הטכניון – מכון טכנולוגי לישראל, הפקולטה להנדסת מכונות

TECHNION—Israel Institute of Technology, Faculty of Mechanical Engineering

Introduction to Control (034040)

tutorial 7

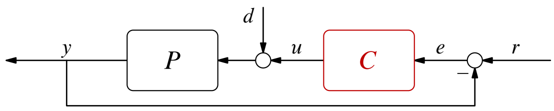


Fig. 1: Unity feedback closed-loop system

Question 1 (self-study)**.** A process with the transfer function $P(s) = 1/((s+1)(s+2))$ is controlled in closed loop with unity feedback as in Fig. 1.

1. Sketch the root locus of the system under a proportional (P) controller, i.e. $C(s) = k_\mathrm{p} > 0$.
2. Find regions of the P controller gain k_p for which the closed-loop response $r \mapsto y$ satisfies the following specifications:
   (a) the overshoot $\mathrm{OS} \le 10\%$,
   (b) the natural frequency $\omega_\mathrm{n} \in [5/3, 8/3]$ [rad/sec].
3. Calculate the steady-state errors to step r and d as functions of the P controller gain k_p. What are the smallest errors under admissible controllers from the previous item?
4. Consider now a proportional-integral (PI) controller of the form $C(s) = k_\mathrm{p}(1 + k_\mathrm{i}/s)$ for some $k_\mathrm{p} > 0$ and $k_\mathrm{i} > 0$. Calculate the steady-state errors to step r and d as functions of the PI controller parameters k_p and k_i.

Solution.

1. In this case
$$G_k(s) = P(s) = \frac{1}{(s+1)(s+2)}.$$
It has two poles (at $s = -2$ and $s = -1$) and no zeros. As the pole excess is 2, there are two asymptotes with $\phi_1 = -\pi/2$ and $\phi_2 = \pi/2$ and with the center of gravity at $\sigma_\mathrm{c} = (-2-1)/2 = -3/2$. The loci are shown by the red and blue lines in Fig. 2.

2. First, the closed-loop system has a second-order transfer function with no zeros. Hence, requirements on the overshoot and the natural frequency can be expressed as appropriate (see Lecture 4) areas on the complex plane. The overshoot depends only on the ratio of the real and imaginary parts of the poles at $-\lambda_\mathrm{r} \pm \mathrm{j}\lambda_\mathrm{i}$. Namely,
$$\mathrm{OS} = \mathrm{e}^{-\pi\lambda_\mathrm{r}/\lambda_\mathrm{i}} \le 0.1 \iff \frac{\lambda_\mathrm{i}}{\lambda_\mathrm{r}} \le \frac{\pi}{\ln 10} \approx 1.36438.$$
Hence, the poles must lie within the sector bounded by radial lines with the slope 1.36438, marked by green dashed line in Fig. 2. The natural frequency ω_n is the absolute value of a root, i.e. $\sqrt{\lambda_\mathrm{r}^2 + \lambda_\mathrm{i}^2}$. Thus, the requirement $\omega_\mathrm{n} \in [5/3, 8/3]$ is translated to a ring between the two green circles in Fig. 2. The region that satisfies both requirements (the overlap between these two areas) is the gray shaded area in Fig. 2.

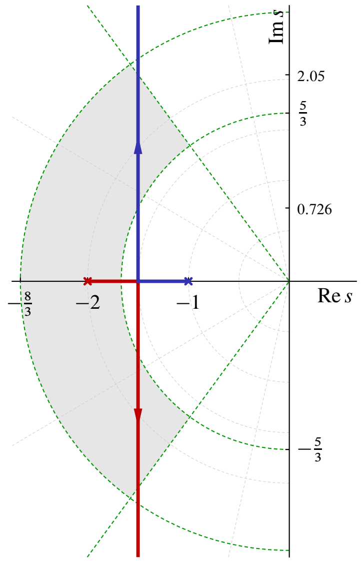


Fig. 2: Root locus plot for Question 1

It can be seen that for small k_p the blue locus is outside the shaded area (as k_p increases, red locus exits it as well). Both loci enter the area at their intersection with the circle of radius 5/3. The point of this intersection can be calculated by Pythagoras,

$$\lambda_\mathrm{i}^2 = \frac{25}{9} - \lambda_\mathrm{r}^2 = \frac{25}{9} - \frac{9}{4} = \frac{181}{9 \cdot 4} \iff \lambda_\mathrm{i} = \frac{\sqrt{19}}{6} \approx 0.726483.$$

