Introduction to Control (034040)

tutorial 9

Question 1. The process

$$P(s)=\frac{12}{(s+2)^2(s+2.5)}$$

is controlled in closed loop (unity feedback) by a proportional controller, $C(s)=k_\mathrm{p}$. Find stabilizing k_p's using the Nyquist stability criterion.

Solution. To determine stabilizing gains k_p using the "floating-point" method, we do the following:

- draw the Nyquist plot for $k_\mathrm{p}=1$,
- find the magnitude of its intersection with the real axis,
- count the encirclements of the points $-1/k_\mathrm{p}+\mathrm{j}0$,
- determine the position of k_p at which the number of encirclements corresponds to the closed-loop stability (0 in our case).

To draw the polar (and Nyquist) plot of $P(\mathrm{j}\omega)$, we find the magnitude and phase as

$$|P(\mathrm{j}\omega)|=\frac{|12|}{|\mathrm{j}\omega+2|^2|\mathrm{j}\omega+2.5|}=\frac{12}{(\omega^2+4)\sqrt{\omega^2+6.25}}$$

and

$$\arg P(\mathrm{j}\omega)=\arg 12-2\arg(\mathrm{j}\omega+2)-\arg(\mathrm{j}\omega+2.5)=-2\arctan\frac{\omega}{2}-\arctan\frac{2\omega}{5}.$$

Since both are monotonically decreasing continuous functions of ω, calculating two limiting cases,

ω	$\lvert P(\mathrm{j}\omega)\rvert$	$\arg P(\mathrm{j}\omega)$
0	1.2	0
∞	0	$-270°$

shall be enough to visualize the polar plot, see the dark blue lines in the plots in Fig. 1. The Nyquist plot is then completed by mirroring the polar with respect to the real axis, see the light blue curves in Fig. 1.

Let us now find the magnitude of the intersections of the polar plot with the real-axis. All finite intersections should satisfy $\tan[\arg P(\mathrm{j}\omega)]=0$, i.e.

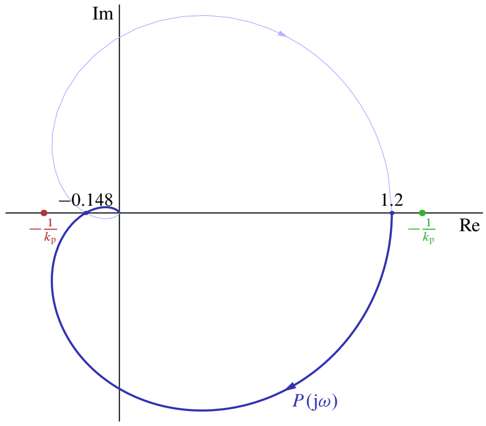
$$\tan\left(-2\arctan\frac{\omega}{2}-\arctan\frac{2\omega}{5}\right)=0.$$

To solve it, remember that $\arctan\alpha\pm\arctan\beta=\arctan\frac{\alpha\pm\beta}{1\mp\alpha\beta}$. Hence, ω satisfies

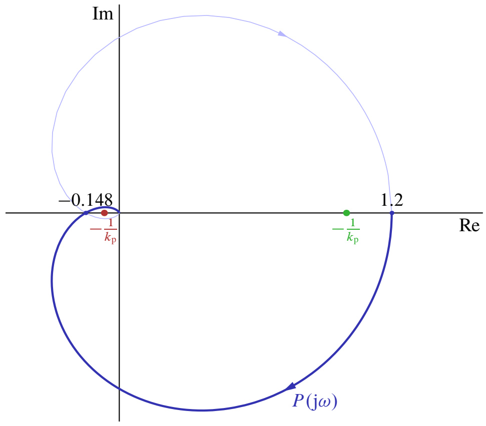
$$0=\tan[\arg P(\mathrm{j}\omega)]=-\tan\left(\arctan\frac{4\omega}{4-\omega^2}+\arctan\frac{2\omega}{5}\right)=-\tan\left(\arctan\frac{2\omega(\omega^2-14)}{13\omega^2-20}\right)$$

or, equivalently,

$$\frac{2\omega(\omega^2-14)}{13\omega^2-20}=0.$$



(a) $k_p = 3$ and $k_p = -0.75$



(b) $k_p = 15$ and $k_p = -1$

Fig. 1: Nyquist plots for Question 1 with different locations of the critical point $-1/k_p$

Thus, we have either $\omega = 0$ (obviously) or $\omega = \pm\sqrt{14} \approx 3.74166$. We consider only the nonnegative solutions, because the gain at the negative solution is the same. Then,

