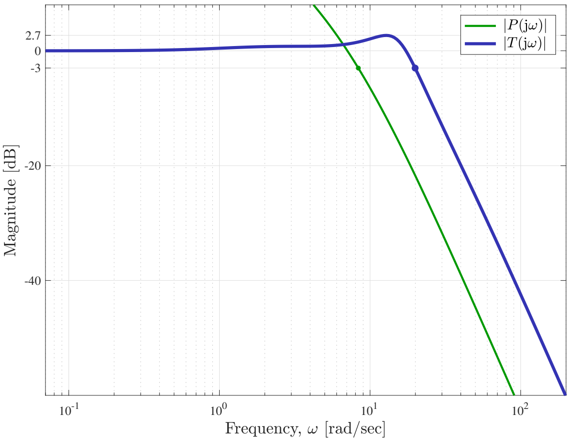
(c) Bode plot of $P(s)kC_{\mathrm{lead}}(s)$

(d) Polar plot of $P(s)kC_{\mathrm{lead}}(s)$

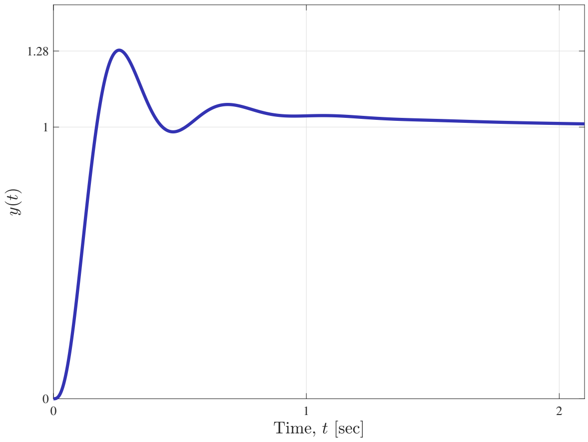
(e) Bode plot of $P(s)C(s)$

(f) Polar plot of $P(s)C(s)$

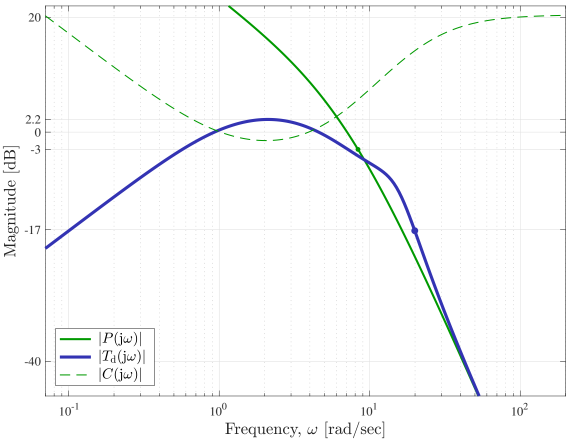
Fig. 2: Q1: Design steps under $\omega_{\mathrm{c}} = 11$ [rad/sec] and $\mu_{\mathrm{ph}} = 45°$



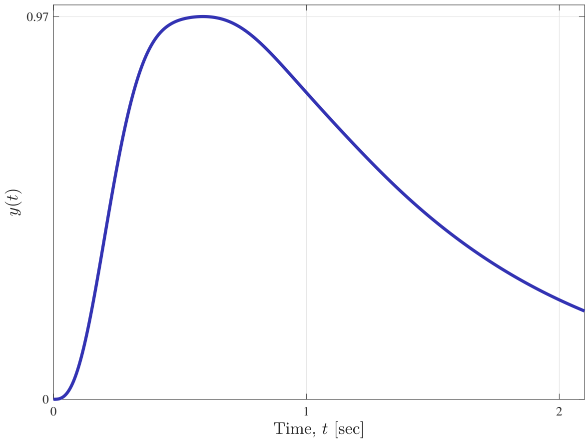
(a) Bode magnitude plot of $T(s)$, actual $\omega_{\mathrm{b}} = 21$ [rad/sec]



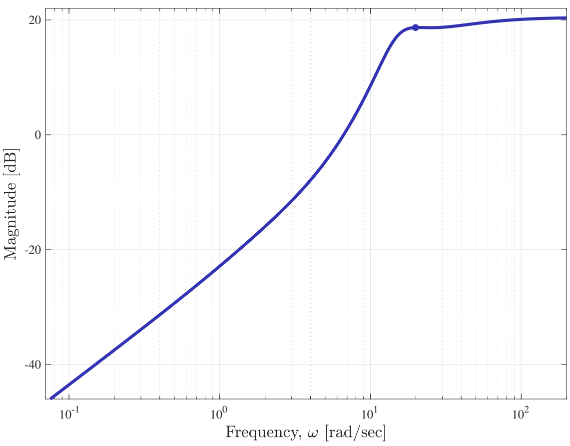
(b) Step response of $T(s)$



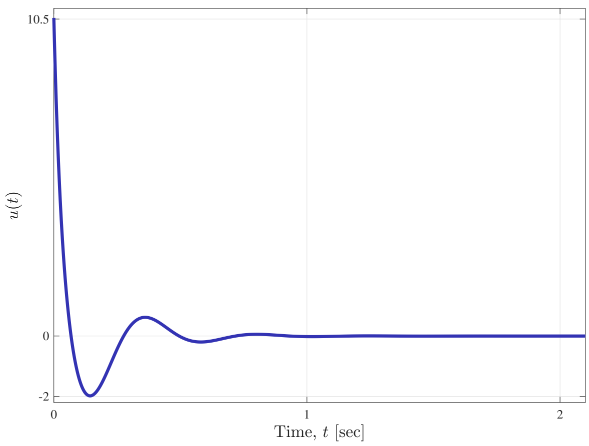
(c) Bode magnitude plot of $T_{\mathrm{d}}(s)$



(d) Step response of $T_{\mathrm{d}}(s)$



(e) Bode magnitude plot of $T_{\mathrm{c}}(s)$



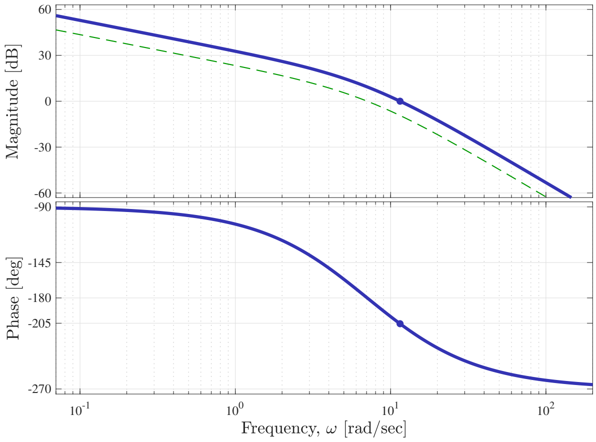
(f) Step response of $T_{\mathrm{c}}(s)$

Fig. 3: Q1: Closed-loop responses under $\omega_{\mathrm{c}} = 11$ [rad/sec] and $\mu_{\mathrm{ph}} = 45°$

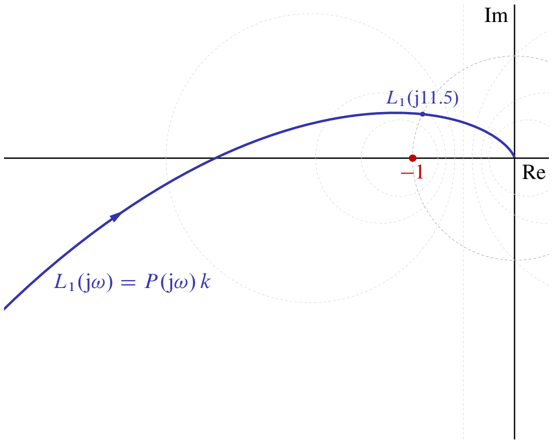
If we repeat the same steps with $\mu_{\rm ph} \geq 35°$, we end up with a close crossover, $\omega_{\rm c} = 11.5$ [rad/sec] and close responses, see Figs. 4 and 5. The controller in this case is

$$C(s) = 2.9303\left(\frac{1.8455s + 11.5}{s + 21.2228}\right)^2 \frac{10s + 11.5}{10s + 0.0989}.$$

