Introduction to Control (034040)

tutorial 8

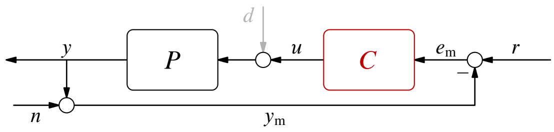


Fig. 1: Unity-feedback control system

Question 1. Consider the unity-feedback system in Fig. 1. Are the requirements below contradictory?

1. $\begin{cases} |E_r(j\omega)| < 0.1|R(j\omega)| & \text{for all } \omega < 10 \\ |E_n(j\omega)| < 0.1|N(j\omega)| & \text{for all } \omega > 1 \end{cases}$

2. $\begin{cases} |E_r(j\omega)| < 0.1|R(j\omega)| & \text{for all } \omega < 1 \\ |E_n(j\omega)| < 0.1|N(j\omega)| & \text{for all } \omega > 10 \end{cases}$

where e_r is the effect of the reference signal r on the tracking error $e := r - y$, e_n is the effect of the measurement noise n on it, and $X(j\omega)$ stands for the value of the spectrum of a signal x at a frequancy ω.

Solution. We know that

$$E_r(s) = S(s)R(s) = \frac{1}{1 + P(s)C(s)} R(s) \quad \text{and} \quad E_n(s) = T(s)N(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} N(s).$$

Hence,

$$|E_r(j\omega)| < \alpha |R(j\omega)| \iff |S(j\omega)| < \alpha$$

and

$$|E_n(j\omega)| < \alpha |N(j\omega)| \iff |T(j\omega)| < \alpha$$

at every given ω. Now, because $S(s) + T(s) \equiv 1$, by the triangle inequality we have that

$$|S(j\omega)| + |T(j\omega)| \geq 1, \quad \forall \omega.$$

Thus,

1. there is a contradiction, because it requires $|S(j\omega)| + |T(j\omega)| < 0.2 < 1$ for all $\omega \in (1, 10)$;
2. there is no contradiction, because constraints on S and T are imposed over non-overlapping frequency ranges.

That's all . . . ▽

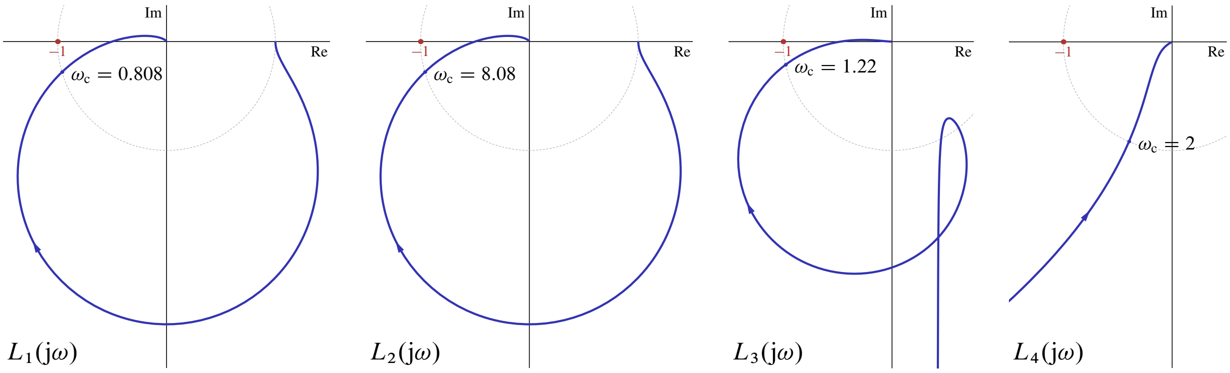


Fig. 2: Polar plots of $L_i(j\omega)$ for Question 2

Question 2. Fig. 2 depicts the polar plots of four loop frequency responses $L_i(j\omega) = P_i(j\omega)C_i(j\omega)$ for $i = 1, \ldots, 4$. These systems are controlled in the standard unity-feedback configuration in Fig. 1.

1. Fig. 3 depicts the Bode plots of the closed-loop complementary sensitivity functions T_k, $k = 1, \ldots, 4$. Relate between k and i.
2. Fig. 4 depicts the step responses of the closed-loop sensitivity functions $S_l(s)$, $l = 1, \ldots, 4$. Relate between l and i.