As k_p further increases, the loci exit the shaded area after intersecting the green dashed radial line. This happens as

$$\frac{\lambda_\mathrm{i}}{\lambda_\mathrm{r}} = \frac{\pi}{\ln 10} \iff \lambda_\mathrm{i} = \frac{3\pi}{2 \ln 10} \approx 2.04656.$$

The corresponding gains k_p can then be calculated by the gain rule:

$$k_\mathrm{p,min} = \frac{1}{|G_k(-1.5 + \mathrm{j}0.726483)|} = \frac{7}{9} \approx 0.777778$$
$$k_\mathrm{p,max} = \frac{1}{|G_k(-1.5 + \mathrm{j}2.04656)|} = \frac{1}{4} + \left(\frac{3\pi}{2 \ln 10}\right)^2 \approx 4.43843$$

Thus, the closed-loop system satisfies the specifications iff

$$0.777778 \le k_\mathrm{p} \le 4.43843.$$

3. We know that

$$e = Sr - T_\mathrm{d} d.$$

Hence, the steady-state error to a step reference signal is

$$e_\mathrm{ss} = |S(0)| = \frac{1}{|1 + P(0)k_\mathrm{p}|} = \frac{2}{2 + k_\mathrm{p}}$$

and that with respect to a step disturbance is

$$e_\mathrm{ss} = |-T_\mathrm{d}(0)| = \frac{|P(0)|}{|1 + P(0)k_\mathrm{p}|} = \frac{1}{2 + k_\mathrm{p}}.$$

They are obviously both decreasing functions of k_p. Hence, the smallest errors are attained under the largest possible k_p, which is $k_\mathrm{p} = 4.43843$. The corresponding errors under step r and d are then

$$e_\mathrm{ss} \approx 0.310635 \quad \text{and} \quad e_\mathrm{ss} \approx 0.155317,$$

respectively.

4. We should know that the presence of an integral action in the controller guarantees zero steady-state errors under step r and d whenever the controller stabilizes the closed-loop system. Thus, all we need is to determine under which combination of k_p and k_i the closed-loop system is stable. To this end, write the characteristic polynomial

$$\chi_\mathrm{cl}(s) = (s+1)(s+2)s + k_\mathrm{p}(s+k_\mathrm{i}) = s^3 + 3s^2 + (k_\mathrm{p}+2)s + k_\mathrm{i}k_\mathrm{p}.$$

Assuming $k_\mathrm{p} > 0$ and $k_\mathrm{i} > 0$, this polynomial is Hurwitz iff

$$3(k_\mathrm{p}+2) > k_\mathrm{i}k_\mathrm{p} \iff (k_\mathrm{i}-3)k_\mathrm{p} < 6$$

If $k_\mathrm{i} \in (0,3]$, this condition holds true for all $k_\mathrm{p} > 0$. If $k_\mathrm{i} > 3$, the closed-loop system is stable iff $0 < k_\mathrm{p} < 6/(k_\mathrm{i}-3)$. Thus, the steady-state error

$$e_\mathrm{ss} = \begin{cases} 0 & \text{if } k_\mathrm{p} < \dfrac{6}{\max\{3, k_\mathrm{i}\} - 3} \\ \infty & \text{otherwise} \end{cases}$$

That's all … ▽

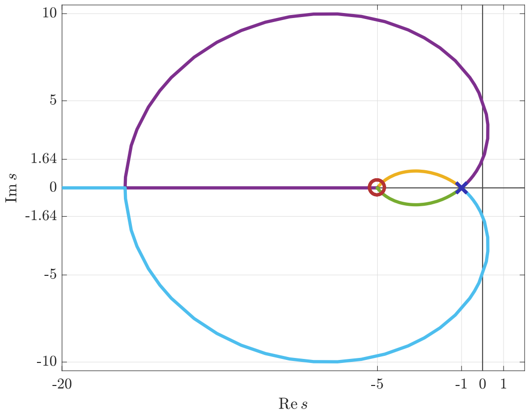


Fig. 3: Root locus system 1

Question 2. Fig. 3 depicts the root-locus of a plant P controlled in the unity-feedback configuration in Fig. 1 with a proportional controller, $C(s) = k_{\mathrm{p}}$.