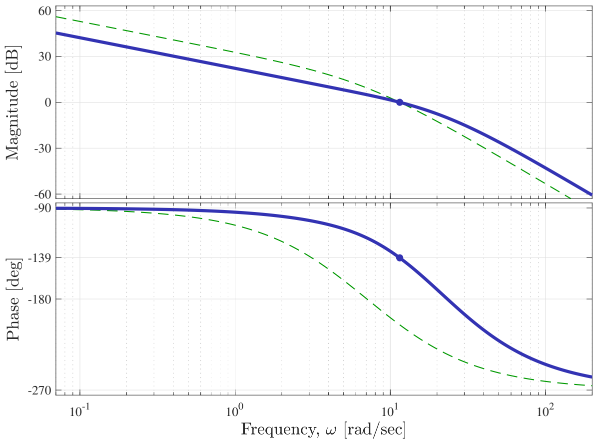
The only visible (and explainable) difference is a higher peak of the complementary sensitivity magnitude (and a higher overshoot of the corresponding step response, 42% vs. 28% in the previous design). This is expectable, as the second design uses a lower phase margin and, as a result, end up with a polar plot closer to the critical point, cf. the polar plots in Figs. 2(f) and 4(f). ▽



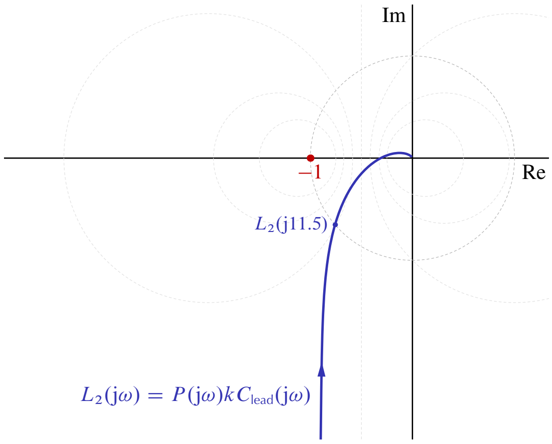
(a) Bode plot of $P(s)k$



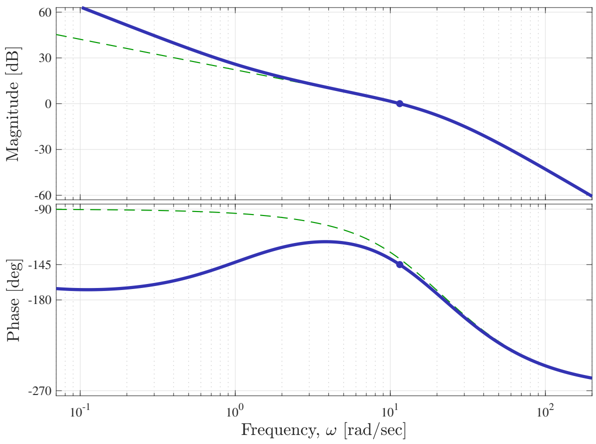
(b) Polar plot of $P(s)k$



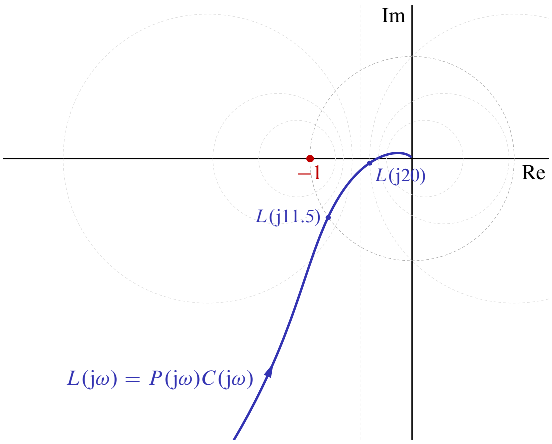
(c) Bode plot of $P(s)kC_{\mathrm{lead}}(s)$



(d) Polar plot of $P(s)kC_{\mathrm{lead}}(s)$

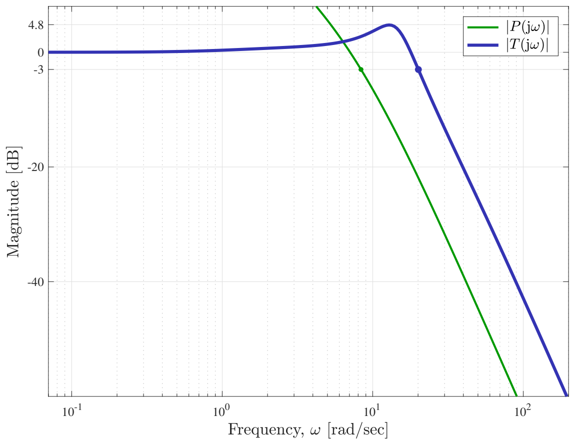


(e) Bode plot of $P(s)C(s)$

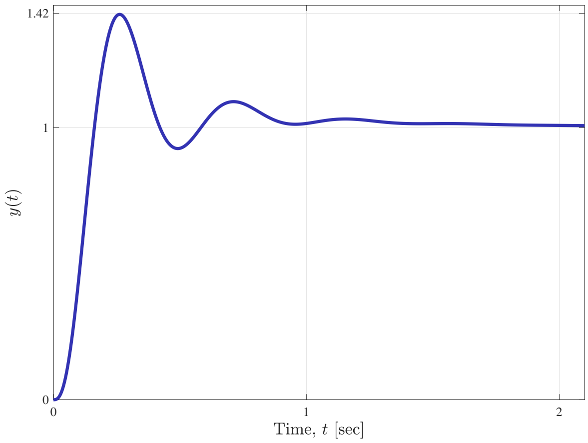


(f) Polar plot of $P(s)C(s)$

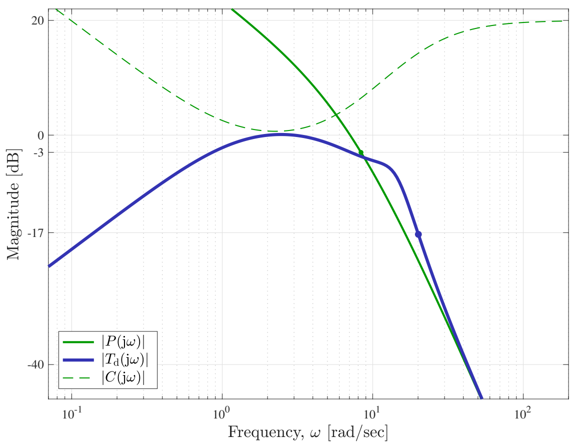
Fig. 4: Q1: Design steps under $\omega_{\mathrm{c}} = 11.5$ [rad/sec] and $\mu_{\mathrm{ph}} = 35°$



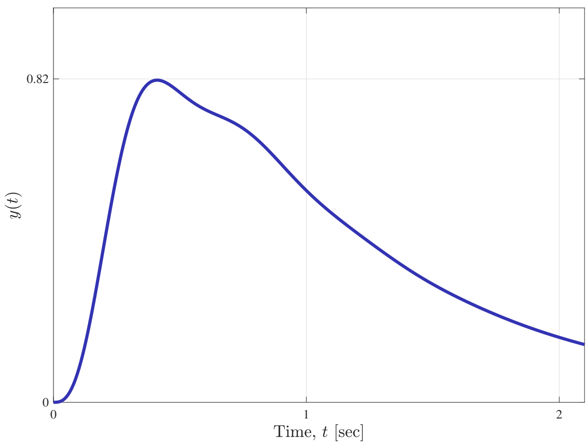
(a) Bode magnitude plot of $T(s)$, actual $\omega_b = 21$ [rad/sec]



