Introduction to Control (034040)

tutorial 6

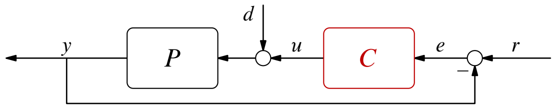


Fig. 1: Unity feedback closed-loop system

Question 1. Consider the unity feedback system in Fig. 1. Sketch (qualitatively) the root-loci of the following systems with respect to the gain k. Is it possible to stabilize the system with $k > 0$?

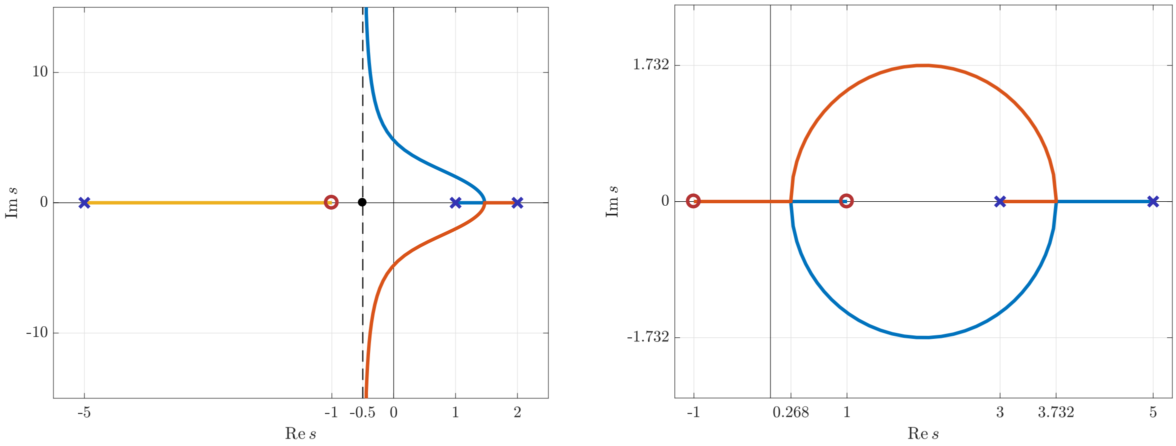
1. $P(s) = \dfrac{s+1}{(s-1)(s-2)(s+5)}$ and $C(s) = k$
2. $P(s) = \dfrac{(s-1)(s+1)}{(s-3)(s-5)}$ and $C(s) = k$
3. $P(s) = \dfrac{1}{(s-1)^3}$ and $C(s) = k\,\dfrac{s-3}{s-7}$
4. $P(s) = \dfrac{1}{s^2}$ and $C(s) = \dfrac{s+k}{s+2}$

Solution. To sketch root-locus plots, we can use the following guidelines:

- construct the transfer function $G_k(s)$ of the root locus form $G_k(s) = -1/k$;
- place poles and zeros of $G_k(s)$ on the pole-zero map;
- draw the loci on the real axis (to the left of an odd number of real poles and zeros of $G_k(s)$);
- if m is the number of zeros of $G_k(s)$, m loci end in these zeros, so sketch this connection (the others will go to infinity via asymptotes);
- find the center of gravity and and angles of the asymptotes and sketch the loci that go to infinity along with the asymptotes.

Now let us study the given systems

1. In this case
$$G_k(s) = P(s) = \frac{s+1}{(s-1)(s-2)(s+5)}$$
and it has three poles (at $s = -5$, $s = 1$, and $s = 2$), a zero (at $s = -1$), and a pole excess of 2. Thus, the real-axis loci are in $[-5, -1]$ (this is the whole locus, it starts at a pole and ends at a zero) and $[1, 2]$ (this contains 2 loci, each starts at the corresponding pole). There are two loci going to infinity with two asymptotes. Their angles are
$$\phi_l = \frac{-\pi + 2\pi l}{2}, \quad l = 0, 1 \quad \Longrightarrow \quad \phi_1 = -\frac{\pi}{2} \quad \text{and} \quad \phi_2 = \frac{\pi}{2}.$$



(a) Item 1

(b) Item 2

Fig. 2: Root locus plots for Question 1

The center of gravity is

$$\sigma_c = \frac{(+1+2-5)-(-1)}{2} = -\frac{1}{2}.$$

This results in the root locus given in Fig. 2(a).

We can see that this closed-loop system (which is unstable in open loop) can be stabilized in closed loop with a proportional controller $C(s) = k$ for a sufficiently high gain k.

Note that in more complicated cases, with a nontrivial combination and proximity of poles and zeros, these rules do not always determine the shape of the root loci unambiguously. In such cases a numerical study will be required. We've actually seen it for a very similar case to item 1 in the lecture. Yet, in many cases the approach described above could help us quickly acquire an intuition regarding the root locus plot and system stability.

2. In this case

$$G_k(s) = P(s) = \frac{(s-1)(s+1)}{(s-3)(s-5)}$$

and it has two poles (at $s = 3$ and $s = 5$), two zero (at $s = -1$ and $s = 1$), and a pole excess of 0. Thus, the real-axis loci are in $[-1, 1]$ (this is where two loci end) and $[3, 5]$ (this is where two loci start). There are no loci going to infinity. The loci start at the poles, approach each other along the real axis, then break away at some point in $[3, 5]$. They then meet again at a break-in point in $[-1, 1]$ and approach the zeros along the real axis. To calculate the breakaway and break-in points, we need to find real solutions to