1. Can $P(s)$ be determined unambiguously?
2. Assuming that the leading coefficient of the numerator of $P(s)$ is positive (the denominator is monic), determine the sign of k_{p} in Fig. 3 and sketch the root-locus plot for the complementary region.
3. Fig. 4 presents the closed-loop step responses for $k_{\mathrm{p}} = 0.093$, $k_{\mathrm{p}} = -0.008$, and $k_{\mathrm{p}} = 1.885$. Match the time-domain response to each of these controller gains. What are the steady-state frequencies of the oscillating responses?

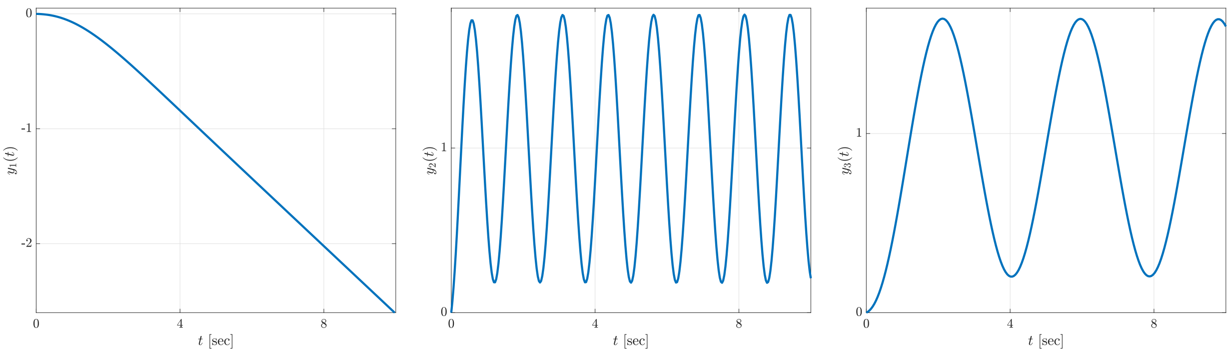


Fig. 4: Step responses of the closed-loop system for different k_{p}

Solution.

1. We observe that 4 loci are leaving the pole at $s = -1$ and 3 loci are entering the zero at $s = -5$. Since the controller is $C(s) = k_{\mathrm{p}}$, we determine that the transfer function of the plant is of the form

$$P(s) = \frac{k(s+5)^3}{(s+1)^4}$$

Because the root-locus is plotted for gains $k_{\mathrm{p}} \in [0, \infty)$, we cannot tell the plant gain k from that of the controller. Hence, we can only determine the locations of poles and zeros of $P(s)$, but not its gain, from root-locus plots.

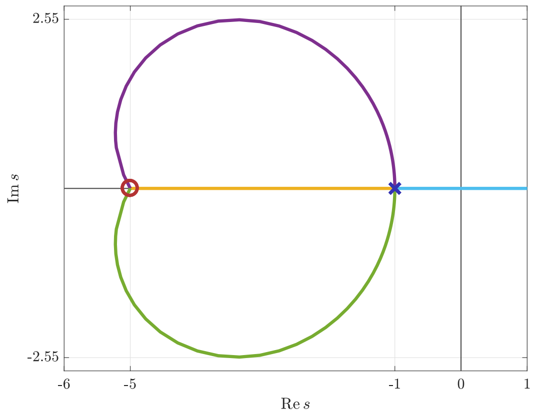


Fig. 5: Root locus for negative gains, $k_p < 0$

2. To tell apart positive and negative gains, we first look at loci on the real axis. Negative gain root-locus will always have a branch going to $+\infty$ (since in that case loci are to the left of an even number of poles and zeros), hence the root-locus in Fig. 3 corresponds to positive gains k_p.

