Introduction to Control (034040)

tutorial 10

Question 1. Fig.1 presents the Bode plot of a system having the transfer function

$$P(s) = \frac{k\mathrm{e}^{-\tau s}}{s+1}$$

for some $k > 0$ and $\tau > 0$. Determine the static gain k and the delay $\tau > 0$.

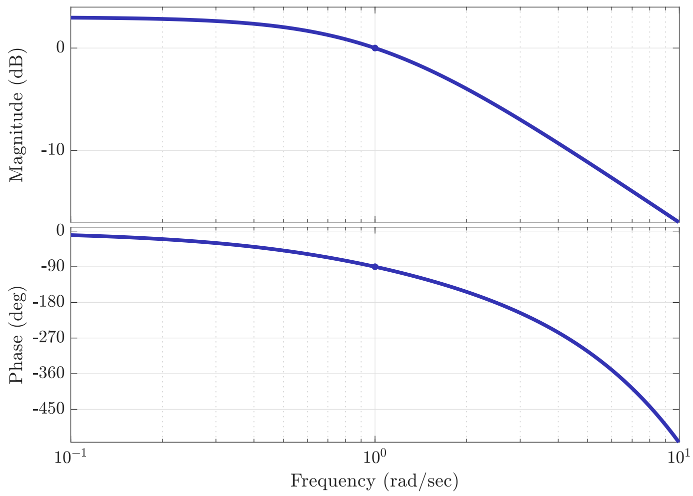


Fig. 1: Bode plot of P in Question 1

Solution. Note that the magnitude plot is not affected by τ, whereas the phase plot is not affected by k.

We know (perhaps) that a first-order system with the transfer function $1/(Ts+1)$ has bandwidth $1/T$ (this, of course, can be routinely calculated). Hence, the magnitude of the frequency response of $1/(s+1)$ at $\omega = 1$ should be $1/\sqrt{2}$. The system in Fig. 1 has it equal to 1 (i.e. 0 dB) at that frequency. Hence, we conclude that

$$k = \sqrt{2} \approx 1.4142.$$

Now, the phase of $k/(s+1)$ at $\omega = 1$ is $-\pi/4$. The phase of the system in Fig. 1 at $\omega = 1$ equals $-90° = -\pi/2$. Hence, the delay adds a phase lag of $\pi/4$ [rad] at $\omega = 1$ [rad/sec]. Hence,

$$\tau = \frac{\pi}{4} \approx 0.7854\,[\text{sec}].$$

That's all … ▽

Question 2. Consider the problem of one-dimensional heat propagation in a semi-infinite rod. Assume that the input is the temperature u at one end and that the output is the temperature y_x at a point along the rod at distance $x > 0$ from the end. Let $\theta(x,t)$ denote the temperature at the position x and time t (so $y_x(t) = \theta(x,t)$). Heat propagation in this system is described by the parabolic partial differential equation

$$\frac{\partial\theta(x,t)}{\partial t} = \alpha\frac{\partial^2\theta(x,t)}{\partial x^2}, \qquad \theta(0,t) = u(t) \tag{1}$$

where $\alpha > 0\,[\mathrm{m}^2/\mathrm{s}]$ is a parameter, known as *thermal diffusivity*, which depends on properties of the rod.

1. Assuming that θ is bounded for every bounded u (which agrees with the physics), derive the transfer function $u \mapsto y_x$ and the corresponding impulse responses. Check the BIBO stability of the system.
2. Construct the Bode and polar plots of the system.
3. Let the system be controlled by a unity-feedback P controller $C(s) = k_\mathrm{p}$.
   (a) Under what $k_\mathrm{p} > 0$ the closed-loop system is stable?
   (b) What is the upper bound on the attainable crossover frequencies under stabilizing k_p's?
   (c) What is the lower bound on the attainable steady-state error?
   (d) What are the stability margins for this loop as functions of x, α, and k_p?
4. Let the system be controlled by a unity-feedback I controller $C(s) = \frac{k_\mathrm{i}}{s}$. Under what $k_\mathrm{i} > 0$ the closed-loop system is stable?

Solution.