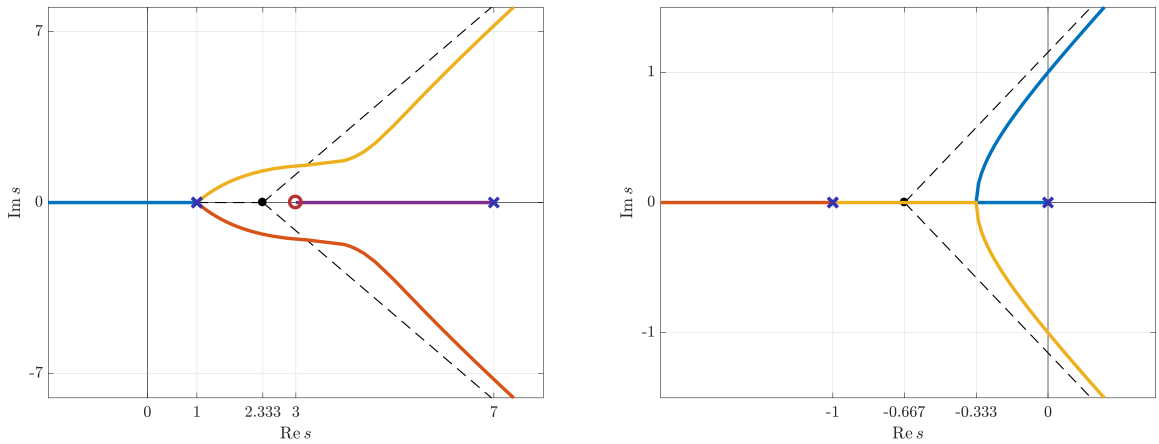
$$0 = \frac{\mathrm{d}G_k(s)}{\mathrm{d}s} = -\frac{8(s^2-4s+1)}{(s-5)^2(s-3)^2} = -\frac{8(s-2-\sqrt{3})(s-2+\sqrt{3})}{(s-5)^2(s-3)^2}.$$

Hence, the breakaway point is $s = 2+\sqrt{3} \approx 3.73205$ (the one in $[3, 5]$) and the break-in point is $s = 2-\sqrt{3} \approx 0.267949$ (the one in $[-1, 1]$). This results in the root locus given in Fig. 2(b).

We can see that this closed-loop system cannot be stabilized by a proportional controller $C(s) = k$, because at every k one locus is always in the RHP.

3. In this case

$$G_k(s) = \frac{1}{(s-1)^3} \cdot \frac{s-3}{s-7} = \frac{s-3}{(s-1)^3(s-7)}$$



(a) Item 3

(b) Item 4

Fig. 3: Root locus plots for Question 1

and it has four poles (three at $s = 1$ and one at $s = 7$), a zero (at $s = 3$), and a pole excess of 3. Thus, the real-axis loci are in $(-\infty, 1]$ (this is the whole locus, it starts at a pole and ends at $-\infty$) and $[3, 7]$ (this is also the whole locus, starting at a pole and ending at a zero). There are three loci that go to infinity with three asymptotes. Their angles are

$$\phi_l = \frac{-\pi + 2\pi l}{3}, \quad l = 0, 1, 2 \quad \Longrightarrow \quad \phi_1 = -\frac{\pi}{3}, \quad \phi_2 = \frac{\pi}{3}, \quad \text{and} \quad \phi_3 = \pi.$$

The center of gravity is

$$\sigma_c = \frac{(+1 + 1 + 1 + 7) - (+3)}{3} = \frac{7}{3} \approx 2.333.$$

This results in the root locus given in Fig. 3(a).

Again, we see that at least tree loci (i.e. at least 3 closed-loop poles) will stay in the RHP for any $k > 0$.

4. To derive the root-locus form in this case, consider the closed-loop characteristic polynomial

$$\chi_{cl}(s) = s + k + s^2(s + 2) = s(s + 1)^2 + k.$$

Hence, $\chi_{cl}(s) = 0$ reads $G_k(s) = -1/k$ for

$$G_k(s) = \frac{1}{s(s + 1)^2},$$

having three poles (two at $s = -1$ and one at $s = 0$), no zeros, and a pole excess of 3. In this case the real-axis loci are in the whole non-positive semi-axis $(-\infty, 0]$ (one whole locus, in $(-\infty, -1]$, and parts of two others in $[-1, 0]$). All loci go to infinity along with asymptotes with the angles

$$\phi_l = \frac{-\pi + 2\pi l}{3}, \quad l = 0, 1, 2 \quad \Longrightarrow \quad \phi_1 = -\frac{\pi}{3}, \quad \phi_2 = \frac{\pi}{3}, \quad \text{and} \quad \phi_3 = \pi.$$

(as in the previous item) and the center of gravity

$$\sigma_c = \frac{(-1 - 1 + 0)}{3} = -\frac{2}{3} \approx -0.667.$$

The breakaway point in this case can be calculated as

$$0 = \frac{\mathrm{d}G_k(s)}{\mathrm{d}s} = -\frac{(3s+1)(s+1)}{s^2(s+1)^4} = -\frac{3s+1}{s^2(s+1)^3} \iff s = -\frac{1}{3} \approx -0.333.$$

Two loci that go along the asymptotes with angles $\pm\pi/3$ end up in the RHP. Hence, they cross the imaginary axis at some finite k. This happens at the points $\pm \mathrm{j}\omega_0$, where ω_0 is determined via the phase rule as follows:

$$\arg G_k(\mathrm{j}\omega) = -\frac{\pi}{2} - 2\arctan(\omega_0) = -\pi \iff \arctan(\omega_0) = \frac{\pi}{4} \iff \omega_0 = 1.$$

The resulting root-locus plot is presented in Fig. 3(b).

We can clearly see that all loci are in the LHP for sufficiently small k's, up to the point where the imaginary axis is crossed. This is the point where (by the gain rule)

$$k = \frac{1}{|G_k(\mathrm{j})|} = |\mathrm{j}(1+\mathrm{j})^2| = 2.$$

Thus, the closed-loop system is stable iff $k \in (0, 2)$.

That's all . . . ∇

Question 2. Consider the system $P(s) = (s-1)/(s-2)$ controlled in closed loop with unity feedback.

1. Can the system be stabilized by a proportional controller of the form $C(s) = k_p$ for some $k_p > 0$?
2. Can the system be stabilized by a *stable* controller, i.e. $C(s) = k_p N_C(s)/D_C(s)$ for some Hurwitz $D_C(s)$?
3. Discuss the requirements for a controller of the form $C(s) = k_p \tilde{C}(s)$ for stabilizing the system. Considering the controller $C(s) = k_p/(s-a)$, for some $k_p > 0$, find the requirements on a
   (a) by analyzing roots of the closed-loop characteristic polynomial,
   (b) by using root-locus principles.
4. Find the range of k_p for which the closed-loop system is stable for the controller in the previous item.

Solution.

1. The root locus form of this case is that with $G_k(s) = P(s)$ and $k = k_p$. It has a RHP pole and and a RHP zero, so by the real axis rule, the whole locus should be located between them (see Fig. 4). Therefore the closed loop is unstable for any $k_p > 0$, meaning the system cannot be stabilized by a proportional controller.