Solution. This question is based on the following relations between open-loop and closed-loop frequency and time responses:

- The open-loop crossover frequency ω_c is related to the closed-loop bandwidth ω_b (of T). Namely, the larger ω_c is, the wider ω_b is.
- The bandwidth of T is related to the speed of transients of its step response and, then to speed of the step response of $S = 1 - T$. Namely, the wider ω_b is, the faster the step response of S, i.e. e, decays.
- The proximity of the frequency response of L to the critical point $-1 + 0j$ is related to the height of resonant peaks of $|T(j\omega)|$. Namely, the closer $L(j\omega)$ to the critical point is, the higher resonant peaks of $|T(j\omega)|$ are.
- Resonant peaks of $|T(j\omega)|$ are related to oscillations / overshoot of its step response. Namely, the sharper resonant peaks of $|T(j\omega)|$ are, the shakier the step response of T, and hence of S, is.
- The loop static gain $L(0)$ is related to the closer-loop steady-state errors to a step reference. Namely, the larger $|L(0)|$ is, the smaller $e_{ss} = |S(0)|$ for the unit step is.

Because both L_3 and L_4 have at least one integrator ($|L(0)| \to \infty$), the corresponding closed-loop static gains $T_k(0) = 1$ (true for $k = 1$ and $k = 4$) and steady-state error $e_{ss} = 0$ (true for $l = 2$ and $l = 4$). To select between them, note that L_3 is much closer to the critical point than L_4. Hence, the corresponding complementary sensitivity should have a smaller resonance peak and the corresponding sensitivity function should have a less oscillatory step response. Hence, we end up with

$$L_3 \leftrightarrow T_4 \leftrightarrow S_2 \quad \text{and} \quad L_4 \leftrightarrow T_1 \leftrightarrow S_4.$$

Now we need to differentiate between the closed-loop responses corresponding to L_1 and L_2. Both systems have $L_i(0) = 1$ (so the same steady state errors) and similar proximity to the critical point (so