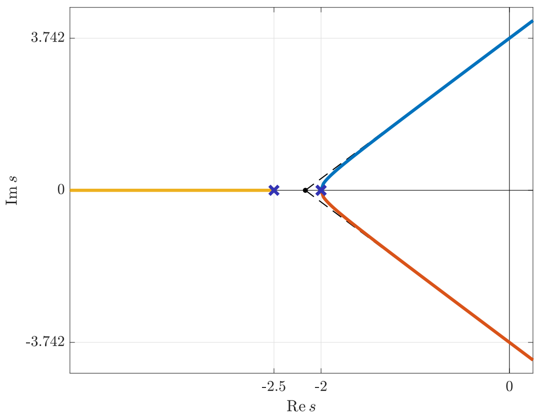
$$G(0) = 1.2 \quad \text{and} \quad G(j\sqrt{14}) = -\frac{4}{27} \approx -0.148148.$$

The next step is to count encirclements of the point $-1/k_p$ on the complex plane. It is clearly seen from Fig. 1(a) that whenever $-1/k_p < -4/27$ or $1/k_p > 6/5$ the Nyquist plot does not encircle the point. If $-1/k_p \in (-4/27, 0)$, we have 2 encirclements (see Fig. 1(b)), while if $1/k_p \in (0, 6/5)$ there is one encirclement. Because the open-loop system is stable, the closed-loop system is stable iff

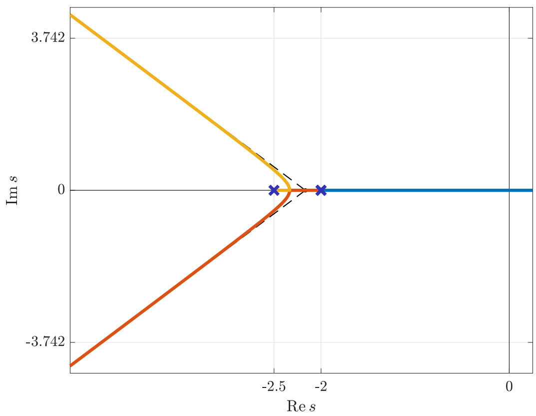
$$k_p \in \left(-\frac{5}{6}, \frac{27}{4}\right) \approx (-0.8333, 6.75).$$

As a matter of fact, if $k_p > 6.75$, the closed-loop system has two unstable poles, whereas if $k_p < -0.8333$, the closed-loop system has one unstable pole.

It may be of interest to see what are the closed-loop poles at the critical gains, where the Nyquist plot intersects the critical points $-1/k_p + j0$? When $k_p \uparrow k_{cr,1} = 0.833$, *two poles* cross to the unstable region,



(a) $k_p > 0$



(b) $k_p < 0$

Fig. 2: Root-locus plots for Question 1

hence for this critical gain there are two complex conjugate poles on the imaginary-axis. Moreover, since for that gain $k_{\text{cr},1}P(\mathrm{j}\omega) = -1$, we can conclude that those $\pm\mathrm{j}\omega$ are the roots of the closed-loop characteristic polynomial (which are zeros of $1 + k_\text{p}P(s)$). Hence, $\pm\mathrm{j}\omega$ are the poles of the closed-loop system. For $k_\text{p} \downarrow k_{\text{cr},2} = -6.75$ *one pole* crosses to the unstable region, meaning that at this critical gain there will be a pole at the origin. This also follows from the fact that $k_{\text{cr},2}P(0) = -1$. These reasonings agree with the root-locus plots in Fig. 2. ▽

Question 2. The process

$$P(s) = \frac{s^2 + 4s/3 + 4}{6s(s^2 + s/2 + 1)}$$

is controlled in closed loop (unity feedback) by a proportional controller, $C(s) = k_\text{p}$, for $k_\text{p} > 0$. Find the number of closed-loop poles in the RHP as a function of k_p using the Nyquist stability criterion.

Solution. Following the same scheme as in Question 1, we find the magnitude and phase of the open loop for unit gain $P(\mathrm{j}\omega)$

$$|P(\mathrm{j}\omega)| = \frac{|-\omega^2 + \mathrm{j}\omega 4/3 + 4|}{|\mathrm{j}6\omega||-\omega^2 + \mathrm{j}\omega/2 + 1|} = \frac{\sqrt{\omega^4 - 56\omega^2/9 + 16}}{6\omega\sqrt{\omega^4 - 7\omega^2/4 + 1}}$$