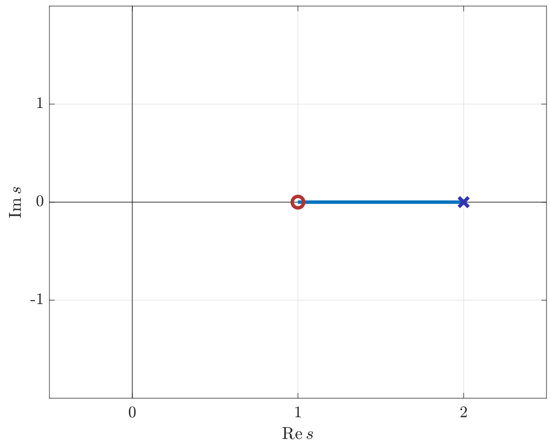


Fig. 4: Root locus for $C(s) = k_p$

2. It is convenient to consider the root locus with respect to the gain k_p. In this case

$$G_k(s) = P(s)\frac{N_C(s)}{D_C(s)} = \frac{(s-1)N_C(s)}{(s-2)D_C(s)} \quad \text{and} \quad k = k_p$$

   This transfer function still has RPH pole (at $s = 2$) and zero (at $s = 1$). Thus, to stabilize the system we have to break the locus in Fig. 4. This could be done by placing *real* poles or zeros of the controller to the right of $s = 1$. But the roots of $D_C(s)$ are assumed to lie in the LHP. Hence, we can only manipulate RHP zeros of the controller, i.e. RPH roots of $N_C(s)$. Our options are

   (i) place zeros in $s \in (1, 2)$: this won't help, because then we will still have the whole locus in the RHP, starting at $s = 2$ and going left to the closest zero;

   (ii) place an even number zeros in $s \in (2, \infty)$: this won't help, because this does not break the locus $2 \to 1$;

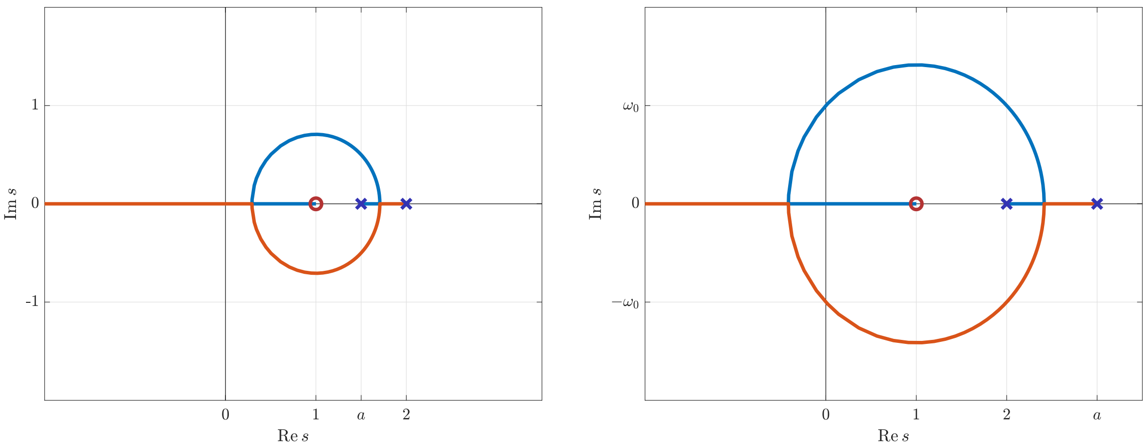
(iii) place an odd number zeros in $s \in (2, \infty)$: this won't help either, because this creates a locus from $s = 2$ to the closest zero to the right of it.

In fact, what we need is an additional *pole* at, say, $s = a > 1$, to break that locus. Then two loci will start at $s = a$ and $s = 2$, meet at some breakaway point between them, and leave the positive real axis. We shall then endeavor to direct those two loci to the LHP by placing a zero there (although it might not be clear yet where exactly and where exactly to put the additional RHP pole).

3. We saw in the previous item that a RHP pole is required in the controller to break the pole-zero sequence, i.e. in the range $s \in (1, \infty)$. For the proposed controller structure this reads $a > 1$. The root locus form is

$$G_k(s) = \frac{s-1}{(s-2)(s-a)} \quad \text{and} \quad k = k_\text{p}.$$

We sketch the loci on the real axis between the two poles and from the RHP zero at $s = 1$ to $-\infty$. The two loci will start at the poles $s = a$ and $s = 2$, meet at some breakaway point between them and leave the positive real axis. These loci then return to the real axis at some break-in point and split, one towards the zero and the other towards $-\infty$. For some choices of a the break-in point is located in the RHP, see Fig. 5(a), in which case one of the loci remains in the RHP for all k_p and the



(a) incorrect choice of a

(b) correct choice of a

Fig. 5: Root locus for $C(s) = k_\text{p}/(s - a)$

closed-loop system is always unstable. We should aim at choosing a so that the break-in point is in the LHP, like that depicted in Fig. 5(b). Below two approaches to this choice are considered:

(a) The (second-order) characteristic polynomial is

$$\chi_\text{cl}(s) = (s-2)(s-a) + k_\text{p}(s-1) = s^2 + (k_\text{p} - 2 - a)s + 2a - k_\text{p}$$

giving $k_\text{p} > 2 + a$ and $k_\text{p} < 2a$. These inequalities are feasible iff $2 + s < 2a$, which yields $a > 2$.

(b) Using root locus rules, we can parametrically find the break-in point and require it to be in the LHP from the relation

$$\frac{\mathrm{d}}{\mathrm{d}s} G_k(s) = \frac{1(s^2 - (a+2)s + 2a) - (s-1)(2s - a - 2)}{(s-2)^2(s-a)^2} = 0.$$

This quadratic equation has two solutions $p_{1,2} = 1 \pm \sqrt{-1+a}$. First, note that breakaway / break-in points exist only if $a > 1$, as expected. We then require that the leftmost point, which must correspond to the break-in in Fig. 5(b), satisfies $1 - \sqrt{-1+a} < 0$ giving $a > 2$, i.e. to the right of the plant pole at $s = 2$.