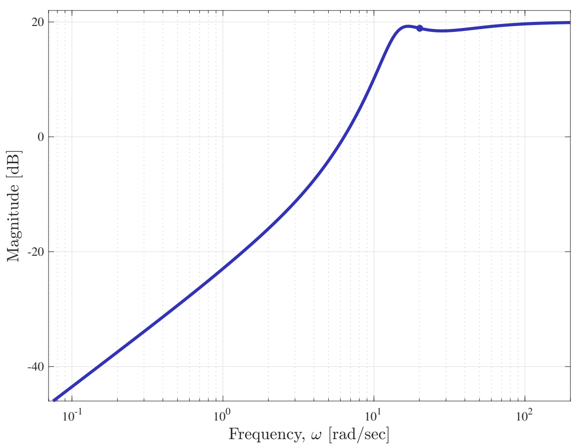
(b) Step response of $T(s)$



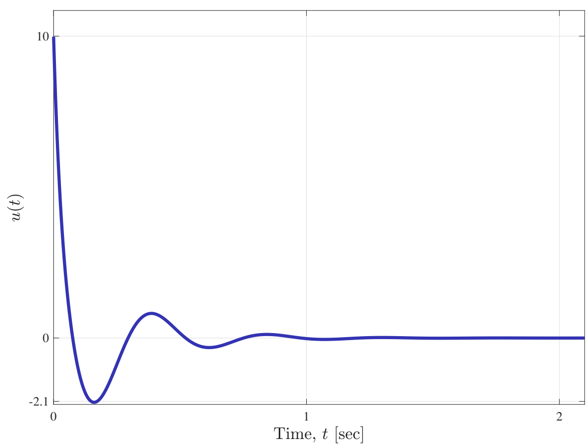
(c) Bode magnitude plot of $T_d(s)$



(d) Step response of $T_d(s)$



(e) Bode magnitude plot of $T_c(s)$



(f) Step response of $T_c(s)$

Fig. 5: Q1: Closed-loop responses under $\omega_c = 11.5$ [rad/sec] and $\mu_{ph} = 35°$

Question 2 (self study)**.** Consider a DC motor like that in Lecture 11 controlled in the unity-feedback scheme. Assume that a loop delay of 1.5 [s] is also present in the loop, i.e. the plant transfer function is now

$$P(s) = \frac{1}{s(s+2)}\,\mathrm{e}^{-1.5s}.$$

Consider the following closed-loop specifications:

- zero steady-state error for a step in r
- zero steady-state error for a step in d
- phase margin $\mu_{\text{ph}} \geq 45°$.

Design stabilizing controllers C for these specifications under the following requirements on the crossover frequency:

1. $\omega_c = 0.05$ [rad/sec],
2. $\omega_c = 0.5$ [rad/sec],
3. $\omega_c = 1$ [rad/sec].

Solution. The controller design in this problem follows literally the steps discussed in Lecture 11. So below we only discuss effects of adding loop delays.

1. The phase lag at the required crossover due to the delay is $0.05 \times 1.5 = 0.075$ [rad] $\approx 4.3°$. This is a very small amount. Moreover, the required crossover is was very small with respect to the plant bandwidth, so no lead compensator was required. The situation does not change now. The resulting plots are presented in Fig. 6 (design steps) and Fig. 5 (closed-loop responses). In fact, the closed-loop plots in Fig. 5 are virtually indistinguishable from those in Slides 26–31 of Lecture 11.

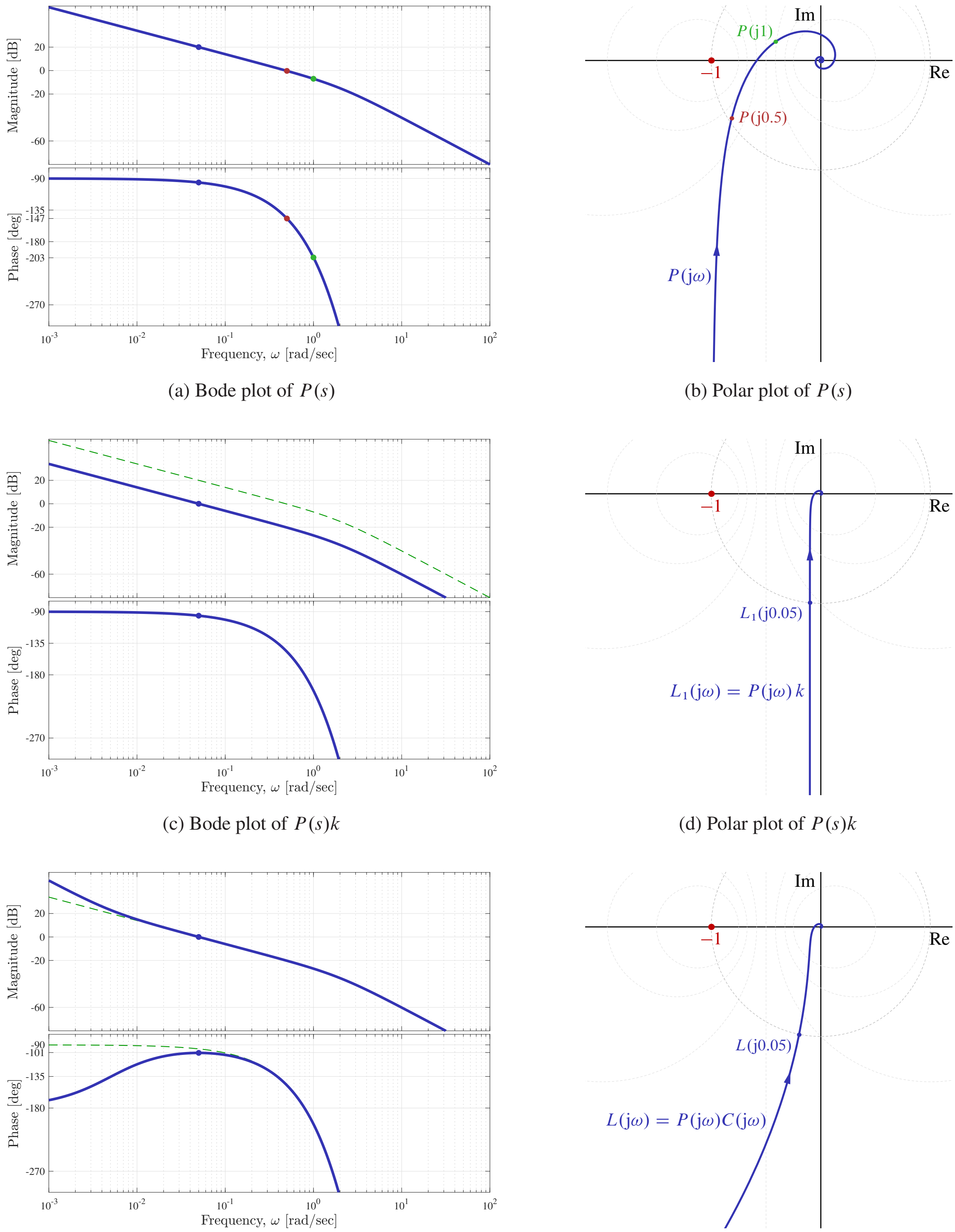
2. Now, the phase lag at the required crossover due to the delay, $0.5 \times 1.5 = 0.75$ [rad] $\approx 43°$ (see also the red dots in Figs. 6(a) and 6(b)), is noticeable. In the delay-free case, we had $\mu_{\text{ph}} \approx 70°$ and no phase lead was required. Now this amount reduces to about 30°, which is below the required phase margin. Hence, a lead controller should be added. Accurate calculations suggest that we need a phase lead of 17.7°, for which a first order lead with $\alpha = 1.87$ is sufficient. The resulting controller is then

   $$C(s) = kC_{\text{lead}}(s)C_{\text{lag}}(s) = 1.0308\,\frac{1.411s + 0.5154}{s + 0.6845}\,\frac{10s + 0.5}{10s},$$

   see the plots in Fig. 8 for design steps. Unlike the delay-free design, now the closed-loop frequency responses have visible resonant peaks at around 0.8 [rad/sec], which is slightly above the crossover frequency. As a result, the closed-loop step responses are quite oscillatory. The reason can be seen at the polar plot in Fig. 8(f), where the plot approaches the critical point after crossing the unit circle at the crossover frequency. This is due to a rapid decay of the loop phase after the crossover frequency, which is clearly visible from the Bode phase plot in Fig. 8(e), aggravated by a relatively slow decay of the gain at those frequencies.