and

$$\begin{aligned}\arg P(\mathrm{j}\omega) &= \arg\Big(-\omega^2 + \mathrm{j}\frac{4\omega}{3} + 4\Big) - \arg(\mathrm{j}6\omega) - \arg\Big(-\omega^2 + \mathrm{j}\frac{\omega}{2} + 1\Big) \\ &= \arctan\frac{4\omega}{3(\omega^2 - 4)} - \arctan\frac{\omega}{2(\omega^2 - 1)} - \frac{\pi}{2},\end{aligned}$$

where to simplify the notation we assume that arctan takes values in $(0, \pi)$ (rather than the conventional $(-\pi/2, \pi/2)$). Also, we may write the frequency response as

$$P(\mathrm{j}\omega) = -\frac{5\omega^2 + 4}{9(4\omega^4 - 7\omega^2 + 4)} - \mathrm{j}\,\frac{2(\omega^2 - 3)(3\omega^2 - 4)}{9\omega(4\omega^4 - 7\omega^2 + 4)}. \qquad (♠)$$

We can still put in the limiting values of ω as

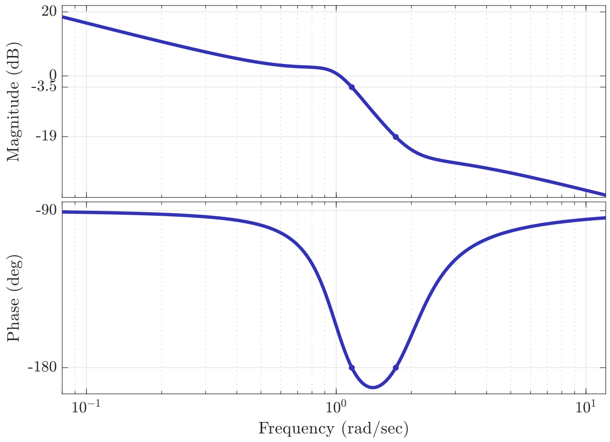
ω	$\lvert P(\mathrm{j}\omega)\rvert$	$\arg P(\mathrm{j}\omega)$	$\operatorname{Re} P(\mathrm{j}\omega)$
0	∞	$-90°$	$-1/9$
∞	0	$-90°$	0

where the real part of the frequency response is required to determine the position of the polar plot at high frequencies. However, we can no longer pre-determine the shape of the polar plot from these numbers, since the magnitude and the phase are not necessarily monotonic. Instead we will first draw the Bode plot, see Fig. 3(a), and then translate it to the polar plot in Fig. 4(a).

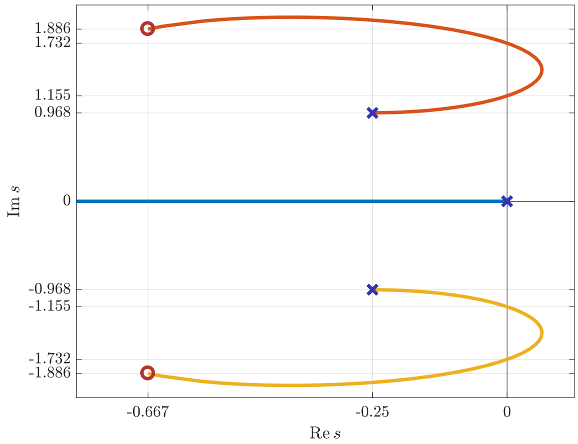
It can be clearly seen from Fig. 3(a) that the phase of $P(\mathrm{j}\omega)$ crosses $-180°$ at two frequencies. These are important points for our analysis, so let us find these frequencies and the corresponding gains. We may use the phase expression to this end, like it was done in Question 1. Yet it is easier to find the real axis crossings from equating the imaginary part of $P(\mathrm{j}\omega)$ in (♠) to zero. The crossing frequencies (the positive ones) are then $\omega = \omega_1 := 2/\sqrt{3} \approx 1.1547$ and $\omega = \omega_2 := \sqrt{3} \approx 1.73205$ and the corresponding gains

$$P(\mathrm{j}\omega_1) = -\frac{2}{3} \quad \text{and} \quad P(\mathrm{j}\omega_2) = -\frac{1}{9}.$$

The polar plot is then depicted in Fig. 4(a). The dashed blue line is the completion of the polar plot to the