4. We now in the position to find the critical gains $k_{\mathrm{cr1}} < k_{\mathrm{p}} < k_{\mathrm{cr2}}$ for which the closed loop is stable.

   The maximal gain k_{cr2} corresponds to the gain for which one locus crosses the origin from left to right in Fig. 5(b). This corresponds to the closed-loop root at $s = 0$. We can then apply the gain rule to find the gain for which a locus crosses the origin:

$$\frac{1}{k_{\mathrm{p}}} = |G_k(0)| = \left|\frac{-1}{(-2)(-a)}\right| = \frac{1}{2a}$$

   hence $k_{\mathrm{p}} < 2a = k_{\mathrm{cr2}}$.

   The minimal gain corresponds to the points at which two loci cross the imaginary axis, say at $s = \pm \mathrm{j}\omega_0$. This gives 2 unknowns: the crossing frequency ω_0 and the critical gain for which the root locus reaches these poles k_{cr1} (a is a parameter). To find these unknowns we first use the phase rule, to find ω_0 (since the phase does not depend on the gain) and then substitute this ω_0 into the gain rule to find k_{cr1}. The phase rule reads

$$\begin{aligned}\arg G_k(\mathrm{j}\omega_0) &= \pi - \arctan\omega_0 - \left(\pi - \arctan\frac{\omega_0}{2} + \pi - \arctan\frac{\omega_0}{a}\right)\\ &= -\pi - \arctan\omega_0 + \arctan\frac{\omega_0}{2} + \arctan\frac{\omega_0}{a} = -\pi + 2\pi n.\end{aligned}$$

   For a given a we could solve this equation numerically. To solve it analytically for a parametric a we shall use (twice) the arctangent summation formula

$$\arctan\alpha \pm \arctan\beta = \arctan\frac{\alpha \pm \beta}{1 \mp \alpha\beta},$$

   which will eventually give $\omega_0 = \pm\sqrt{a-2}$. The gain rule reads then

$$\begin{aligned}\frac{1}{k_{\mathrm{p}}} &= |G_k(\mathrm{j}\sqrt{a-2})| = \left|\frac{\mathrm{j}\sqrt{a-2}-1}{(\mathrm{j}\sqrt{a-2}-2)(\mathrm{j}\sqrt{a-2}-a)}\right| = \frac{\sqrt{a-2+1}}{\sqrt{a-2+4}\sqrt{a-2+a^2}}\\ &= \frac{\sqrt{a-1}}{\sqrt{(a+2)^2(a-1)}} = \frac{1}{a+2},\end{aligned}$$

   whence the minimal critical gain $k_{\mathrm{p}} > a + 2 = k_{\mathrm{cr1}}$.

   Note that the derived k_{cr1} and k_{cr2} agree with the bounds on the stabilizing k_{p} derived via the analysis of the characteristic polynomial above.

That's all ... ▽

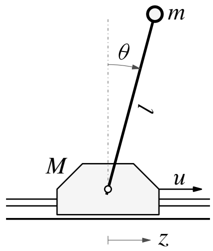


Fig. 6: Inverted pendulum on a cart for Question 3

Question 3 (self-study)**.** Consider an inverted pendulum which consists of a point mass m on a mass-less rod of length l installed on a cart of mass M. An external force $u(t)$ is acting on the cart. The controlled output is assumed to be the acceleration of the cart, $y(t) = \ddot{z}(t)$. We know from Tutorial 6 that the transfer function of this system, linearized around the pendulum "up" position is (here g is the standard gravity)

$$P(s) = \frac{ls^2 - g}{Mls^2 - (M+m)g} = \frac{0.1(s-3.13)(s+3.13)}{(s-3.834)(s+3.834)}, \; P(s) = \frac{ls^2 - g}{Mls^2 - (M+m)g} = \frac{1}{M}\frac{s^2-\gamma^2}{s^2-\mu^2\gamma^2},$$

where

$$\gamma := \sqrt{\frac{g}{l}} > 0 \quad \text{and} \quad \mu := \sqrt{1 + \frac{m}{M}} > 1.$$

Consider the control of this system by the unity-feedback system like that in Fig. 1.

1. Is it possible to stabilize this system by a proportional controller, i.e. by $C(s) = k_{\rm p}$ for some $k_{\rm p} > 0$? Explain via root-locus arguments.

2. Is it possible to stabilize this system by a *stable* controller having the positive high-frequency gain, i.e. such that
$$C(s) = \frac{k_{\rm p} N_C(s)}{D_C(s)} = \frac{k_{\rm p}(s^m + +b_{m-1}s^{m-1} \cdots + b_1 s + b_0)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{1}$$
for $m \le n$, $k_{\rm p} > 0$, and Hurwitz $D_C(s)$? Explain via root-locus arguments. What property this controller lacks to stabilize the system?

3. Consider now
$$C(s) = \frac{k_{\rm p}(s + \alpha - \mu\gamma)(s + \mu\gamma)}{(s - \alpha - \gamma)(s + \gamma)} \tag{2}$$
for $k_{\rm p} > 0$ and $\alpha > \mu\gamma$ (it cancels *stable* pole and zero of $P(s)$ and has a LHP zero at $s = -\alpha + \mu\gamma$ and a RHP pole at $s = \alpha + \gamma$). Under what conditions on $k_{\rm p}$ and α this controller stabilizes the plant? Explain via analyzing roots of the closed-loop characteristic polynomial.

4. Do the same, but using root-locus arguments. Find then $k_{\rm p}$ and α such that the closed-loop system has all its poles at $s = -\gamma$.

5. Simulate the closed-loop system under the the pulse input disturbance $d(t) = \mathbb{1}(t-0.5) - \mathbb{1}(t-2.5)$ with $m = 5$ [kg], $M = 10$ [kg], $l = 1$ [m], and $g = 9.8$ [m/sec^2] for the parameters $k_{\rm p}$ and α chosen in the previous item.

6. Simulate the closed-loop system in the case when the standard gravity in the plant model is actually is $g = 9.80665$ [m/sec^2] (affecting γ), whereas the controller is still designed assuming $g = 9.8$ [m/sec^2] (i.e. pole-zero cancellations between the plant and the controller are not exact).