   In principle, this situation can be alleviated, although the requires tools go beyond the scope of this course. Just to provide a flavor of what can be done, still within the cascade design philosophy of loop shaping, consider the addition to the designed controller another component,

   $$C_{\text{snotch}}(s) = 1 + \frac{1.15(s^2 + 0.25)}{s^2 + 1.5s + 2.25} = \frac{2.15s^2 + 1.5s + 2.5375}{s^2 + 1.5s + 2.25},$$



(a) Bode plot of $P(s)$

(b) Polar plot of $P(s)$

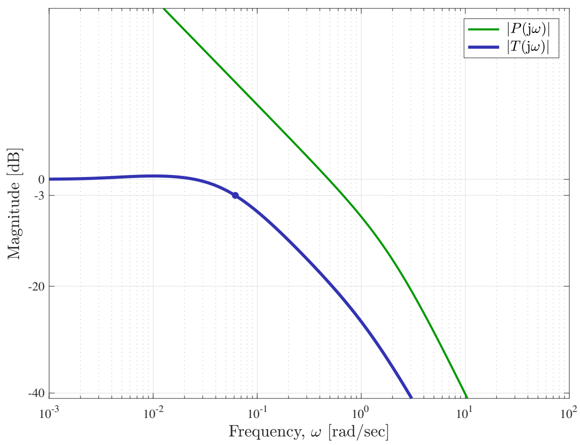
(c) Bode plot of $P(s)k$

(d) Polar plot of $P(s)k$

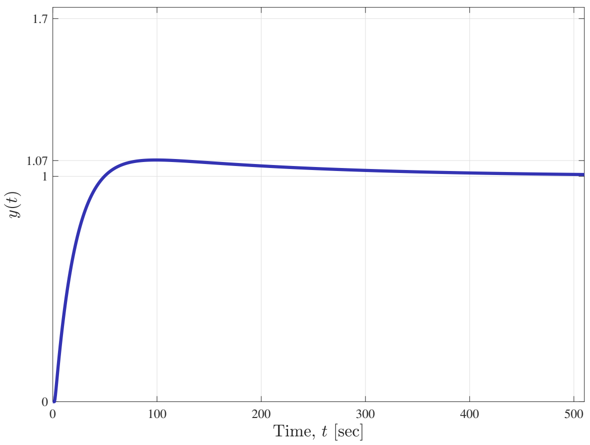
(e) Bode plot of $P(s)kC_{\text{lag}}(s)$

(f) Polar plot of $P(s)kC_{\text{lag}}(s)$

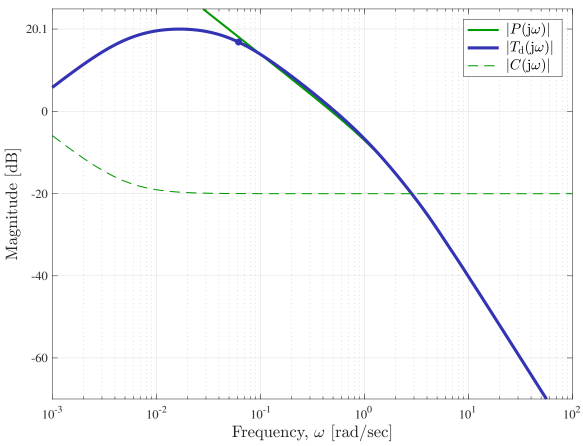
Fig. 6: Q2: Design steps with $\omega_c = 0.05$ [rad/sec]



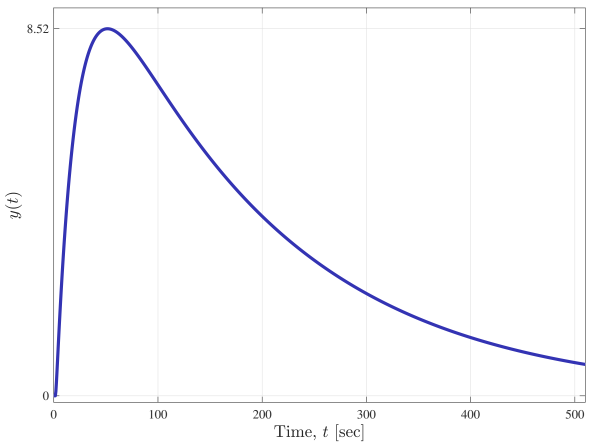
(a) Bode magnitude plots of $T(s)$ and $P(s)$



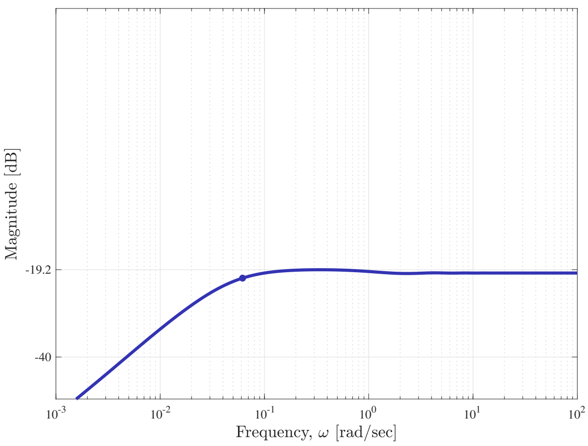
(b) Step response of $T(s)$



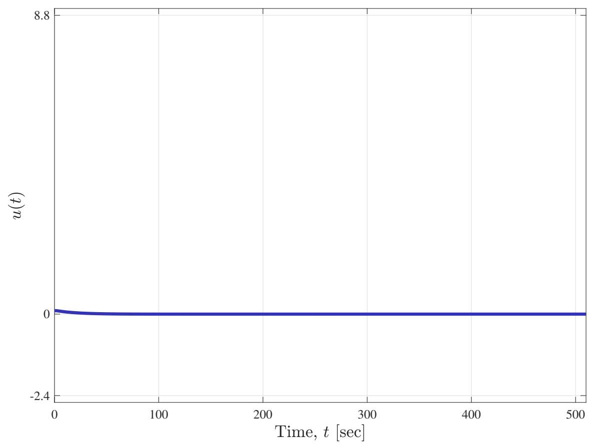
(c) Bode magnitude plot of $T_{\mathrm{d}}(s)$



(d) Step response of $T_{\mathrm{d}}(s)$

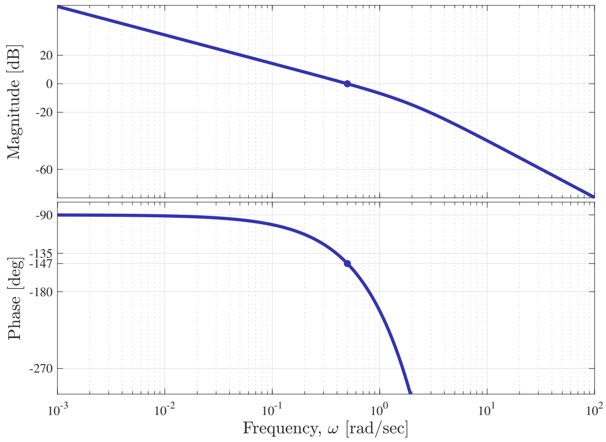


(e) Bode magnitude plot of $T_{\mathrm{c}}(s)$

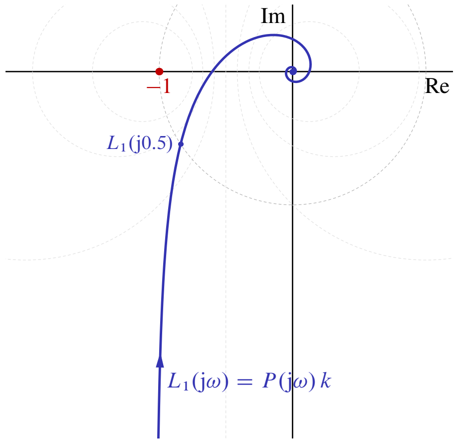


(f) Step response of $T_{\mathrm{c}}(s)$ (peaks 0.1 at $t = 0$)

