הטכניון – מכון טכנולוגי לישראל, הפקולטה להנדסת מכונות

TECHNION—Israel Institute of Technology, Faculty of Mechanical Engineering

Introduction to Control (00340040)

tutorial 5

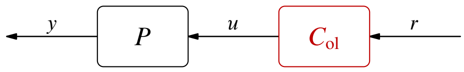


Fig. 1: Open-loop control system

Question 1. Consider the open-loop control system in Fig. 1 for the plant

$$P(s) = \frac{0.2s + 1}{(0.5s + 1)(s^2 + 0.2s + 1)}.$$

1. Can this plant be controlled via the use of a reference model? If it can, what conditions must be met by the reference model to guarantee the internal stability of the control system?

2. Consider a second-order reference model $T_{\text{ref}} : r \mapsto y$ with the transfer function

$$T_{\text{ref}}(s) = \frac{\omega_{\text{n}}^2}{s^2 + \sqrt{2}\omega_{\text{n}} s + \omega_{\text{n}}^2}$$

   (this is the second-order low-pass Butterworth filter, whose magnitude frequency response satisfies $|T_{\text{ref}}(\text{j}\omega)| = 1/\sqrt{1 + \omega^4/\omega_{\text{n}}^4}$ and whose bandwidth $\omega_{\text{b}} = \omega_{\text{n}}$). What will be the resulting C_{ol}? Is it admissible?

3. Plot the step responses of y and u and the magnitude bode plots of the reference model and the plant itself for $\omega_{\text{n}} \in \{0.4, 1, 3, 10\}$.

Solution.

1. For a reference model to be used in open-loop control systems three conditions must be met:

   (a) the reference model must be stable,

   (b) all nonminimum-phase zeros (RHP zeros) of the plant must be zeros of the reference model,

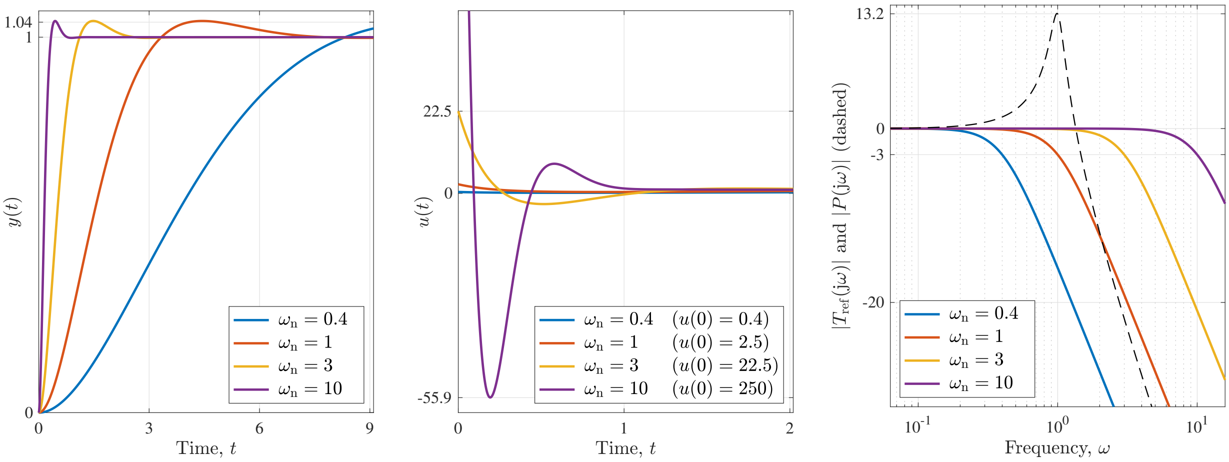
   (c) the pole excess of the reference model must be larger than or equal to that of the plant.

   In our case the plant has no nonminimum-phase zeros, so this condition is always met, and its pole excess is 2. Therefore, every stable reference model whose pole excess is at least 2 is admissible.

2. The open-loop controller will have the transfer function

$$C_{\text{ol}}(s) = \frac{T_{\text{ref}}(s)}{P(s)} = \frac{\omega_{\text{n}}^2(0.5s + 1)(s^2 + 0.2s + 1)}{(s^2 + \sqrt{2}\omega_{\text{n}} s + \omega_{\text{n}}^2)(0.2s + 1)}.$$

   The controller is bi-proper, and has all of its poles in the OLHP. Thus, the controller is BIBO stable. The plant is also BIBO stable, which means the control system is internally stable.