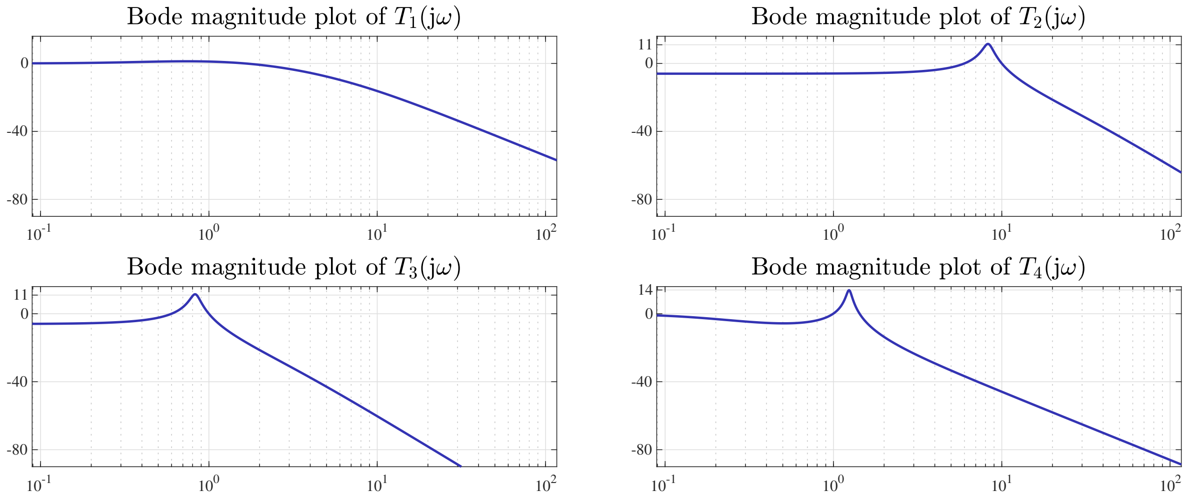


Fig. 3: Bode magnitude plots of $T_k(j\omega)$ for Question 2

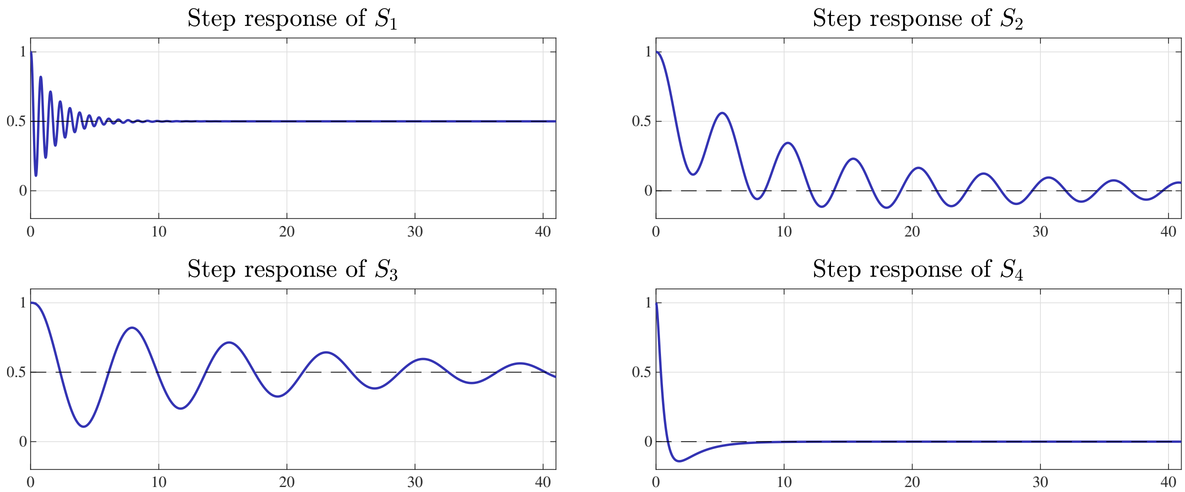


Fig. 4: Step responses of S_l for Question 2

compatible oscillations). At the same time, the crossover frequency of $L_2(j\omega)$ is an order of magnitude larger than that of $L_1(j\omega)$. Hence, we expect from the closed-loop systems corresponding to L_2 to have a wider bandwidth and a faster time response. This yields

$$L_1 \leftrightarrow T_3 \leftrightarrow S_3 \quad \text{and} \quad L_2 \leftrightarrow T_2 \leftrightarrow S_1.$$

That's all … ∇

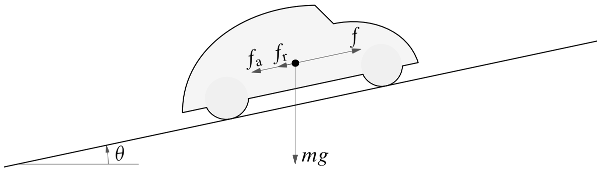


Fig. 5: System for Question 3

Question 3. Consider yet again the problem of cruise control for a vehicle, like that depicted in Fig. 5. The problem here is to maintain the vehicle velocity v at prespecified levels by changing the driving force f generated by the engine. As we already know (Tutorials 3 and 5), the linearized (around some given velocity v_{eq}) model of this system is

$$y = \frac{1}{ms + \alpha v_{\text{eq}}} u$$

where m is the mass of the vehicle, α is a constant depending on the density of air, the frontal area of the car, and a shape-dependent aerodynamic drag coefficient, and the deviation variables $y := v - v_{\text{eq}}$ and $u := f - 0.5\alpha v_{\text{eq}}^2 - mg(\sin\theta + C_r \cos\theta)$, where C_{r} is the (dimensionless) rolling resistance coefficient and θ is the actual road slope. In all numerical calculations of what follows we assume the following numerical values:

α	C_{r}	g	m_0	θ_0	v_{eq}
1 [kg/m]	0.01	9.8 [m/sec^2]	1000 [kg]	12° $\approx$ 0.20944 [rad]	80 [km/h] $\approx$ 22.2222 [m/sec]

Assume that the system is controlled by in the unity-feedback scheme, like that in Fig. 1 by a proportional controller $C(s) = k_{\text{p}} > 0$

1. Normalize the control signal and the output such that the plant has the unit static gain.
2. Find the bandwidth of the normalized plant and of the closed-loop complementary sensitivity function T (as function of k_{p}).
3. Find the control effort at $t = 0$. Express the closed-loop bandwidth $\omega_{\text{b},T}$ as a function of $\bar{u}(t \to 0)$ and the bandwidth of the normalized plant.
4. Draw the Bode plots of $\bar{T}(\text{j}\omega)$, $\bar{P}(\text{j}\omega)$, and $\bar{T}_{\text{c}}(\text{j}\omega)$ for $k_{\text{p}} \in \{500, 2000, 6000\}$. Explain the results of section 2 based on the resulting plots.