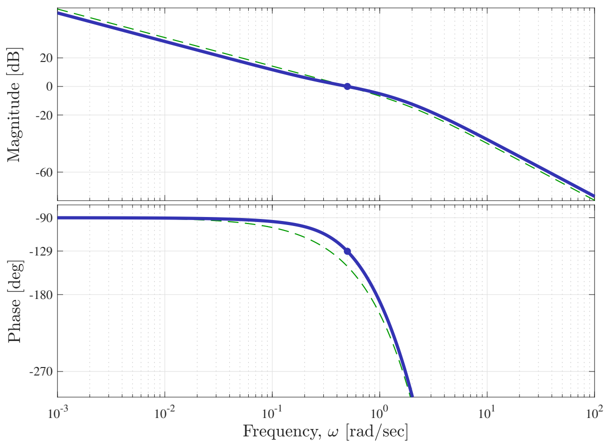
Fig. 7: Q2: Closed-loop responses under $\omega_{\mathrm{c}} = 0.05$ [rad/sec]



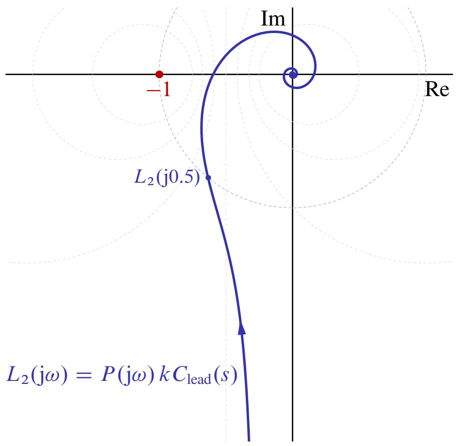
(a) Bode plot of $P(s)k$



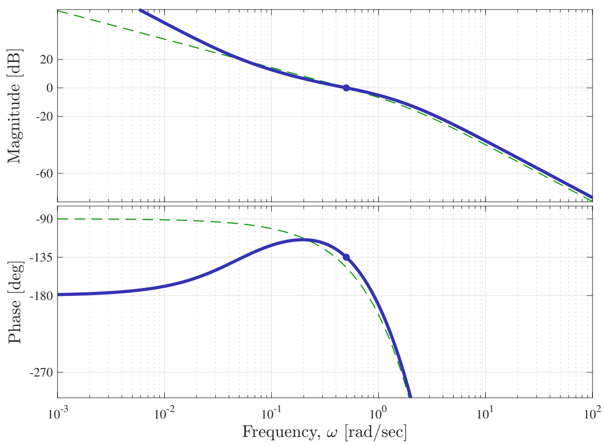
(b) Polar plot of $P(s)k$



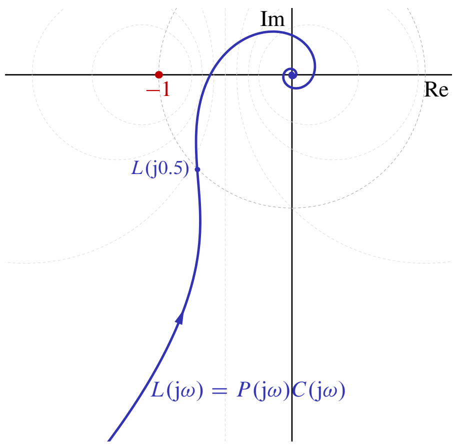
(c) Bode plot of $P(s)kC_{\mathrm{lead}}(s)$



(d) Polar plot of $P(s)kC_{\mathrm{lead}}(s)$

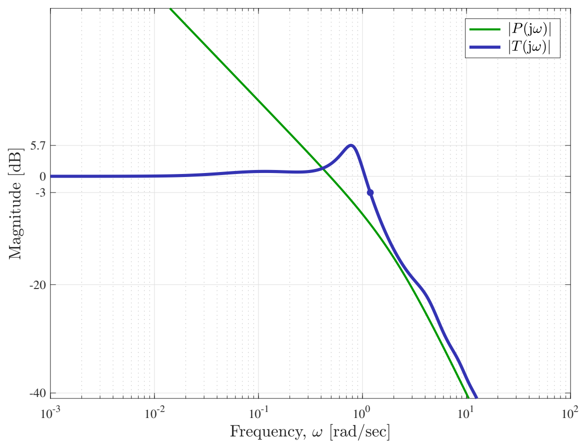


(e) Bode plot of $P(s)kC_{\mathrm{lead}}(s)C_{\mathrm{lag}}(s)$

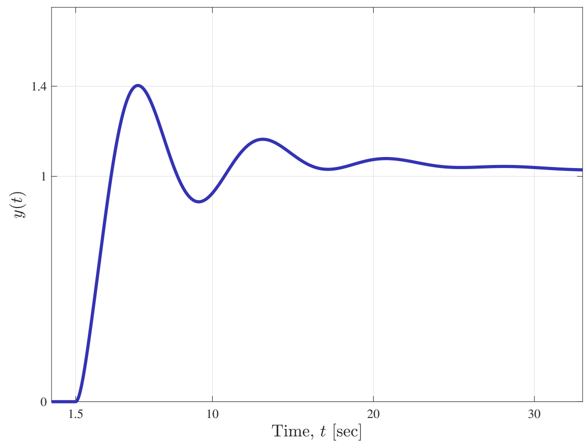


(f) Polar plot of $P(s)kC_{\mathrm{lead}}(s)C_{\mathrm{lag}}(s)$

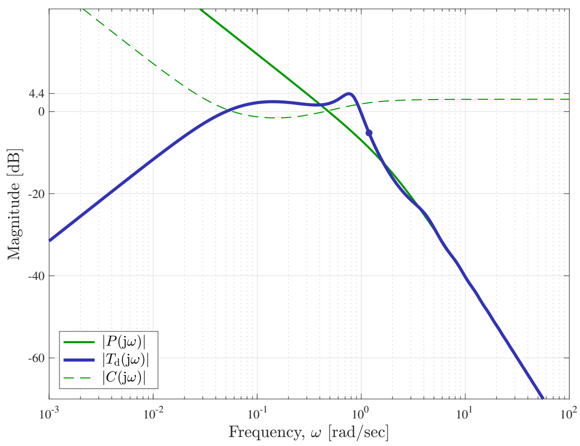
Fig. 8: Q2: Design steps with $\omega_{\mathrm{c}} = 0.5$ [rad/sec]



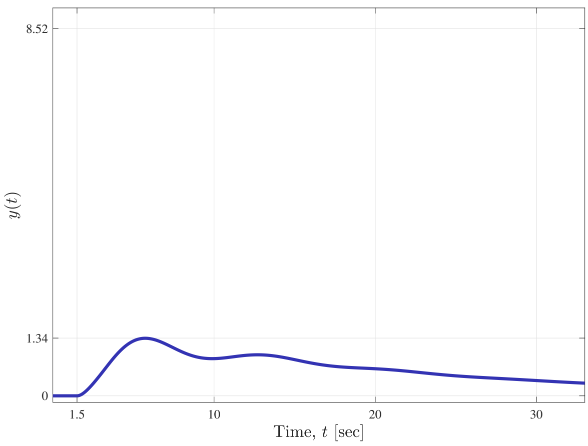
(a) Bode magnitude plots of $T(s)$ and $P(s)$