(a) step response of T_{ref} (b) step response of C_{ol} (c) frequency responses of T_{ref} and P

Fig. 2: Step responses of closed-loop systems

3. The required plots are presented in Fig. 2. We can see a trend caused by changes of the natural frequency ω_n of T_{ref}. As ω_n grows, so does the bandwidth of T_{ref}, which is exactly ω_n. This implies that the rise times of the step response of T_{ref} decreases as ω_n increases. At the same time, the overshoot in Fig. 2(a) is not affected by changes in ω_n. This is because the overshoot depends only on the damping ratio, viz.

$$\text{OS} = \mathrm{e}^{-\pi\zeta/\sqrt{1-\zeta^2}} \cdot 100\%,$$

for second-order systems without zeros.

The increase of the bandwidth also increases the initial value of the control signal u. This can be supported by the Initial Value Theorem,

$$u(0) = \lim_{s\to\infty} s \cdot C_{ol}(s) \cdot R(s) = \lim_{s\to\infty} s \cdot \frac{\omega_n^2(0.5s+1)(s^2+0.2s+1)}{(s^2+\sqrt{2}\omega_n s+\omega_n^2)(0.2s+1)} \cdot \frac{1}{s} = 2.5\omega_n^2.$$

The growing control effort as a function of ω_n can also be seen via the magnitude frequency-response plots in Fig. 2(c). Assume that the control signal is normalized, in the sense that the bound between small and large control signals is 1. This can always be attained via scaling. In this case, the relation between uncontrolled (i.e. that of P) and controlled (i.e. that of T_{ref}) bandwidths is often a good indication of the required control effort. Specifically, the control effort grows if the controlled bandwidth exceeds that of the plant itself. This is intuitive, because the increase of the bandwidth of the controlled response with respect to that of the uncontrolled system implies that the controller accelerates the response to r. This naturally requires a larger control signals.

That's all … ▽

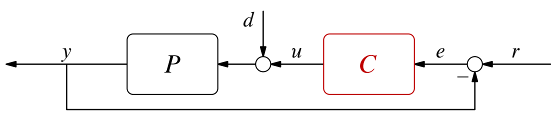


Fig. 3: Unity feedback closed-loop system

Question 2. A plant with the transfer function

$$P(s) = \frac{1}{s(s+1)(s+2)}$$

is controlled in the unity feedback scheme with static (proportional) controllers of the form $C(s) = k_{\mathrm{p}}$, see Fig. 3.

1. Derive the four closed-loop transfer functions (the Gang of Four) for this system. What signals in Fig. 3 each of them connects? What is the closed-loop characteristic polynomial? Under what controller gains the closed-loop system is internally stable?
2. Let $k_{\mathrm{p}} \in \{1, 4, 7\}$. Plot the responses of each closed-loop system to a unit step. Explain the differences between the responses for different values of k_{p}.

Solution.

1. Remember that the four closed-loop transfer functions are
   - $T(s) = \dfrac{P(s)C(s)}{1+P(s)C(s)}$ (complementary sensitivity), connects $r \mapsto y$ and $d \mapsto -u$
   - $S(s) = \dfrac{1}{1+P(s)C(s)} = 1 - T(s)$ (sensitivity), connects $r \mapsto e$
   - $T_{\mathrm{c}}(s) = \dfrac{C(s)}{1+P(s)C(s)}$ (control sensitivity), connects $r \mapsto u$
   - $T_{\mathrm{d}}(s) = \dfrac{P(s)}{1+P(s)C(s)}$ (disturbance sensitivity), connects $d \mapsto y$

   With our choices of P and C we have:

$$\begin{aligned}
T(s) &= \frac{\frac{1}{s(s+1)(s+2)}k_{\mathrm{p}}}{1+\frac{1}{s(s+1)(s+2)}k_{\mathrm{p}}} = \frac{k_{\mathrm{p}}}{s^3+3s^2+2s+k_{\mathrm{p}}}\\
S(s) &= \frac{1}{1+\frac{1}{s(s+1)(s+2)}k_{\mathrm{p}}} = \frac{s(s+1)(s+2)}{s^3+3s^2+2s+k_{\mathrm{p}}}\\
T_{\mathrm{c}}(s) &= \frac{k_{\mathrm{p}}}{1+\frac{1}{s(s+1)(s+2)}k_{\mathrm{p}}} = \frac{k_{\mathrm{p}}s(s+1)(s+2)}{s^3+3s^2+2s+k_{\mathrm{p}}}\\
T_{\mathrm{d}}(s) &= \frac{\frac{1}{s(s+1)(s+2)}}{1+\frac{1}{s(s+1)(s+2)}k_{\mathrm{p}}} = \frac{1}{s^3+3s^2+2s+k_{\mathrm{p}}}
\end{aligned}$$