Solution.

1. A natural choice is to normalize y by its equilibrium value, v_{eq}, in which case the choice of the normalization factor of u is unambiguous,

$$\bar{y} := \frac{1}{v_{\text{eq}}} y \quad \text{and} \quad \bar{u} := \frac{1}{\alpha v_{\text{eq}}^2} u.$$

The normalized plant $\bar{u} \mapsto \bar{y}$ is then

$$\bar{P}(s) = \frac{1}{v_{\text{eq}}} \cdot P(s) \cdot \alpha v_{\text{eq}}^2 = \frac{\alpha v_{\text{eq}}}{ms + \alpha v_{\text{eq}}}.$$

Its static gain $\bar{P}(0) = 1$, indeed. This shall facilitate a fair comparison between the bandwidths of the plant and the closed-loop system T. Because we normalized y and u, we must also normalize r

and thus $C(s)$. The control signal has the same units as the output so we will normalize by the same quantity:

$$\bar{r} := \frac{1}{v_{\mathrm{eq}}} r \implies \bar{C}(s) = \frac{1}{\alpha v_{\mathrm{eq}}^2} \cdot C(s) \cdot v_{\mathrm{eq}} = \frac{k_{\mathrm{p}}}{\alpha v_{\mathrm{eq}}}.$$

2. The bandwidth of a first-order system having no zeros is the inverse of its time constant, i.e.

$$\omega_{\mathrm{b},P} = \frac{1}{m/(\alpha v_{\mathrm{eq}})} = \frac{\alpha v_{\mathrm{eq}}}{m}.$$

The complementary sensitivity function $\bar{r} \mapsto \bar{y}$ is

$$T(s) = \bar{T}(s) = \frac{k_{\mathrm{p}}}{ms + \alpha v_{\mathrm{eq}} + k_{\mathrm{p}}}.$$

This is still a first-order system, but its static gain $T(0) \neq 1$. Hence, the bandwidth of the frequency response of T is smaller than the reciprocal of its time constant. To find the bandwidth, note that

$$|\bar{T}(\mathrm{j}\omega)|^2 = \frac{k_{\mathrm{p}}^2}{k_{\mathrm{p}}^2 + 2k_{\mathrm{p}}\alpha v_{\mathrm{eq}} + \alpha^2 v_{\mathrm{eq}}^2 + m^2\omega^2}.$$

This is a monotonically decreasing function of ω, so the bandwidth is the positive solution to the equation $|\bar{T}(\mathrm{j}\omega)|^2 = 1/2$. Two situations are possible. It might happen that $|\bar{T}(0)|^2 \leq 1/2$, for which the bandwidth is obviously zero. This happens if $k_{\mathrm{p}} \leq (1+\sqrt{2})\alpha v_{\mathrm{eq}} \approx 53.649$. For larger gains $|\bar{T}(0)|^2 > 1/2$ and the bandwidth is always nonzero. It can be verified that

$$\omega_{\mathrm{b},T} = \frac{1}{m}\sqrt{k_{\mathrm{p}}^2 - 2k_{\mathrm{p}}\alpha v_{\mathrm{eq}} - \alpha^2 v_{\mathrm{eq}}^2}$$

in this case. Evidently, the increase of $k_{\mathrm{p}} > 0$ widens the controlled bandwidth. Moreover, it the closed-loop bandwidth exceeds that of the open-loop plant under $k_{\mathrm{p}} > (1+\sqrt{3})\alpha v_{\mathrm{eq}} \approx 60.712$.