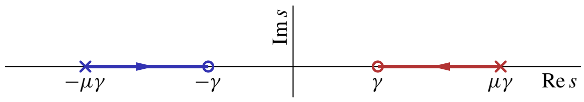


Fig. 7: Root locus for the proportional $C(s)$

Solution. The root-locus form, which is the basis for all constructions / reasonings, is

$$\frac{\beta_m s^m + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} = G_k(s) = -\frac{1}{k}, \tag{RLF}$$

where k is a parameter running from 0 to $+\infty$ and $G_k(s)$ is a given proper transfer function. Below we use arguments corresponding to the case $\beta_m > 0$.

1. The root locus form (RLF) in this case is that with $G_k(s) = P(s)$ and $k = k_\text{p}$. It has two real poles and two real zeros, see Fig. 7. By the real axis rule, the loci will be located between the RHP pole and zero and between the LHP pole and zero (starts at poles and ends at zeros, as always). It is clearly seen that one locus (the red one) is located entirely in the RHP. Hence, the closed-loop system is stable for no k_p, i.e. the pendulum cannot be stabilized by a proportional controller.

2. It is convenient to consider root locus with respect to the gain k_p. In this case

   $$G_k(s) = P(s)\frac{N_C(s)}{D_C(s)} = \frac{1}{M}\frac{(s^2-\gamma^2)N_C(s)}{(s^2-\mu^2\gamma^2)D_C(s)} \quad \text{and} \quad k = k_\text{p}.$$

   This transfer function still has RPH pole (at $s = \mu\gamma$) and zero (at $s = \gamma$). Thus, to stabilize the system we have to break the red locus in Fig. 7. This could be done by placing *real* poles or zeros of the controller to the right of $s = \gamma$. But the roots of $D_C(s)$ are assumed to lie in the LHP. Hence, we can only manipulate RHP zeros of the controller, i.e. RPH roots of $N_C(s)$. Our options are

   (a) place zeros in $s \in (\gamma, \mu\gamma)$: this won't help, because then we will still have the whole locus in the RHP, starting at $s = \mu\gamma$ and going left to the closest zero;

   (b) place an even number zeros in $s \in (\mu\gamma, \infty)$: this won't help, because this does not break the locus $\mu\gamma \to \gamma$;

   (c) place an odd number zeros in $s \in (\mu\gamma, \infty)$: this won't help either, because this creates a locus from $s = \mu\gamma$ to the closest zero to the right of it.

   In fact, what we need is an additional *pole* at, say, $s_0 > \gamma$, to break that locus. Then two loci will start at $s = s_0$ and $s = \mu\gamma$, meet at some breakaway point between them, and leave the positive real axis. We shall then endeavor to direct those two loci to the LHP by placing a zero there (although it might not be clear yet where exactly and where exactly to put the additional RHP pole).

3. The characteristic polynomial in this case is

   $$\begin{aligned}\chi_\text{cl}(s) &= (s^2-\gamma^2)\cdot k_\text{p}(s+\alpha-\mu\gamma)(s+\mu\gamma) + M(s^2-\mu^2\gamma^2)\cdot(s-\alpha-\gamma)(s+\gamma)\\ &= (s+\gamma)(s+\mu\gamma)\big(k_\text{p}(s-\gamma)(s+\alpha-\mu\gamma) + M(s-\mu\gamma)(s-\alpha-\gamma)\big)\\ &= (s+\gamma)(s+\mu\gamma)\\ &\quad\times\big((k_\text{p}+M)s^2 + ((\alpha-\gamma-\mu\gamma)k_\text{p} - (\alpha+\gamma+\mu\gamma)M)s + ((\alpha+\gamma)M\mu - (\alpha-\mu\gamma)k_\text{p})\gamma\big)\end{aligned}$$

   It clearly has two stable roots at $s = -\gamma$ and $s = -\mu\gamma$ and two more roots of the polynomial

   $$\chi_1(s) = (k_\text{p}+M)s^2 + ((\alpha-\gamma-\mu\gamma)k_\text{p} - (\alpha+\gamma+\mu\gamma)M)s + ((\alpha+\gamma)M\mu - (\alpha-\mu\gamma)k_\text{p})\gamma,$$

whose location we have to determine. Because the leading coefficient of $\chi_1(s)$ is positive (under our assumptions), the roots of $\chi_1(s)$ are in the LHP iff the other coefficients are positive as well. In other words, we have to guarantee that

$$(\alpha-\gamma-\mu\gamma)k_{\mathrm{p}} > (\alpha+\gamma+\mu\gamma)M \quad \text{and} \quad (\alpha-\mu\gamma)k_{\mathrm{p}} < (\alpha+\gamma)M\mu.$$

The first inequality above requires $\alpha > (\mu+1)\gamma$ and then

$$\frac{\alpha+\gamma+\mu\gamma}{\alpha-\gamma-\mu\gamma}M < k_{\mathrm{p}} < \frac{\alpha+\gamma}{\alpha-\mu\gamma}M\mu. \qquad (\spadesuit)$$

This condition is not empty iff

$$\frac{\alpha+\gamma+\mu\gamma}{\alpha-\gamma-\mu\gamma} < \frac{\alpha+\gamma}{\alpha-\mu\gamma}\mu \iff \frac{\alpha}{\gamma} > \frac{\mu^2+1}{\mu-1} > \mu+1$$

(the last inequality is true because $(\mu+1)(\mu-1) = \mu^2-1 < \mu^2+1$). Thus, there are stabilizing k_{p} and α iff α is sufficiently large, in a sense that

$$\alpha > \frac{\mu^2+1}{\mu-1}\gamma. \qquad (\clubsuit)$$

Admissible k_{p}'s are then easy to choose from ($\spadesuit$).