(a) Bode plot of $P(s)$



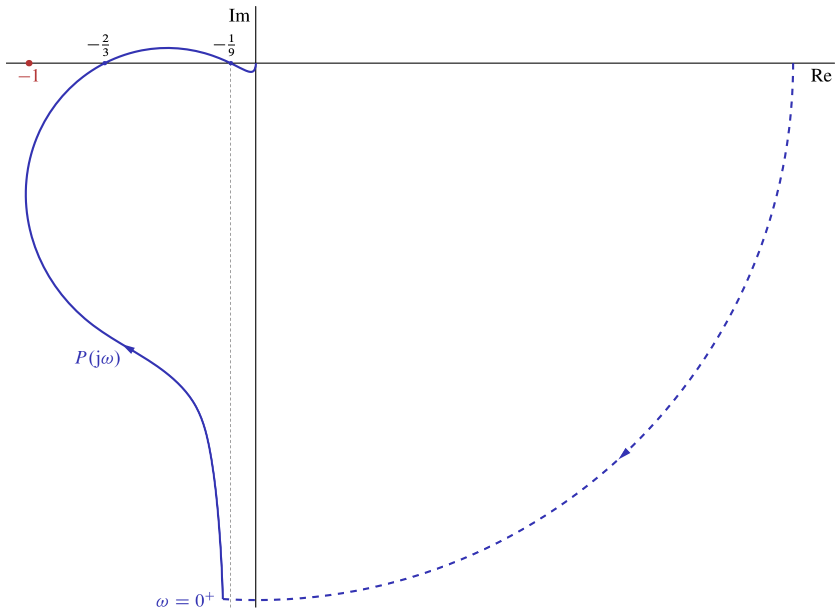
(b) Root locus with $G_k(s) = P(s)$ ($b_m > 0$ here)

Fig. 3: Auxiliary plots for Question 2

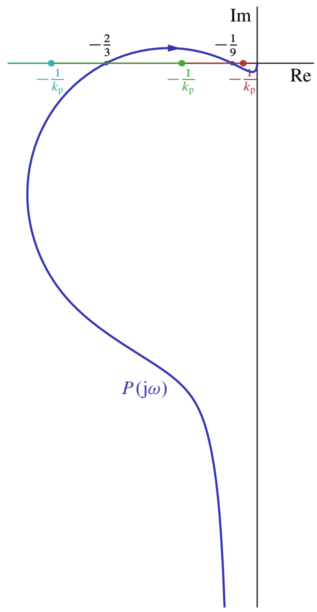
(positive half) of the Nyquist plot due to the presence of one integrator in $P(s)$. It is the an arc, having the angle of 90° and whose radius approach ∞.

It should be clear from the plot in Fig. 4(a) that the part of the Nyquist plot in the positive real half-plane does not affect the stability analysis. Hence, we may consider only the polar plot shown in Fig. 4(b). It is also convenient in this case to count encirclements via counting the crossings the ray $(-\infty, 1/k_\mathrm{p}]$ by the polar plot. Each downward crossing is counted as positive, whereas each upward crossing is counted as negative. The net number of crossings equals then the number of counterclockwise encirclements around the critical point $-1/k_\mathrm{p} + \mathrm{j}0$ by the Nyquist plot of $P(\mathrm{j}\omega)$.

We can see from the plot in Fig. 4(b) that there are 3 different situations:



(a) The polar plot of $P(\mathrm{j}\omega)$ (the plot completion is due to an integrator)



(b) Rays for various k_p

Fig. 4: Polar plots for Question 2

1. if $-1/k_{\rm p} < -2/3$, there is no crossings, hence no encirclements,
2. if $-2/3 < -1/k_{\rm p} < -1/9$, there is "$-1$" crossing of the ray by the polar plot, hence 2 clockwise encirclements of the critical point by the Nyquist plot,
3. if $-1/9 < -1/k_{\rm p} < 0$, there are 2 crossings, one downward and another one upward, hence no encirclements overall.

Hence, as the open-loop system is treated as stable, the closed-loop system is stable iff

$$k_{\rm p} \in (0, 1.5) \cup (9, \infty)$$

and in the interval $(1.5, 9)$ the closed-loop system has two RHP poles.

The corresponding root-locus plot is shown in Fig. 7. Its crossings of the imaginary axis, at the points $\pm\sqrt{3}$ and $\pm 2/\sqrt{3}$, can be explained by the same arguments as those in Question 1. ▽

Question 3. Use the Nyquist stability criterion to determine the stability of the unity-feedback closed-loop system with

1. $L_1(s) = \dfrac{1}{s^2}$
2. $L_2(s) = \dfrac{\sqrt{3}s + 1}{s^2(s + \sqrt{3})}$
3. $L_3(s) = \dfrac{s + \sqrt{3}}{s^2(\sqrt{3}s + 1)}$

Explain the difference.