3. The control sensitivity transfer function $\bar{r} \mapsto \bar{u}$ is

$$\bar{T}_{\mathrm{c}}(s) = \frac{k_{\mathrm{p}}/(\alpha v_{\mathrm{eq}})\,(ms + \alpha v_{\mathrm{eq}})}{ms + \alpha v_{\mathrm{eq}} + k_{\mathrm{p}}}$$

By the Initial Value Theorem, the control effort at the initial time is

$$\bar{u}(0) = \lim_{s\to\infty} sT_{\mathrm{c}}(s)\frac{1}{s} = \frac{k_{\mathrm{p}}}{\alpha v_{\mathrm{eq}}}.$$

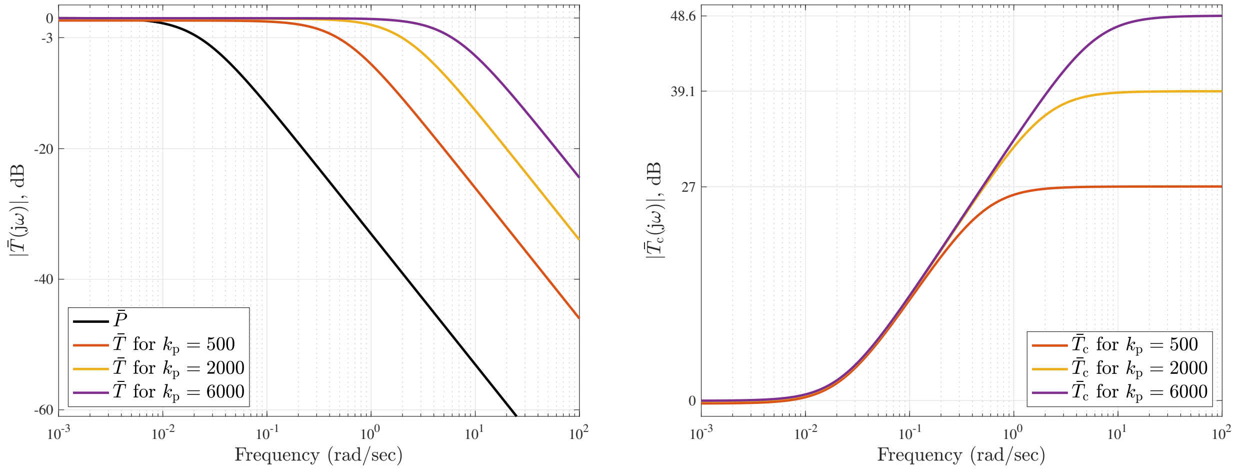
Therefore, the closed loop bandwidth (we assume hereafter that $\bar{u}(0) = k_{\mathrm{p}}/(\alpha v_{\mathrm{eq}}) > 1+\sqrt{2}$, so that the closed-loop bandwidth is nonzero) is

$$\begin{aligned}\omega_{\mathrm{b},T} &= \frac{1}{m}\sqrt{k_{\mathrm{p}}^2 - 2k_{\mathrm{p}}\alpha v_{\mathrm{eq}} - \alpha^2 v_{\mathrm{eq}}^2} = \frac{1}{m}\sqrt{\alpha^2 v_{\mathrm{eq}}^2[\bar{u}(0)]^2 - 2\alpha^2 v_{\mathrm{eq}}^2\bar{u}(0) - \alpha^2 v_{\mathrm{eq}}^2} \\ &= \omega_{\mathrm{b},P}\sqrt{[\bar{u}(0)]^2 - 2\bar{u}(0) - 1} = \omega_{\mathrm{b},P}\sqrt{(\bar{u}(0)-1)^2 - 2},\end{aligned}$$

whence

$$\bar{u}(0) = 1 + \sqrt{2 + \left(\frac{\omega_{\mathrm{b},T}}{\omega_{\mathrm{b},P}}\right)^2}.$$

Thus, the increase of the closed-loop bandwidth with respect to that of the plant itself gives rise to an increase of the initial control amplitude, at $t = 0$. This is intuitive, if we want to get a wider bandwidth for the closed-loop system, we should pay with a higher control effort.



(a) Closed loop system $\bar{T}$

(b) Control sensitivity $\bar{T}_c$

Fig. 6: Bode plots for Question 3

4. The results are presented in Fig. 6. Expectably, the further apart $\omega_{b,P}$ and $\omega_{b,T}$ are, the higher the magnitude of $|\bar{T}_c(j\omega)|$ at high frequencies is. Not only does this give higher control effort at time $t = 0$, it also creates a higher sensitivity to noise, which is typically concentrated in high frequencies.

That's all . . . ∇