4. It is again convenient to consider root locus with respect to the gain k_{p}. In this case

$$G_k(s) = P(s)\frac{(s+\alpha-\mu\gamma)(s+\mu\gamma)}{(s-\alpha-\gamma)(s+\gamma)} = \frac{1}{M}\frac{(s+\alpha-\mu\gamma)(s-\gamma)}{(s-\alpha-\gamma)(s-\mu\gamma)} \quad \text{and} \quad k = k_{\mathrm{p}}$$

(it should be remembered that there are two additional closed-loop poles, at $s=-\gamma$ and $s=-\mu\gamma$, but they do not move as k_{p} changes). From the reasonings of item 2, the controller pole should be to the right of the plant unstable zero and the controller zero should be in the LHP. Hence, the assumption $\alpha > \mu\gamma$. The root locus for this $G_k(s)$ should then look like that depicted in Fig. 8. We can see that the zero of the controller is chosen to be at $s = \mu\gamma-\alpha$ to keep symmetry in the pole-zero map (which pays off in calculations below). The breakaway ($s = s_1$) and break-in ($s = s_2$) points are the *real* solutions to

$$\frac{\mathrm{d}}{\mathrm{d}s}G_k(s) = \frac{-2s^2+2(1+\mu)\gamma s-(\alpha+\gamma-\alpha\mu+\gamma\mu^2)\gamma}{M(s-\alpha-\gamma)^2(s-\gamma\mu)^2}\alpha = 0.$$

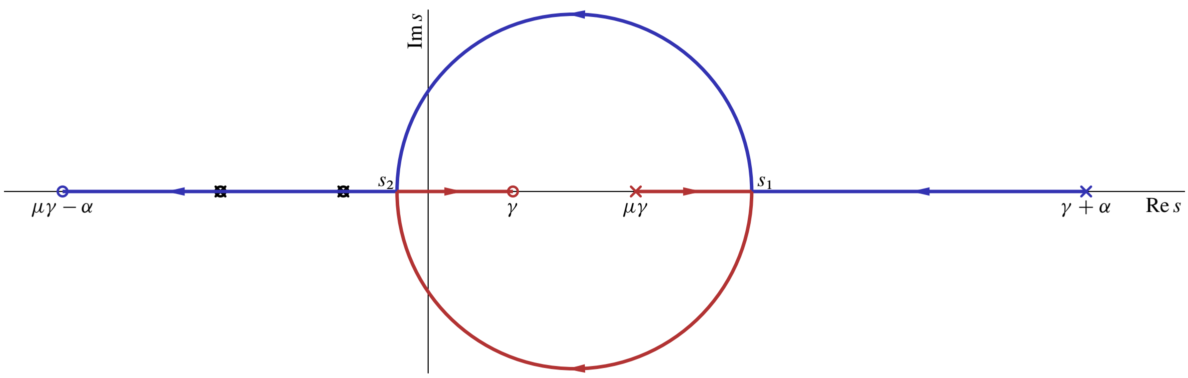


Fig. 8: Root locus for $C(s)$ in (2)

This is a quadratic equation, with two solutions

$$s_{1,2} = \frac{1}{2}\Big((1+\mu)\gamma \pm \sqrt{\gamma(\mu-1)(2\alpha+\gamma-\gamma\mu)}\Big). \tag{$\diamondsuit$}$$

These solutions are real iff

$$\gamma(\mu-1)(2\alpha+\gamma-\gamma\mu) \geq 0 \iff \alpha \geq \frac{\mu-1}{2}\gamma \tag{$\heartsuit$}$$

(remember, $\gamma > 0$ and $\mu > 1$). The break-in point is the smallest solution. In order to have loci in the LHP, we need that smallest solution to be negative then. In other words, we need

$$(1+\mu)\gamma - \sqrt{\gamma(\mu-1)(2\alpha+\gamma-\gamma\mu)} < 0 \iff \alpha > \frac{\mu^2+1}{\mu-1}\gamma,$$

which is exactly ($\clubsuit$). This condition implies ($\heartsuit$), because

$$\frac{\mu^2+1}{\mu-1} > \frac{\mu-1}{2} \iff 2\mu^2+2 > \mu^2+1-2\mu \iff \mu^2+2\mu+1 = (\mu+1)^2 > 0.$$

Our next step is to determine the minimal and the maximal admissible gains. The former is the gain at which the loci in Fig. 8 cross the imaginary axis and the latter is the gain at which the red locus crosses the origin. Crossing the origin is easy. Indeed, all we need is the gain rule

$$\frac{1}{k_{\rm p}} = |G_k(0)| = \left|-\frac{1}{M}\frac{(\alpha-\mu\gamma)\gamma}{(\alpha+\gamma)\mu\gamma}\right| = \frac{\alpha-\mu\gamma}{(\alpha+\gamma)M\mu},$$

which is the inverse of the upper bound in ($\spadesuit$). To calculate the gain at crossing the imaginary axis, we need to calculate the point, say $s = j\omega_0$, at which that happens. To this end, consider the pole-zero map of the system in Fig. 9. The angles there verify

$$\cot\phi_1 = -\frac{\gamma}{\omega_0}, \quad \cot\psi_1 = -\frac{\mu\gamma}{\omega_0}, \quad \cot\phi_2 = \frac{\alpha-\mu\gamma}{\omega_0}, \quad \text{and} \quad \cot\psi_2 = -\frac{\alpha+\gamma}{\omega_0}.$$

Hence,

$$\cot(\phi_1-\psi_2) = \frac{1+\cot\phi_1\cot\psi_2}{\cot\phi_1-\cot\psi_2} = \frac{\omega_0^2+\gamma(\alpha+\gamma)}{\alpha\omega_0},$$
$$\cot(\phi_2-\psi_1) = \frac{1+\cot\phi_2\cot\psi_1}{\cot\phi_2-\cot\psi_1} = \frac{\omega_0^2-\mu\gamma(\alpha-\mu\gamma)}{\alpha\omega_0}$$

and then

$$\cot(\phi_1+\phi_2-\psi_1-\psi_2) = \frac{\alpha^2\omega_0^2-(\omega_0^2+\gamma(\alpha+\gamma))(\omega_0^2-\mu\gamma(\alpha-\mu\gamma))}{\omega_0\alpha(2\omega_0^2+\gamma\alpha+\gamma^2-\mu\gamma\alpha+\mu^2\gamma^2)}.$$

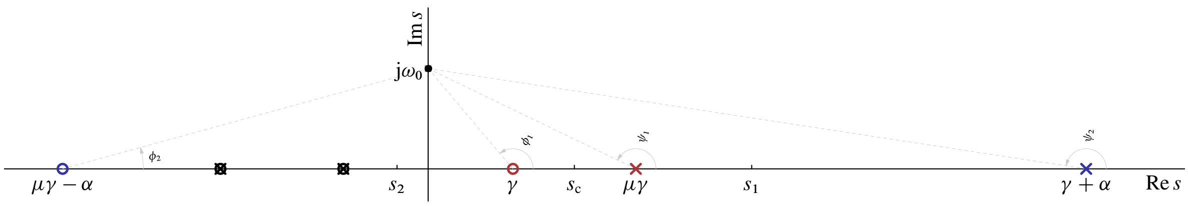


Fig. 9: Angles at root locus for $C(s)$ in (2)