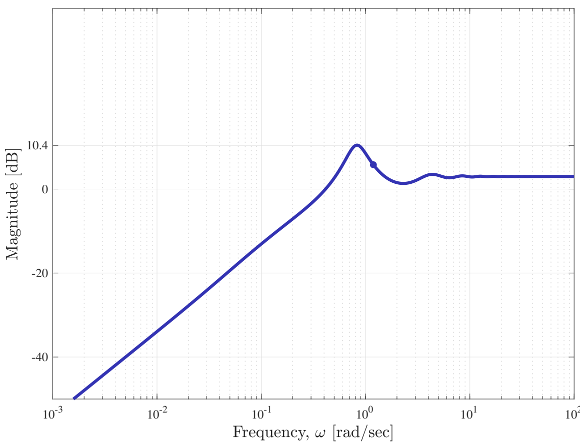
(b) Step response of $T(s)$



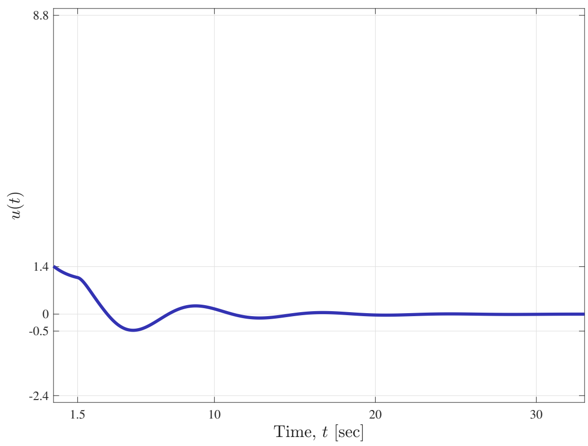
(c) Bode magnitude plot of $T_\mathrm{d}(s)$



(d) Step response of $T_\mathrm{d}(s)$

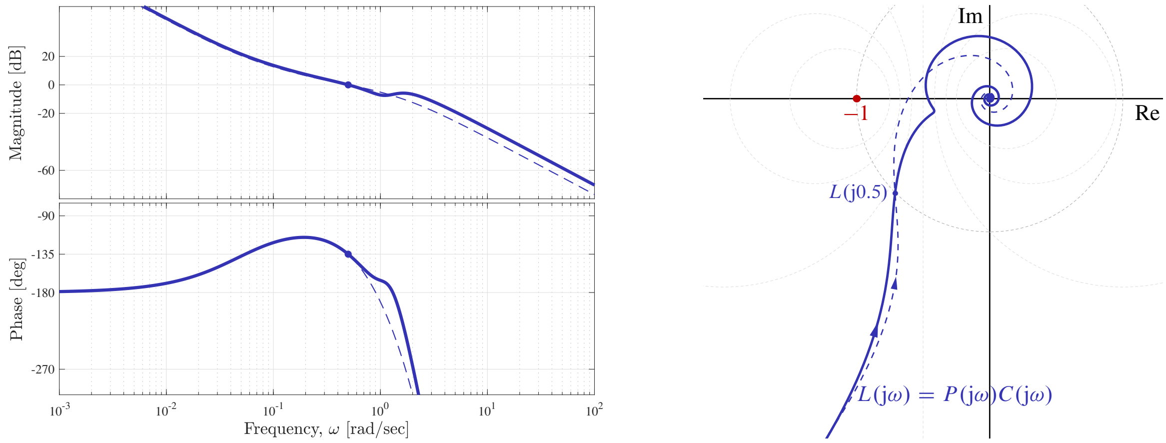


(e) Bode magnitude plot of $T_\mathrm{c}(s)$



(f) Step response of $T_\mathrm{c}(s)$

Fig. 9: Q2: Closed-loop responses under $\omega_\mathrm{c} = 0.5\,[\mathrm{rad/sec}]$



(a) Bode plot of $P(s)kC_{\text{lead}}(s)C_{\text{snotch}}(s)C_{\text{lag}}(s)$ (b) Polar plot of $P(s)kC_{\text{lead}}(s)C_{\text{snotch}}(s)C_{\text{lag}}(s)$

Fig. 10: Q2: Design steps with $\omega_c = 0.5$ [rad/sec]

which verifies $C_{\text{snotch}}(\mathrm{j}0.5) = 1$ (i.e. it does not alter the loop frequency response at the crossover frequency). This system, known as *skewed notch*, is yet another lead element having its maximal phase lead at $\omega = 1.4$ [rad/sec]. Its addition results in the loop gain presented in Fig. 10. It might not be that pretty, but it moves the polar plot a bit away from the critical point by reducing the gain immediately after the crossover frequency and adding a phase lead there. The closed-loop responses are then presented in Fig. 11. We can see that the resonant peaks diminished and, as a result, the overshoot of the reference step response reduced by a factor of 2 (20% vs. 40% before).

3. This crossover requirement was not studied in Lecture 11, as in the delay-free case it is not substantially different from the previous case. But the presence of the delay renders the situation different. It can be seen from Fig. 6(a) (the green dot) that the phase of the plant at the required crossover $\omega = 1$ is about $-202.5°$ (the phase lag at this frequency due to the delay alone is $1 \times 1.5 = 1.5$ [rad] $\approx 86°$). Taking into account the phase margin requirement and the need to compensate for the required lag controller, we need a phase lead of

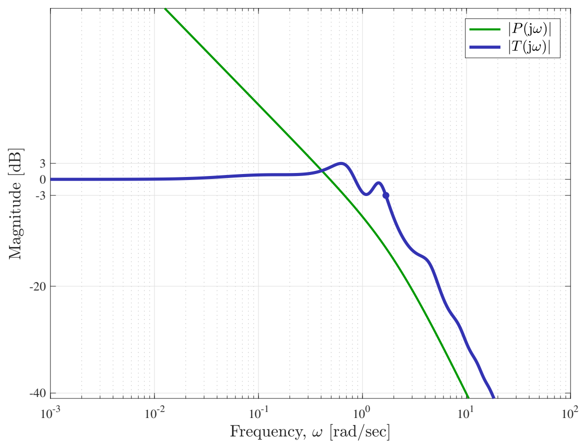
$$\phi = -180° + 202.5° + \mu_{\text{ph}} + 5.7° = 73.2°,$$

which is close to what we needed in Question 1. Thus, we may again use a second-order lead, similar to (♠):

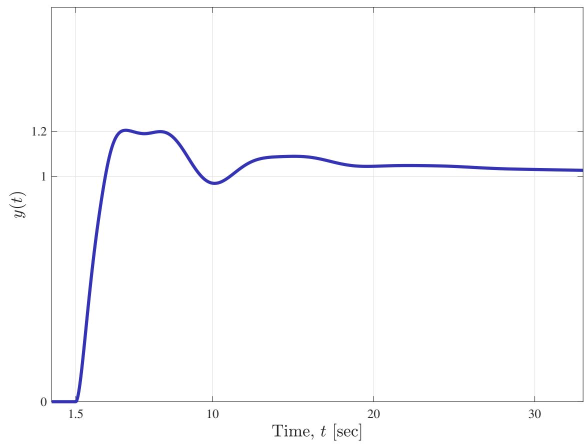
$$C_{\text{lead}}(s) = \left(\frac{2s+1}{s+2}\right)^2, \qquad (♣)$$

This is not a problem per se. The problem is that the phase of the plant drops rapidly after $\omega = 1$, whereas the magnitude does not (unlike the system in Question 1). The addition of (♣), which has a positive magnitude slope around the crossover (see Slide 8 of Lecture 11), renders the resulting loop gain around ω_c almost flat, see the magnitude plot in Fig. 12(e). Combined with the phase lag, we then end up with the polar plot in Fig. 12(f), which, despite a large phase margin, is extremely close to the critical point (at about $\omega = 1.41$ [rad/sec]). As a result, the closed-loop system has a pair of very lightly damped poles, which show up in the resonant peaks in Fig. 13 and oscillatory step responses there. This design cannot be regarded successful and it might not be possible to fix that. This demonstrates that loop delays impose severe limitation on the achievable crossover.