Solution. The transfer functions in the second and third items can be presented as

$$L_2(s) = L_1(s)G(s) \quad \text{and} \quad L_3(s) = L_1(s)/G(s),$$

where

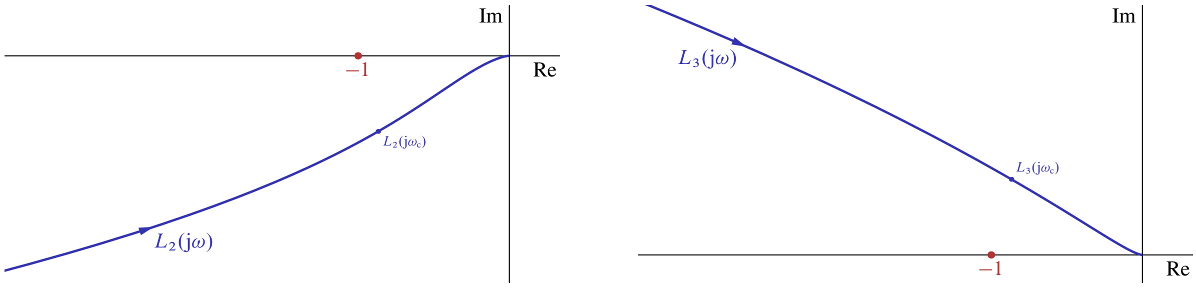
$$G(s) = \frac{\sqrt{3}s + 1}{s + \sqrt{3}}. \tag{♣}$$

It is then natural to investigate $L_1(s)$ first, and then to see how the multiplication and division by $G(s)$ alters it.

1. The polar plot of $L_1(\mathrm{j}\omega)$ is simple, because

   $$L_1(\mathrm{j}\omega) = -\frac{1}{\omega^2}.$$

   This is a real strictly decreasing function of ω. Its Bode magnitude plot is a straight line decaying with a slope of -40 dB/dec and its phase plot is a horizontal line at the $-180°$ level. The polar plot is clearly the negative real semi-axis, crossing the critical point at $\omega = \omega_{\rm c} = 1$ rad/sec. This means that the closed-loop system is unstable. It should also be clear that a way to stabilize this system is to cause the polar plot to shift downwards in the crossover region, i.e. add a phase lead at $\omega \approx \omega_{\rm c}$.



(a) $L_2(\mathrm{j}\omega)$ (b) $L_3(\mathrm{j}\omega)$

Fig. 5: Polar plots for Question 3

2. Let us analyze then the frequency response of $G(s)$ given by (♣). This can be done either analytically (this is a first-order system, after all) or via the Bode (even asymptotic Bode) arguments. Let's start with the latter.

   It follows from the discussion in the previous item that we should be mostly interested in $\arg G(\mathrm{j}\omega)$ at the frequency where $|L_2(\mathrm{j}\omega)| = 1$. The denominator of $G(s)$ always add phase lag (negative phase), which starts with $0°$ at $\omega = 0$ and monotonically drops to $-90°$ as $\omega \to \infty$. In the log-scale of the Bode diagram the phase decreases is symmetric, around $\arg G(\mathrm{j}\omega) = -45°$ at the corner frequency $\omega = \omega_1 = \sqrt{3}$. The numerator of $G(s)$ always add phase lead (positive phase). This lead monotonically increases from $0°$ to $90°$ as ω increases and it is symmetric around $\omega = \omega_2 = 1/\sqrt{3}$. Because $\omega_2 < \omega_1$, the superposition of the phases is *always positive*. In other words, $\arg G(\mathrm{j}\omega) > 0$ for all $\omega > 0$. This conclusion is suported by the analytic expression

$$\arg G(\mathrm{j}\omega) = \arctan(\sqrt{3}\omega) - \arctan\frac{\omega}{\sqrt{3}} = \arctan\frac{\sqrt{3}\omega - \omega/\sqrt{3}}{1+\omega^2} = \arctan\frac{2\omega}{\sqrt{3}(1+\omega^2)} > 0.$$

   As a matter of fact, with $\omega_{\mathrm{c}} = 1$ we have:

$$|G(\mathrm{j}\omega_{\mathrm{c}})| = \frac{|1+\mathrm{j}\sqrt{3}|}{|\sqrt{3}+\mathrm{j}|} = \frac{2}{2} = 1,$$

   which means that the addition of $G(s)$ to the double integrator does not alter the crossover frequency.

   The fact that the phase of $G(\mathrm{j}\omega)$ is positive implies that the closed-loop system is stable. This is attested by the polar plot in Fig. 5(a) (don't forget to add a semi-circle at low frequency, because of a double integrator).

3. The phase of $1/G(\mathrm{j}\omega)$ is the negation (additive inverse) of that of $G(\mathrm{j}\omega)$. Hence, it is always negative and the closed-loop system is unstable, see Fig. 5(b) (again, stability analysis requires to add a semicircle).

That's all … ▽