We know that at the imaginary axis the phase rule says that $\phi_1+\phi_2-\psi_1-\psi_2 \equiv \pi \pmod{2\pi}$, which implies that the denominator above must be zero. One possibility, $\omega_0 = 0$, was already discussed. The other possibility is

$$\omega_0^2 = \frac{\gamma((\mu-1)\alpha - (\mu^2+1)\gamma)}{2} > 0.$$

Then the magnitude

$$|G_k(\mathrm{j}\omega_0)|^2 = \frac{1}{M^2}\frac{(\omega_0^2+(\alpha-\mu\gamma)^2)(\omega_0^2+\gamma^2)}{(\omega_0^2+(\alpha+\gamma)^2)(\omega_0^2+\mu^2\gamma^2)} = \left(\frac{1}{M}\frac{\alpha-\gamma-\mu\gamma}{\alpha+\gamma+\mu\gamma}\right)^2,$$

from which the gain rule yields the lower bound in (♠).

Finally, it is quite evident from Fig. 8 that for every stabilizing k_p there is only one LHP point where two poles are equal. This happens in the break-in point, $s = s_2$, where s_2 is given by the "–" part of (◇). Thus, all we need is to resolve $s_2 = -\gamma$ in α, which yields

$$\alpha = \frac{\gamma(\mu^2+2\mu+5)}{\mu-1}. \tag{$*$}$$

The gain k_p can then be determined from the gain rule:

$$\frac{1}{k_\mathrm{p}} = |G_k(s_2)| = \left|\frac{1}{M}\frac{(s_2+\alpha-\mu\gamma)(s_2-\gamma)}{(s_2-\alpha-\gamma)(s_2-\mu\gamma)}\right| = \frac{4}{M(\mu+1)^2}.$$

Hence, we end up with

$$k_\mathrm{p} = \frac{M(\mu+1)^2}{4}. \tag{$**$}$$

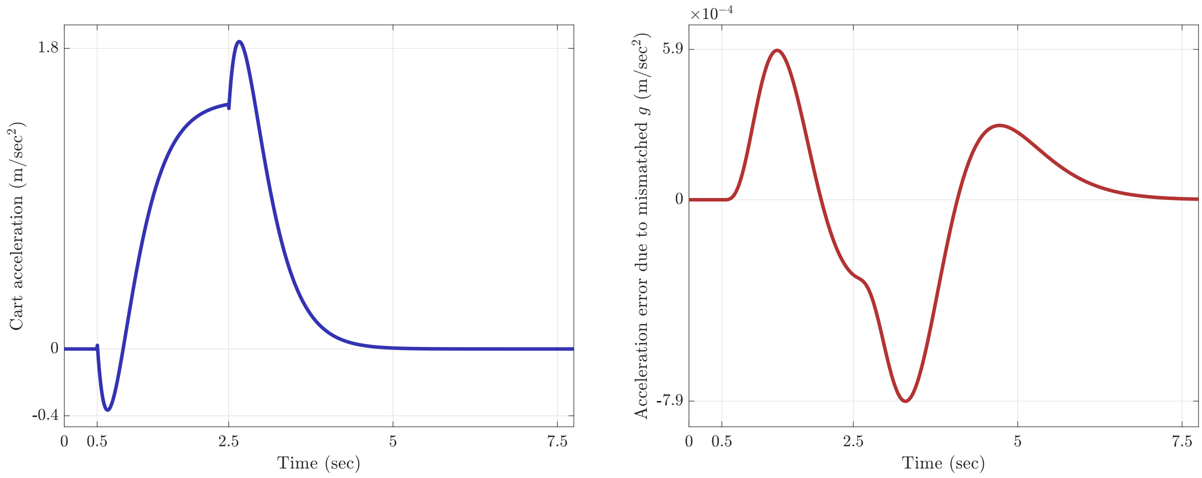
As a matter of fact, the controller in this case is

$$C(s) = \frac{M(\mu+1)^2}{4}\frac{(s+\mu\gamma)(s+\gamma(3\mu+5)/(\mu-1))}{(s+\gamma)(s-\gamma(\mu^2+3\mu+4)/(\mu-1))}$$

and the resulting closed-loop transfer functions are

$$T(s) = \frac{(\mu+1)^2(\mu-1)}{\mu^3+\mu^2+3\mu-5}\frac{(s-\gamma)(s+\gamma(3\mu+5)/(\mu-1))}{(s+\gamma)^2},$$
$$T_\mathrm{d}(s) = \frac{4(\mu-1)}{M(\mu^3+\mu^2+3\mu-5)}\frac{(s-\gamma)(s-\gamma(\mu^2+3\mu+4)/(\mu-1))}{(s+\gamma)(s+\mu\gamma)},$$
$$T_\mathrm{c}(s) = \frac{M(\mu+1)^2(\mu-1)}{\mu^3+\mu^2+3\mu-5}\frac{(s^2-\mu^2\gamma^2)(s+\gamma(3\mu+5)/(\mu-1))}{(s+\gamma)^3}.$$

They are all stable, of course. Note that $T(s)$ is a second-order system (rather than forth), because one plant pole at $s = -\gamma$ is canceled by the controller, one controller pole at $s = -\mu\gamma$ is canceled by the plant, and these cancellations do not show up in $T(s)$. Then, $T_\mathrm{d}(s)$ is also a second-order system, with one pole at $s = -\gamma$ and another one at $s = -\mu\gamma$. The latter is the (stable) pole of the plant, canceled by the controller. Still, plant poles canceled by the controller are present in $T_\mathrm{d}(s)$. Two additional poles at $s = -\gamma$ of the closed-loop characteristic polynomial are then canceled by a zero of $P(s)$ and a pole of $C(s)$. This double cancellation is, in a sense, accidental. If we picked other stable closed-loop poles, $T_\mathrm{d}(s)$ would be a third-order transfer function. The control sensitivity, $T_\mathrm{c}(s)$, has a triple pole at $s = -\gamma$. The third one is the pole of $C(s)$, which cancels the zero of the plant. This kind of cancellations is not visible in $T(s)$ and $T_\mathrm{d}(s)$, but does show up in $T_\mathrm{c}(s)$.