   To sketch the root-locus for negative gains, we go through standard steps. First, we determine that still $G_k(s) = P(s)$. Then we place 4 poles at $s = -1$ and 3 zeros at $s = -5$. We draw the loci on the real axis to the left of an *even* number of poles and zeros. In our case we have zero (which is even) poles and zeros to the left of points in $(-1, \infty)$, 4 poles to the left of $(-5, -1)$, and 7 poles and zeros to the left of $(-\infty, -5)$. Hence, the loci on the real axis will be in $(-5, \infty)$.

   The next step is to determine the behavior as $k_p \to -\infty$. The pole excess is 1, so 3 poles will end up in the zeros (we shall now draw two more branches additionally to the one on the real axis), and one pole will go to infinity along with the asymptote at the positive real axis ($\phi_1 = 0$). The resulting plot is shown in Fig. 5.

3. First, remember that for each gain k_p the 4 poles of the closed loop are positioned in a specific 4 points on each loci. Next, studying the time-responses we recognize that y_1 is the step response of a system containing an integrator (a pole at the origin), while y_2 and y_3 are the step responses of systems containing imaginary-axis poles (non-decaying oscillations). In all cases, the other poles of closed-loop systems must be stable, as their contributions to time responses vanish with time.

   Comparing the root-locus plots in Figs. 3 and 5, we can see that a pole at the origin is only possible for negative controller gains. Hence, we have that

$$y_1 \leftrightarrow k_p = -0.008.$$

   The positive gain root-locus in Fig. 3 has two imaginary-axis crossings: at some lower gain its loci cross at $s_{1,2} \approx \pm j1.64$ and then at some higher gain they cross again at $s_{3,4} \approx \pm j5$. When we have one of those critical gains, two poles of the closed-loop system are positioned on the imaginary axis, while the other loci are in the LHP. Hence, the response (in steady state) will oscillate without decay at a frequency corresponding to the crossing point. As slower oscillations (those with a lower imaginary part) occur at a lower gain and faster oscillations occur at a higher gain, we must have that

$$y_2 \leftrightarrow k_p = 1.885 \quad \text{and} \quad y_3 \leftrightarrow k_p = 0.093$$

   (their frequencies are $\omega \approx 5$ [rad/sec] and $\omega \approx 1.64$ [rad/sec], respectively).

That's all . . . ▽

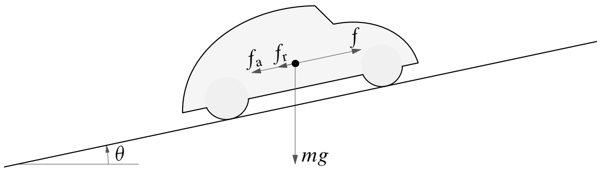


Fig. 6: System for Question 3

Question 3. Consider again the problem of cruise control for a vehicle, like that depicted in Fig. 6. The problem here is to maintain the vehicle velocity v at prespecified levels by changing the driving force f generated by the engine. As we already know (Tutorials 3 and 5), the linearized (around some given velocity v_{eq}) model of this system is

$$y = \frac{1}{ms + \alpha v_{\text{eq}}}(u + d),$$

where m is the actual mass of the vehicle, α is a constant depending on the density of air, the frontal area of the car, and a shape-dependent aerodynamic drag coefficient, and the deviation variables $y := v - v_{\text{eq}}$ and $u := f - 0.5\alpha v_{\text{eq}}^2 - m_0 g(\sin\theta_0 + C_{\text{r}}\cos\theta_0)$, where C_{r} is the (dimensionless) rolling resistance coefficient, m_0 is the *assumed* mass of the vehicle and θ_0 is the *assumed* slope angle of the road. The disturbance signal, which accounts for inaccuracies in the assumed mass and angle, satisfies

$$d(t) = d_0 := m_0 g(\sin\theta_0 + C_{\text{r}}\cos\theta_0) - mg(\sin\theta + C_{\text{r}}\cos\theta),$$

where θ is the actual road slope. In all numerical calculations of what follows we assume the following numerical values:

α	C_{r}	g	m_0	θ_0	v_{eq}
1 [kg/m]	0.01	9.8 [m/sec^2]	1000 [kg]	$12° \approx 0.20944$ [rad]	80 [km/h] ≈ 22.2222 [m/sec]

Assume that the system is controlled by in the unity-feedback scheme, like that in Fig. 1. We saw in Tutorial 6 that a proportional controller is not capable to attain a required new velocity v_{new}. Consider now another controller, PI:

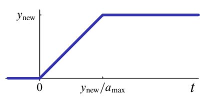
$$C(s) = k_{\text{p}}\Big(1 + \frac{k_{\text{i}}}{s}\Big) \tag{1}$$

for some parameters $k_{\text{p}} > 0$ and $k_{\text{i}} > 0$.

1. Sketch the root-locus plot for the system with respect to the proportional gain k_{p}. How the choice of k_{i} affects it? Under what k_{p} and k_{i} the closed-loop system is stable?

2. What is the steady-state velocity in this case under

$$r_v(t) = \begin{cases} 0 & \text{if } t \le 0 \\ a_{\max} t & \text{if } 0 \le t \le y_{\text{new}}/a_{\max} \\ y_{\text{new}} & \text{if } t \ge y_{\text{new}}/a_{\max} \end{cases} = \tag{2}$$



   for the peak acceleration $a_{\max} = 0.5$ [m/s^2] and $y_{\text{new}} = 10$ [km/h] $= 25/9 \approx 2.78$ [m/sec] under $m = m_0$ and $\theta = \theta_0$?

3. How the choices of k_{p} and k_{i} affect the steady-state error in general, under $m \ne m_0$ and $\theta \ne \theta_0$? Simulate the response of the linearized closed-loop system to r as in (2) under the nominal car mass and a step change of the road slope at $t = 13$ [sec] to $\theta = 13° \approx 0.2269$ [rad] under $k_{\text{p}} \in \{500, 1000, 5000\}$ and $k_{\text{i}} \in \{0.0222, 0.1331\}$.

4. Analyze the nonlinear system with the unity feedback closed-loop PI controller as in (1) (the nonlinear plant dynamics are $m\dot{v} = f - 0.5\alpha v^2 - mg(\sin\theta + C_r\cos\theta)$). What is its steady-state response to the reference signal in (2)?

Solution.

1. The root-locus form in this case has

$$G_k(s) = \frac{1}{ms + \alpha v_{eq}} \cdot \frac{s + k_i}{s} = \frac{s + k_i}{s(ms + \alpha v_{eq})}.$$

It has two real poles (at $s = 0$ and $s = -\alpha v_{eq}/m$) and one real zero (at $s = -k_i$). There is one asymptote with $\phi_0 = \pi$, which goes to infinity along the negative real axis. The loci occupy the following parts of the real axis:

$$\big(-\infty, -\max\{k_i, \alpha v_{eq}/m\}\big) \cup \big(-\min\{k_i, \alpha v_{eq}/m\}, 0\big),$$

all in its negative part. Then the following situations are possible:

(a) If $k_i = \alpha v_{eq}/m$, the root-locus plot effectively coincides with that of a single integrator, $1/s$. The only difference is that we always have an additional closed-loop pole at $s = -\alpha v_{eq}/m$ (canceled pole of the plant). In this case, the closed-loop system is stable for all $k_p > 0$.