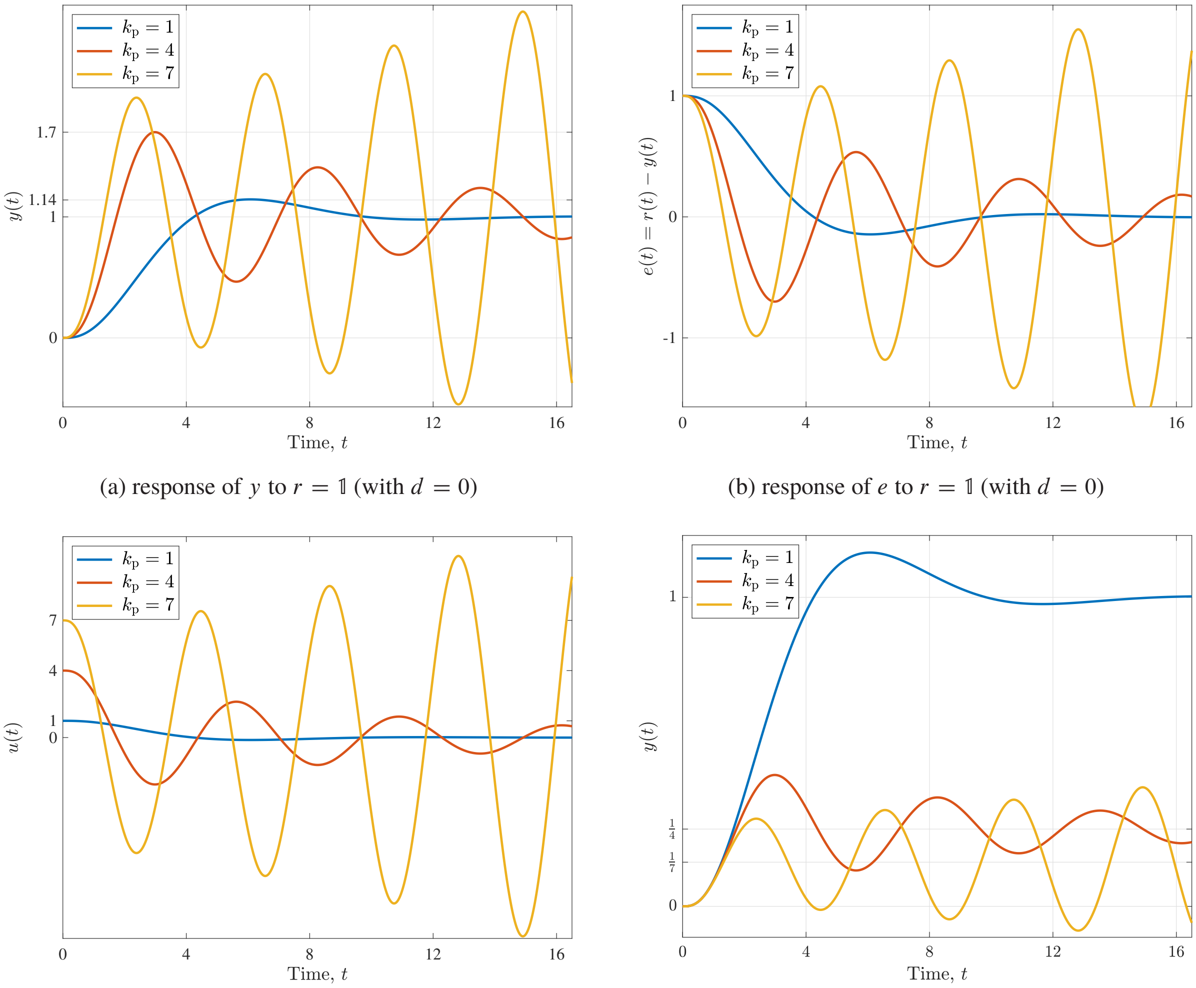
   These are all 3-order proper transfer functions. The closed-loop characteristic polynomial is

$$\chi_{\mathrm{cl}}(s) = k_{\mathrm{p}} + s(s+1)(s+2) = s^3 + 3s^2 + 2s + k_{\mathrm{p}}.$$