(a) Cart acceleration under nominal g

(b) Mismatch in y due to mismatch in g

Fig. 10: Disturbance response of the cart acceleration, $\ddot{z}(t)$

5. Substituting our data to $(*)$ and $(**)$, we end up with $\alpha \approx 12.3737$ and $k_\mathrm{p} \approx 124.658$, so that the controller from (2) is

$$P(s) = \frac{0.1(s-3.1305)(s+3.1305)}{(s-3.834)(s+3.834)} \quad \text{and} \quad C(s) = \frac{12.3737(s+3.834)(s+120.824)}{(s-127.789)(s+3.1305))}.$$

The disturbance sensitivity transfer function in this case is

$$T_\mathrm{d}(s) = \frac{0.044695(s-127.789)(s-3.13)}{(s+3.834)(s+3.13)},$$

as expected. Its response to the pulse $d(t) = \mathbb{1}(t-0.5) - \mathbb{1}(t-2.5)$ is presented in Fig. 10(a). We can see that the response first raises a bit, then has an undershoot, then raises until $t = 2.5$, and then mirrors the initial response in the other direction. The initial raise is worth emphasizing. It is because there is an *even number* of RHP zeros (namely, 2).

6. Here we have

$$P(s) = \frac{0.1(s-3.1316)(s+3.1316)}{(s-3.835)(s+3.835)} \quad \text{and} \quad C(s) = \frac{12.3737(s+3.834)(s+120.824)}{(s-127.789)(s+3.1305))}$$

(the controller remains the same, the plant changes a bit). We can see that there are no cancellations now. Indeed, the disturbance sensitivity becomes

$$T_\mathrm{d}(s) = \frac{0.044695(s-127.789)(s-3.1316)(s+3.132)(s+3.1305)}{(s+4.21)(s+2.569)(s^2+6.446s+10.88)},$$

which is a *fourth-order* system (the order of the plant plus that of the controller). Yet its response to the pulse disturbance is virtually indistinguishable from that in the nominal case. Fig. 10(b) presents the mismatch (difference) between two responses. It is more than three orders of magnitude smaller than the response of $\ddot{z}(t)$. In other words,

- feedback can render the system less sensitive to modeling inaccuracies

(open-loop control failed in this case, cf. Tutorial 6).

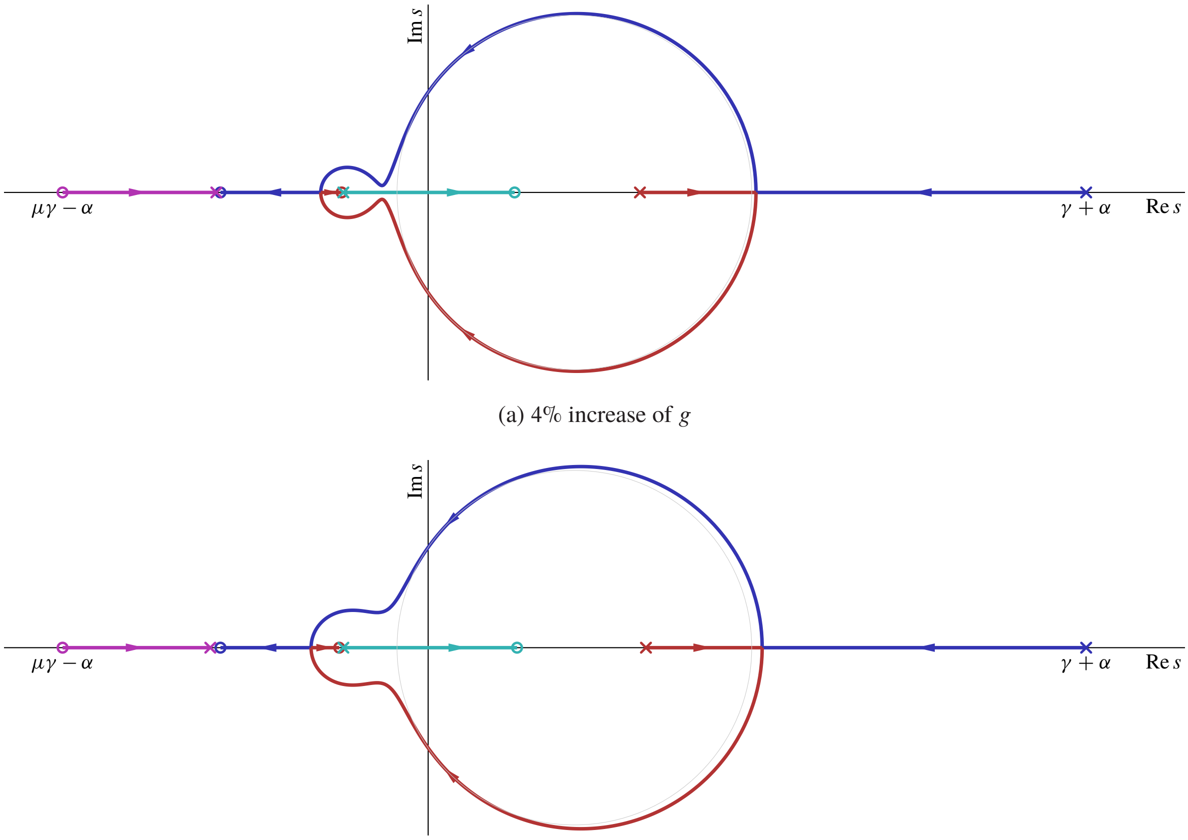


Fig. 11: Effect of the increase of g on root-locus plots

Just out of curiosity, the changes of the root locus with respect to slight increases of the standard gravity are shown in Fig. 11. The changes are not negligible around the break-in points. This is because the increase of g moves the stable zero of the plant to the left of the pole of the controller at $s = -\gamma$ (that attempted to cancel that zero). Then, the locus that went left along the real axis after the break-in point in Fig. 8 can no longer do that, because it cannot end in a pole. To end in a zero, the locus needs an additional "maneuver." If the actual g slightly decreased, the plot in Fig. 8 would remain virtually the same, with additions of two "holes" between each black poles-zero pair, which would separate.

That's all . . . ∇