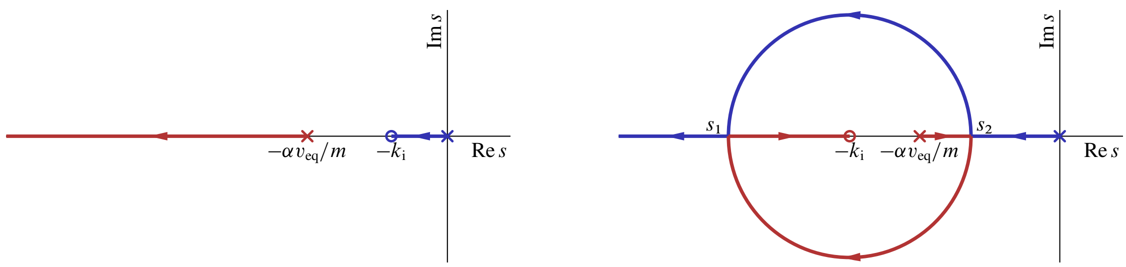
(b) If $0 < k_i < \alpha v_{eq}/m$, the interval $(-k_i, 0)$ contains the whole locus, as it has a pole at one its end, and a zero at the other. Then the interval $(-\infty, -\alpha v_{eq}/m)$ is also the whole locus, starting at $-\alpha v_{eq}/m$ and going to $-\infty$. The resulting plot is presented in Fig. 7(a). As all loci are in the LHP, the closed-loop is stable for all such k_i under all $k_p > 0$.

(c) If $k_i > \alpha v_{eq}/m$, the interval $(-\alpha v_{eq}/m, 0)$ contains two loci (poles at both its ends), as so does the interval $(-\infty, -k_i)$ (zero at one and and asymptote at the other). The loci must then connect through the "non-real" parts of the complex plane. In other words, we must have breakaway and break-in points. To calculate them, we need real points s belonging to the root-locus plot such that

$$\frac{d}{ds} G_k(s) = -\frac{ms^2 + 2mk_i s + \alpha v_{eq} k_i}{s^2(ms + \alpha v_{eq})^2} = 0.$$

The solutions to this equation are

$$s_{1,2} = -k_i \pm k_i \sqrt{1 - \alpha v_{eq}/(mk_i)}.$$



(a) The case $0 < k_i < \alpha v_{eq}/m$

(b) The case $k_i > \alpha v_{eq}/m$

Fig. 7: Root-locus plots for Question 3

Because the discriminant is always positive in our case, there are two real solutions. The point s_1 (which corresponds to "$-$") is clearly located to the left of $-k_\mathrm{i}$, so it belongs to the root-locus and is a break-in point. The point s_2 must be larger than $-\alpha v_\mathrm{eq}/m$ to belong to loci. We have:

$$s_2 > -\alpha v_\mathrm{eq}/m \iff \sqrt{1-\alpha v_\mathrm{eq}/(mk_\mathrm{i})} > 1-\alpha v_\mathrm{eq}/(mk_\mathrm{i}),$$

which is true because $1-\alpha v_\mathrm{eq}/(mk_\mathrm{i}) \in (0,1)$. Thus, s_2 also belongs to the root locus plot and is a breakaway point. The resulting plot is presented in Fig. 7(b). All loci are again in the LHP, so the closed-loop is stable for all such k_i under all $k_\mathrm{p} > 0$.

Thus, the closed-loop system is stable for all $k_\mathrm{p} > 0$ and $k_\mathrm{i} > 0$.

2. Because the closed-loop system is stable for all $k_\mathrm{p} > 0$ and $k_\mathrm{i} > 0$ and because we have an integral action in the controller, the steady-state error must be zero, i.e. we must have

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} y(t) + v_\mathrm{eq} = y_\mathrm{new} + v_\mathrm{eq} =: v_\mathrm{new}.$$

   Of course, we may also show that via calculating steady-state values from the Final Value Theorem.

3. The situation here does not change. As the stability of the closed-loop system does not depend on m and θ, any $k_\mathrm{p} > 0$ and $k_\mathrm{i} > 0$ and the integral action in the controller yield the same result as above. The simulations are presented in Figs. 8.

   Note that $k_\mathrm{i} = 0.0222 = \alpha v_\mathrm{eq}/m$ in this case, so we cancel the pole of the plant by a zero of the controller. The velocity response to the reference signal is then driven by the complementary sensitivity function, which is a first-order system in this case (with a pole at $\{-0.5,-1,-5\}$ for different k_p's). The disturbance response, on the other hand, is driven by T_d, where the canceled pole of the plant at $s = -0.0222$ is still a pole. This pole is slow, relatively to the other pole of the closed-loop system. This is why the decay of the disturbance response is slow.