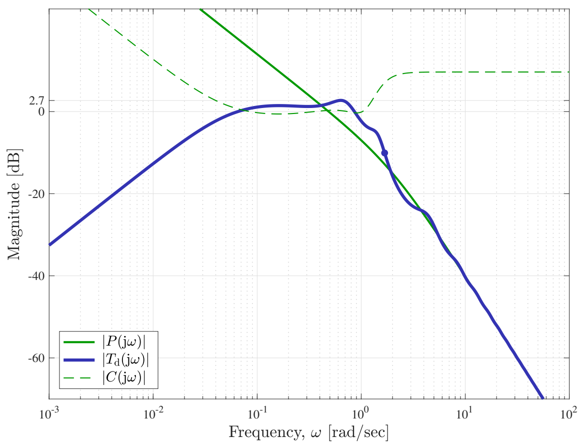
As a matter of fact, a further increase of the required ω_c will lead to an unstable closed-loop system. This is because the loop gain slope around the crossover will be positive. You may check that yourselves … ▽



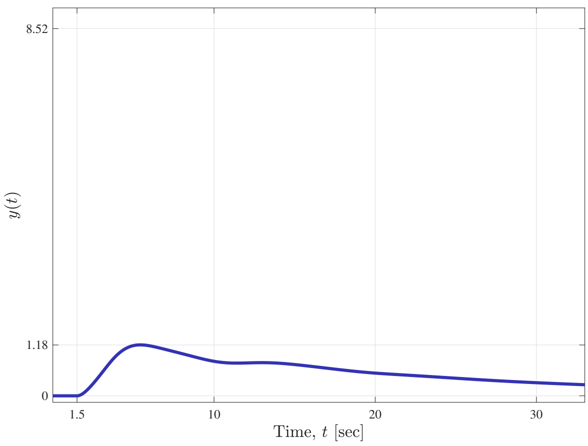
(a) Bode magnitude plots of $T(s)$ and $P(s)$



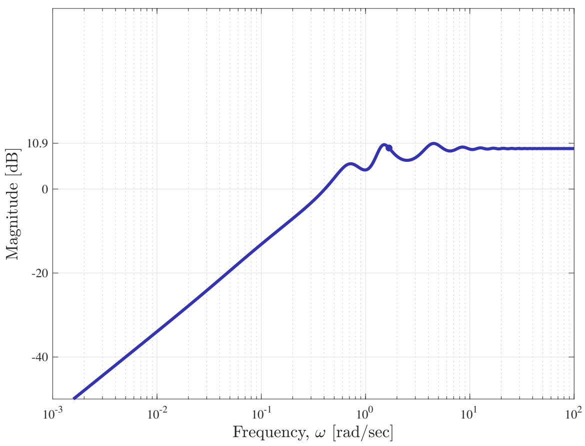
(b) Step response of $T(s)$



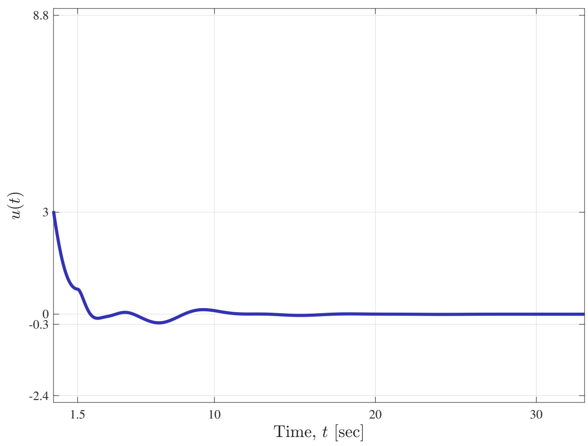
(c) Bode magnitude plot of $T_\mathrm{d}(s)$



(d) Step response of $T_\mathrm{d}(s)$

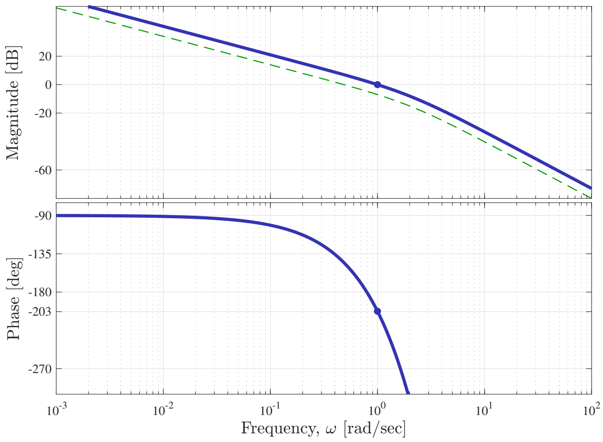


(e) Bode magnitude plot of $T_\mathrm{c}(s)$

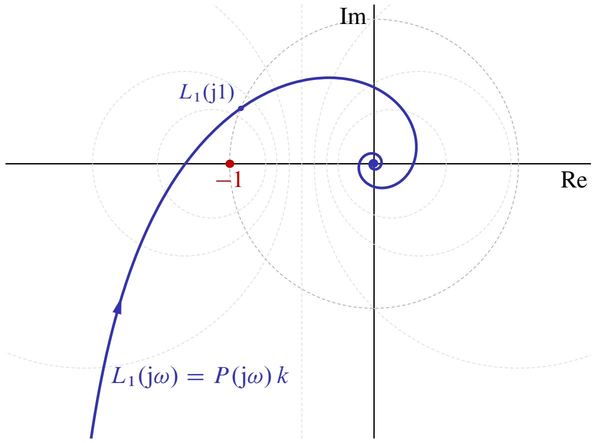


(f) Step response of $T_\mathrm{c}(s)$

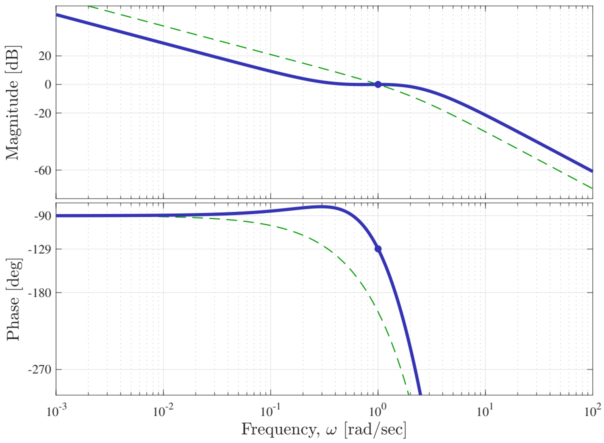
Fig. 11: Q2: Closed-loop responses under $\omega_\mathrm{c} = 0.5$ [rad/sec], with the addition of a skew notch