1. In the Laplace domain, PDE (1) reads

$$s\,\Theta(x,s) = \alpha\frac{\partial^2\Theta(x,s)}{\partial^2 x}.$$

This is a second-order ODE, whose solution is

$$\Theta(x,s) = A_1(s)\mathrm{e}^{\sqrt{s/\alpha}x} + A_2(s)\mathrm{e}^{-\sqrt{s/\alpha}x}, \quad \Theta(0,s) = U(s)$$

for some A_1 and A_2 independent of x (by $\sqrt{s/\alpha}$ we understand the *principal* square root of a complex number s/α here). As we assumed that the temperature is a bounded function of x, the choice $A_1(s) \equiv 0$ appears natural (it can be justified rigorously, but the math of it goes beyond the scope of the course). Then the boundary condition yields $A_2(s) = U(s)$ and the solution reads

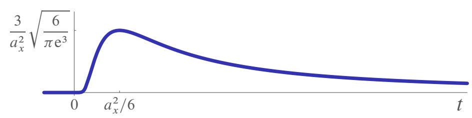
$$\Theta(x,s) = \mathrm{e}^{-\sqrt{s/\alpha}x}U(s).$$

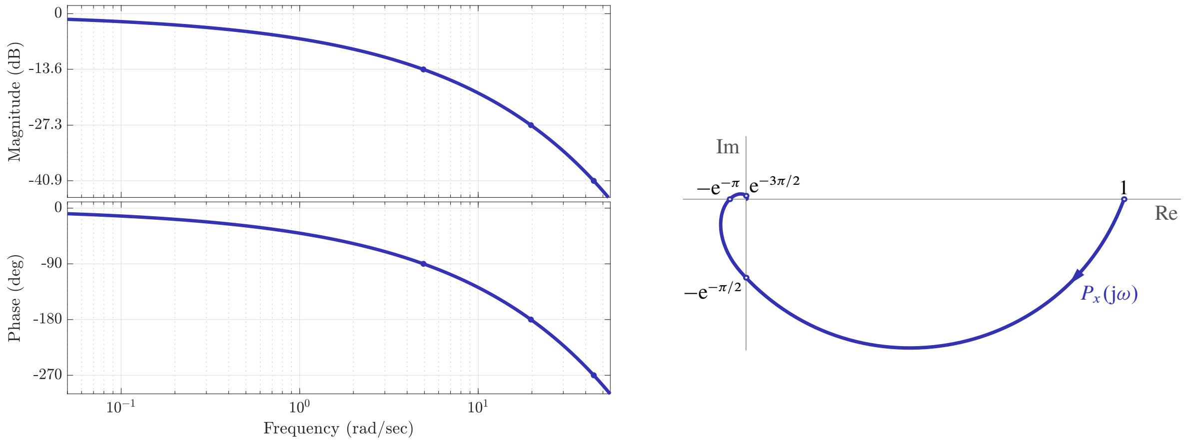
Hence, the transfer function of the system is

$$P_x(s) = \mathrm{e}^{-a_x\sqrt{s}}, \quad \text{where } a_x := \frac{x}{\sqrt{\alpha}} > 0. \tag{♠}$$

Note that a_x is a monotonically increasing function of x. This $P_x(s)$ is irrational, meaning that the system is infinite dimensional (i.e. it has no finite-dimensional state vector). Its impulse response (calculated by Wolfram Mathematica) satisfies

$$p_x(t) = (\mathfrak{L}^{-1}\{P_x(s)\})(t) = \frac{a_x}{2\sqrt{\pi}}\frac{\mathrm{e}^{-(a_x/2)^2/t}}{t^{3/2}} =$$





(a) Bode plot (b) Polar plot

Fig. 2: Frequency-response plots of $P_x(s)$ in Question 2 (here $a_x = 1$)

(note that there is no pure delay in the plot, it just grows extremely slow at the beginning). Because

$$\int_0^\infty |p_x(t)|\mathrm{d}t = \int_0^\infty p_x(t)\mathrm{d}t = 1 < \infty, \quad \forall x > 0$$

the system is BIBO-stable (remember Lecture 3, Slide 27).