   Then $k_\mathrm{i} = 0.1331 = 6\alpha v_\mathrm{eq}/m$, which corresponds to the root locus in Fig. 7(b). As k_p grows, one closed-loop pole moves left and another one approaches the zero at $s = -k_\mathrm{i}$. The reference signal response for k_p's around the break-in point s_1 has a dominant zero, which gives rise to overshoot. At larger k_p's this zero is effectively canceled by a closed-loop pole and its effect is negligible. The disturbance response still decays slowly though, because the closed-loop pole approaching $s = -k_\mathrm{i}$ is not canceled there.

4. The PI controller of (1) in the time domain can be expressed as

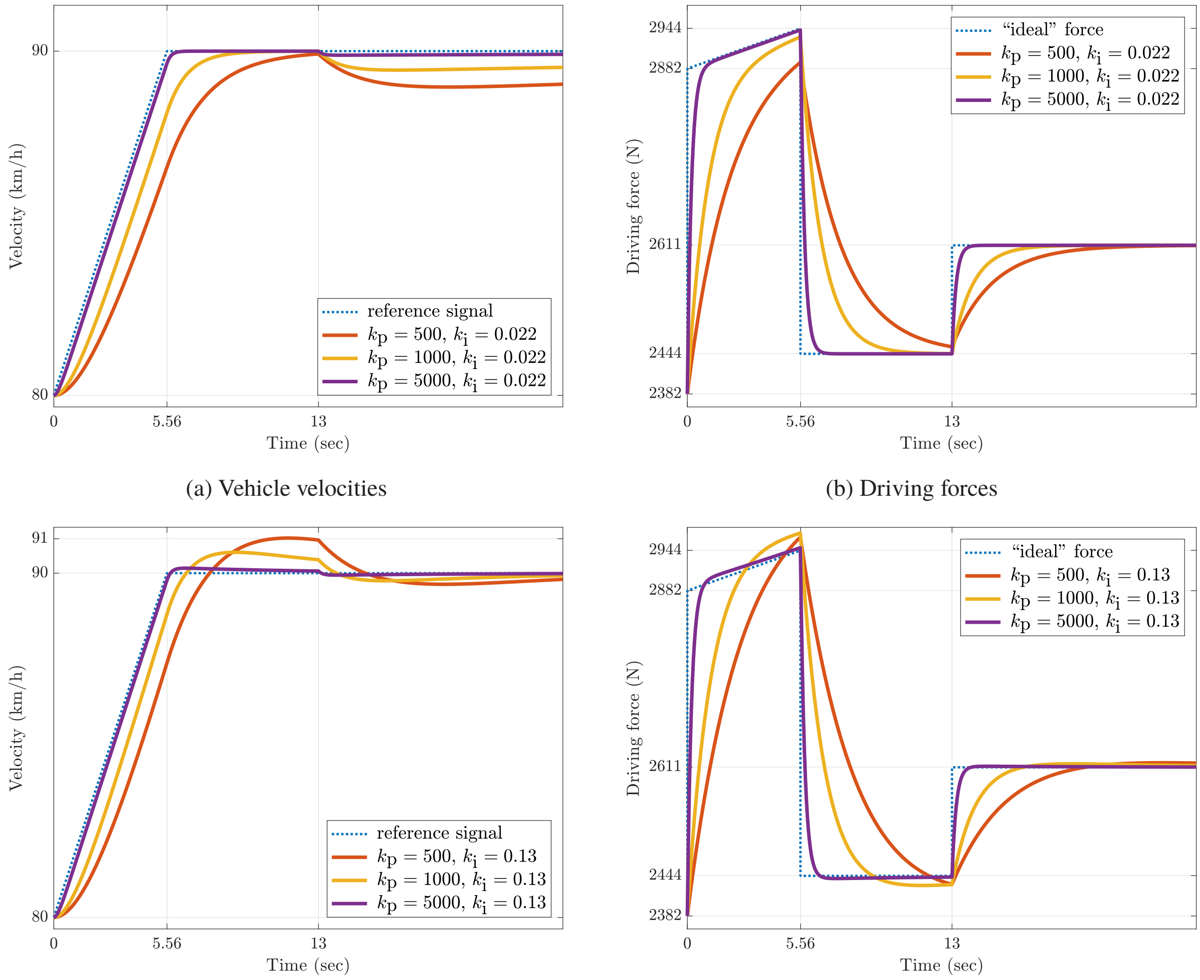
$$\begin{cases} \dot{x}_\mathrm{c}(t) = k_\mathrm{i} e(t) \\ u(t) = k_\mathrm{p} x_\mathrm{c}(t) + k_\mathrm{p} e(t), \end{cases}$$