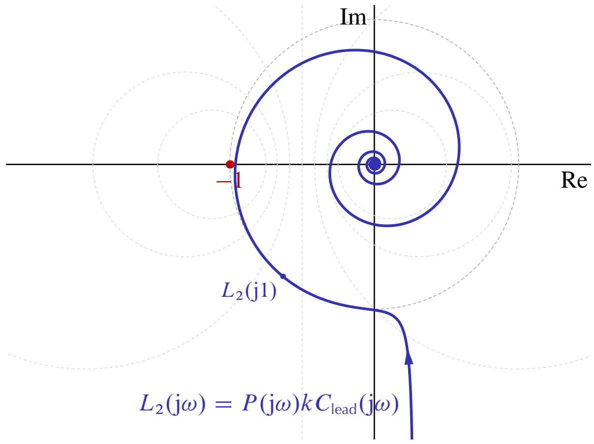
(a) Bode plot of $P(s)k$



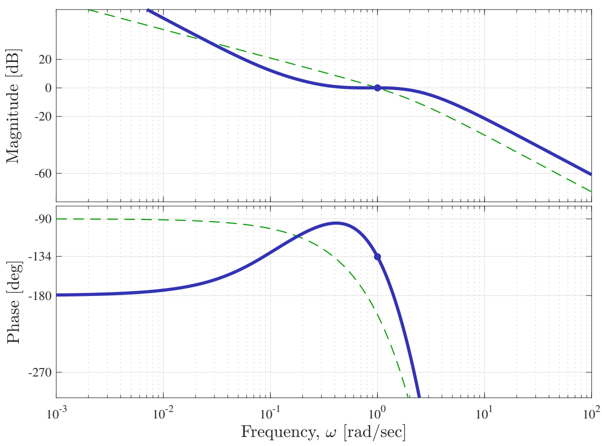
(b) Polar plot of $P(s)k$



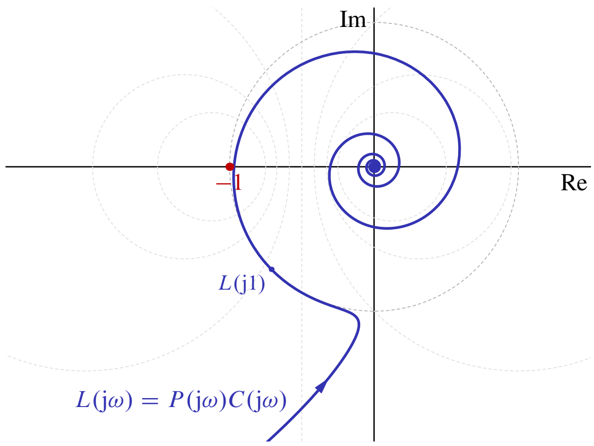
(c) Bode plot of $P(s)kC_{\mathrm{lead}}(s)$



(d) Polar plot of $P(s)kC_{\mathrm{lead}}(s)$

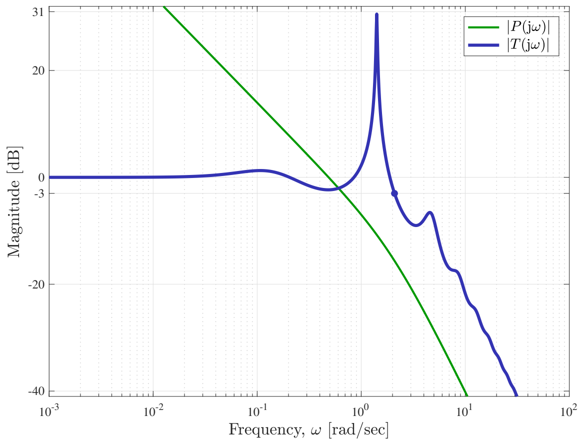


(e) Bode plot of $P(s)kC_{\mathrm{lead}}(s)C_{\mathrm{lag}}(s)$

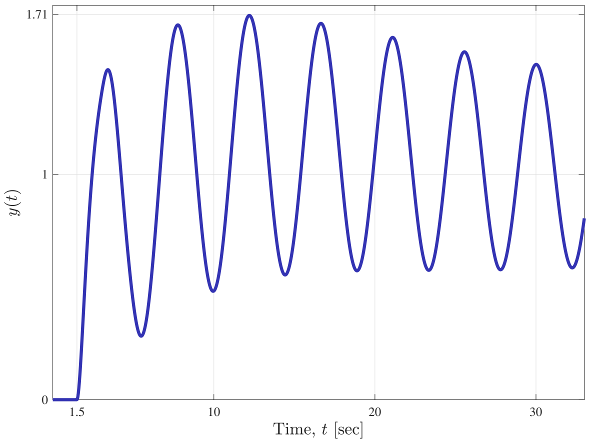


(f) Polar plot of $P(s)kC_{\mathrm{lead}}(s)C_{\mathrm{lag}}(s)$

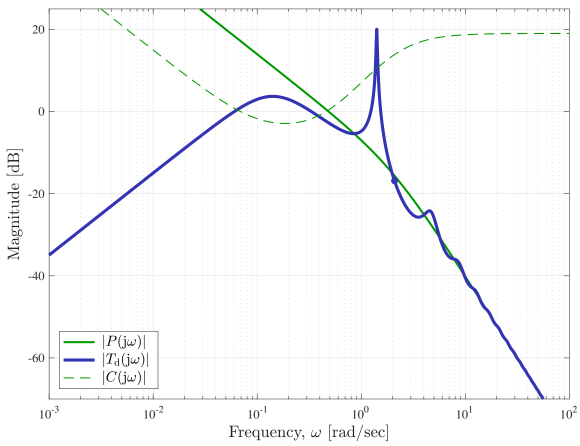
Fig. 12: Q2: Design steps with $\omega_c = 1$ [rad/sec]



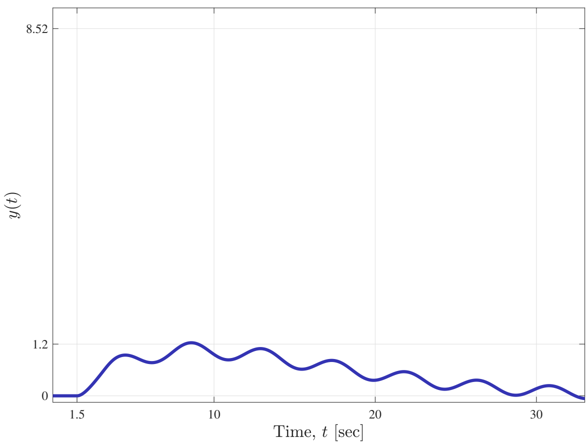
(a) Bode magnitude plots of $T(s)$ and $P(s)$



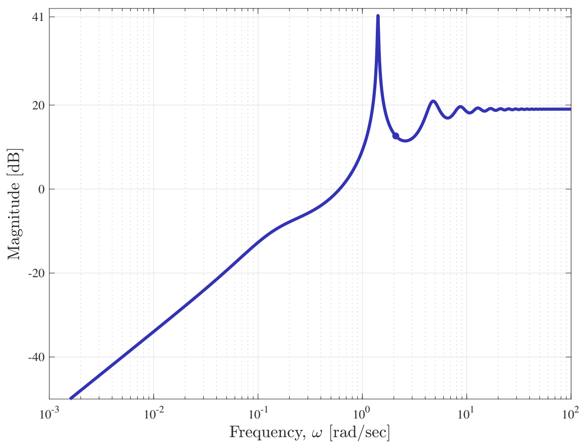
(b) Step response of $T(s)$



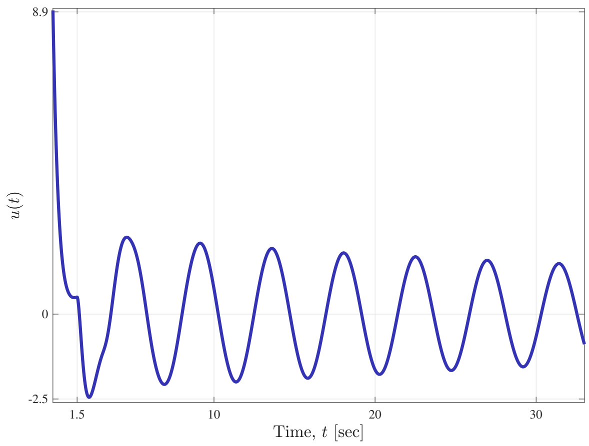
(c) Bode magnitude plot of $T_\mathrm{d}(s)$



(d) Step response of $T_\mathrm{d}(s)$



(e) Bode magnitude plot of $T_\mathrm{c}(s)$



(f) Step response of $T_\mathrm{c}(s)$

Fig. 13: Q2: Closed-loop responses under $\omega_\mathrm{c} = 1$ [rad/sec]