2. To construct frequency response plots we need to derive the frequency response of the system. To this end, note that

$$\sqrt{\mathrm{j}} = (\mathrm{e}^{\mathrm{j}\pi/2})^{1/2} = \mathrm{e}^{\mathrm{j}\pi/4} = \frac{1+\mathrm{j}}{\sqrt{2}}$$

(the principal square root, again). Hence, for $P_x(s)$ given by (♠) we have that

$$P_x(\mathrm{j}\omega) = \mathrm{e}^{-a_x\sqrt{\mathrm{j}\omega}} = \mathrm{e}^{-a_x(1+\mathrm{j})\sqrt{\omega/2}} = \mathrm{e}^{-a_x\sqrt{\omega/2}}\mathrm{e}^{-\mathrm{j}a_x\sqrt{\omega/2}}.$$

Hence,

$$|P_x(\mathrm{j}\omega)| = \mathrm{e}^{-a_x\sqrt{\omega/2}} = -\frac{20\,a_x}{\ln 10}\sqrt{\frac{\omega}{2}}\ [\mathrm{dB}] \quad \text{and} \quad \arg P_x(\mathrm{j}\omega) = -a_x\sqrt{\frac{\omega}{2}}\ [\mathrm{rad}].$$

Both are decaying functions[1] of ω. The Bode plot of this system is presented in Fig. 2(a). It was generated by the Matlab code

```
ax = 1;
w  = logspace(log10(0.05),log10(55),101);
Px = frd(exp(-ax*sqrt(i*w)),w);          % frd: frequency response data
bode(Px,w)
```

(plus some bells and whistles for labels). Important for constructing the polar plot, as well as for determining stability margins, is to find the points of intersection with the real and imaginary axes. Such crossings happen whenever $\arg P_x(\mathrm{j}\omega) = -m\pi/2$ for some $m \in \mathbb{Z}_+$. If m is odd, we have an

[1] And decaying functions of x, as a matter of fact. This implies that the further we measure from the point where the control input is applied, the slower the change can be seen.

imaginary axis crossing, whereas if m is even, the plot crosses the real axis. This condition holds at frequencies that satisfy

$$a_x\sqrt{\frac{\omega}{2}} = \frac{m}{2}\pi \iff \omega = \omega_m := \frac{m^2\pi^2}{2a_x^2}. \qquad (\clubsuit)$$

Then,

$$|P_x(\mathrm{j}\omega_m)| = \mathrm{e}^{-a_x\sqrt{\omega_m/2}} = \mathrm{e}^{-m\pi/2} = \{1, 0.20788, 0.043214, 0.008983, \ldots\}$$

(independent of a_x). With these points determined, the polar plot takes the form shown in Fig. 2(b).

3. (a) Inspecting the polar plot in Fig. 2(b) and taking into account that $P_x(s)$ has no poles (hence, no RHP poles), it is clear that the closed-loop system is stable iff $-1/k_\mathrm{p} < -\mathrm{e}^{-\pi}$, which yields the following stability condition (independent of x and α):

$$k_\mathrm{p} < \mathrm{e}^{\pi} \approx 23.1407,$$

(b) The loop transfer function in this case is $L(s) = k_\mathrm{p}P_x(s)$. The crossover frequency ω_c is the frequency at which

$$|L(\mathrm{j}\omega)| = |k_\mathrm{p}P_x(\mathrm{j}\omega)| = k_\mathrm{p}\mathrm{e}^{-a_x\sqrt{\omega/2}} = 1.$$

This equation is solvable in real ω iff $k_\mathrm{p} \geq 1$ and then the solution is *unique*, i.e.

$$\omega_\mathrm{c} = 2\left(\frac{\ln k_\mathrm{p}}{a_x}\right)^2, \quad \text{whenever } k_\mathrm{p} \geq 1,$$

which is an increasing function of k_p, as expected. The upper bound on attainable crossover frequencies over *stabilizing* P controllers is then attained at the minimal destabilizing k_p, i.e. we have that

$$\omega_\mathrm{c} < \frac{2\pi^2}{a_x^2} = \frac{2\alpha\pi^2}{x^2}.$$

This bound is a decreasing function of x.

(c) We know that the steady-state error to a unit step in the reference for y_x is

$$e_\mathrm{ss} = \frac{1}{|1 + k_\mathrm{p}P_x(0)|} = \frac{1}{1 + k_\mathrm{p}}.$$

This is a decreasing function of k_p. Hence,

$$e_\mathrm{ss} > \frac{1}{1 + \mathrm{e}^{\pi}} \approx 0.0414238.$$

(d) The gain margin, μ_g, defined only if the closed-loop system for a given k_p is stable, is in this case the minimal factor for which $\mu_\mathrm{g}L(\mathrm{j}\omega) = \mu_\mathrm{g}k_\mathrm{p}P_x(\mathrm{j}\omega) = -1$. This happens at the first phase crossover frequency $\omega_\phi = \omega_2$, where ω_2 is a particular case (for $m = 2$) of the frequency defined by ($\clubsuit$). Hence,

$$\mu_\mathrm{g} = \begin{cases} \mathrm{e}^{\pi}/k_\mathrm{p} & \text{if } 0 < k_\mathrm{p} < \mathrm{e}^{\pi} \\ \text{not defined} & \text{if } k_\mathrm{p} \geq \mathrm{e}^{\pi} \end{cases},$$

which is independent of x and α and is a decreasing function of k_p. To calculate the phase margin, we need the phase at the crossover frequency, ω_c:

$$\arg P_x(\mathrm{j}\omega_\mathrm{c}) = -a_x\sqrt{\frac{\omega_\mathrm{c}}{2}} = -\ln k_\mathrm{p}.$$