   where x_c is the state vector of the controller (in fact, $x_\mathrm{c}(t) = k_\mathrm{i}\int_0^t e(\tau)\mathrm{d}\tau$ or $X_\mathrm{c}(s) = (k_\mathrm{i}/s)E(s)$). Combining these equations with those of the car $m\dot{v} = f - 0.5\alpha v^2 - mg(\sin\theta + C_\mathrm{r}\cos\theta)$ and setting

$$f(t) = u(t) + \frac{\alpha}{2}v_\mathrm{eq}^2 + m_0 g(\sin\theta_0 + C_\mathrm{r}\cos\theta_0) \quad \text{and} \quad e(t) = r(t) - v(t) + v_\mathrm{eq},$$

   we end up with the combined dynamics

$$\begin{aligned} \begin{bmatrix} m\dot{v}(t) \\ \dot{x}_\mathrm{c}(t) \end{bmatrix} &= \begin{bmatrix} u(t) + \frac{\alpha}{2}v_\mathrm{eq}^2 + m_0 g(\sin\theta_0 + C_\mathrm{r}\cos\theta_0) - \frac{\alpha}{2}v^2(t) - mg(\sin\theta + C_\mathrm{r}\cos\theta) \\ k_\mathrm{i}(r(t) + v_\mathrm{eq} - v(t)) \end{bmatrix} \\ &= \begin{bmatrix} k_\mathrm{p} x_\mathrm{c}(t) - k_\mathrm{p}(v(t) - r(t) - v_\mathrm{eq}) - \frac{\alpha}{2}(v^2(t) - v_\mathrm{eq}^2) + d_0 \\ -k_\mathrm{i}(v(t) - r(t) - v_\mathrm{eq}) \end{bmatrix} \end{aligned}$$



(a) Vehicle velocities

(b) Driving forces

(c) Vehicle velocities

(d) Driving forces

Fig. 8: Velocity responses to r_v from (2) under changed slope ("ideal" force is that under which $v \equiv r_v$)

The steady-state response (assuming stability[1]) under $\lim_{t\to\infty} r(t) = y_{\rm new}$ corresponds to $\dot{v} = 0$ and $\dot{x}_{\rm c} = 0$, i.e.

$$\begin{bmatrix} k_{\rm p}x_{\rm c} - \frac{\alpha}{2}(v^2 - v_{\rm eq}^2) - k_{\rm p}(v - v_{\rm new}) + d_0 \\ -k_{\rm i}(v - v_{\rm new}) \end{bmatrix} = 0,$$

where $v_{\rm new} = y_{\rm new} + v_{\rm eq}$. Hence, the unique equilibrium of the closed-loop system is

$$v = v_{\rm new} \quad \text{and} \quad k_{\rm p}x_{\rm c} = \frac{\alpha}{2}(v_{\rm new}^2 - v_{\rm eq}^2) - d_0.$$

This implies that the equilibrium is always at the required $v = v_{\rm new}$. As a matter of fact, the driving force at the equilibrium is

$$f = \frac{\alpha}{2}v_{\rm new}^2 + mg(\sin\theta + C_{\rm r}\cos\theta),$$

which is the equilibrium force in the open-loop analysis, exactly as it should be.

That's all … ▽

[1]Although we have no tools to analyze stability of nonlinear systems, we assume that the closed-loop system is stable.