The phase margin, which equals $\pi + \arg P_x(j\omega_c)$ whenever there is exactly one crossover frequency and the closed-loop system is stable, is then

$$\mu_{ph} = \begin{cases} \infty & \text{if } 0 < k_p < 1 \\ \pi - \ln k_p & \text{if } 1 \le k_p < e^{\pi} \\ \text{not defined} & \text{if } k_p \ge e^{\pi} \end{cases}$$

and it also is independent of x and α and is a decreasing function of k_p. Finally, the delay margin in this case is

$$\mu_d = \frac{a_x^2\,\mu_{ph}}{2\ln^2 k_p},$$

because there is at most one crossover frequency and $\lim_{\omega\to\infty}|k_p P_x(j\omega)| = 0 < 1$. As $a_x = x/\sqrt{\alpha}$, μ_d increases as the distance x from the actuator to the sensor increases and as the thermal diffusivity decreases. μ_d is also a decreasing function of k_p.

4. The loop transfer function is $L(s) = P_x(s)k_i/s$ in this case. Its frequency response (mind that $1/(j\omega) = (1/\omega)e^{-j\pi/2}$)

$$L(j\omega) = \frac{k_i e^{-a_x\sqrt{\omega/2}}}{\omega}\,e^{-j(\pi/2+a_x\sqrt{\omega/2})} = -\frac{k_i e^{-a_x\sqrt{\omega/2}}}{\omega}\bigl(\sin a_x\sqrt{\omega/2} + j\cos a_x\sqrt{\omega/2}\bigr).$$

Thus, we end up with

$$|L(j\omega)| = \frac{k_i}{\omega}\,e^{-a_x\sqrt{\omega/2}} \quad\text{and}\quad \arg L(j\omega) = -\frac{\pi}{2} - a_x\sqrt{\frac{\omega}{2}}$$

The loop gain is still a monotonically decreasing function of ω, but now it starts at ∞ at $\omega = 0$ and vanishes as $\omega \to \infty$, see Fig. 3. Note that as $\omega \to 0$, both real and imaginary parts of $L(j\omega)$

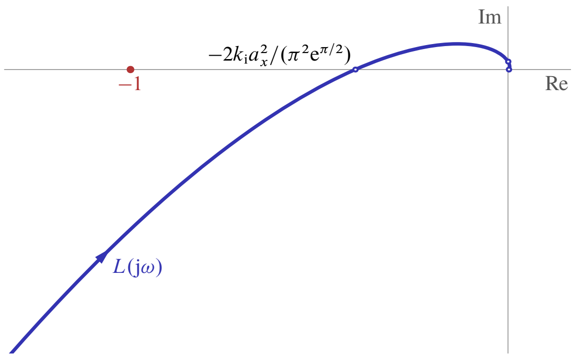


Fig. 3: Polar plot of $P_x(s)k_i/s$ in Question 2

approach $-\infty$. But the real part does it slower, at a ratio of $1/\sqrt{\omega}$, vs. $1/\omega$ for the imaginary part.

We can see from Fig. 3 that the most important frequency from the closed-loop stability point of view is $\omega = \omega_1$ defined by (♣). This is because an integrator adds exactly 90° of phase lag, so the phase crossover is now the frequency at which $\arg P_x(j\omega) = -\pi/2$. In other words, now $\omega_\phi = \pi^2/(2a_x^2)$ and the corresponding gain $|L(j\omega_\phi)| = 2k_i a_x^2 \pi^{-2} e^{-\pi/2}$. The closed-loop system is then stable iff

$$|L(j\omega_\phi)| < 1 \iff k_i < \frac{\pi^2 e^{\pi/2}}{2a_x^2} \approx \frac{23.7388}{a_x^2}.$$

This upper bound decreases as x increases.

That's all … ▽