   The closed-loop system is internally stable iff $\chi_{\mathrm{cl}}(s)$ is Hurwitz. The latter happens iff

$$(k_{\mathrm{p}} > 0) \wedge (3 \cdot 2 > 1 \cdot k_{\mathrm{p}}) \iff \boxed{0 < k_{\mathrm{p}} < 6}.$$

2. Step responses of the closed-loop systems are presented in Fig. 4.



(a) response of y to $r = \mathbb{1}$ (with $d = 0$)

(b) response of e to $r = \mathbb{1}$ (with $d = 0$)

(c) response of u to $r = \mathbb{1}$ (with $d = 0$)

(d) response of y to $d = \mathbb{1}$ (with $r = 0$)

Fig. 4: Step responses of closed-loop systems

- Fig. 4(a) depicts the step responses of T, which may be interpreted as the regulated output y under the unit step reference $r = \mathbb{1}$, for different controller gains. The response for $k_p = 7$ diverges, which agrees with the stability condition derived in the previous item. Two other responses are stable and both converge to the same steady-state value, $y_{ss} = 1$. This agrees with the fact that the static gain $T(0) = 1$ regardless k_p.

  The transient response under $k_p = 1$ is relatively smooth, with OS $\approx$ 14%, but slow. The response under $k_p = 4$ is faster (shorter rise time), but has a substantially larger overshoot, about 70%. This could be seen via the pole location of their transfer functions,

$$T(s) = \frac{1}{(s + 0.338 \pm \mathrm{j}0.562)(s + 2.325)} \quad \text{and} \quad T(s) = \frac{4}{(s + 0.102 \pm \mathrm{j}1.192)(s + 2.796)}.$$

  Both have a pair of underdamped poles, with $\lambda_r/\lambda_i \approx 0.6$ and $\lambda_r^2 + \lambda_i^2 \approx 0.656^2$ for $k_p = 1$ and $\lambda_r/\lambda_i \approx 0.085$ and $\lambda_r^2 + \lambda_i^2 \approx 1.196^2$ for $k_p = 4$. The third, real, poles are substantially further away from the imaginary axis. This suggests that the underdamped poles are dominant in both cases. Indeed, their location should correspond to overshoots (the formula is $\mathrm{e}^{-\pi(\lambda_r/\lambda_i)}$) of $\approx$ 15% and $\approx$ 76%, respectively, which are close to what we actually have. The actual overshoots are a bit lower than those expected from the dominant poles, which is always the case when a real pole is added. The speed of transients also agrees with the natural frequencies of the dominant pole, with faster transients corresponding to a larger ω_n.

- Fig. 4(b) presents the step response of S, which is the error signal e under the unit step in r. The transfer function $S(s)$ has 3 zeros, one of which (that at the origin) is *dominant*. Hence, we do not have tools to deduce properties of transients directly from the pole location in this case. Just note that because $e = r - y$ and $r(t) = 1$ for all $t > 0$, the transients in Fig. 4(b) are merely shifted and sign-inverted versions of those in Fig. 4(a). Also note that lightly-damped poles of $S(s)$ still result in oscillations of the response.

  The static gain of the sensitivity function $S(0) = 0$ for all k_p. Hence, if the system is stable, then the response always converges to 0. This can be clearly seen in Fig. 4(b).

- Fig. 4(c) illustrates the step responses of T_c, which is the control signal u under the unit step in r. This is merely a scaled version of S in this case, so most qualitative conclusions are the same. Just note that the initial value of the control signal, $u(0)$, increases as the controller gain k_p grows. This can be seen via the Initial Value Theorem, which says that

  $$\lim_{t\downarrow 0} u(t) = \lim_{s\to\infty} s \cdot U(s) = \lim_{s\to\infty} s \cdot T_c(s)R(s) = \lim_{s\to\infty} s \cdot T_c(s)\frac{1}{s} = \lim_{s\to\infty} T_c(s) = k_p,$$

  which agrees with what we have in Fig. 4(c).

- Fig. 4(d) shows the step response of T_d, which is the response of y to a step in d. Because this T_d is a scaled T, by a factor of $1/k_p$, all transient properties are the same as in complementary sensitivity case. The steady state response is now a function of the controller gain. Specifically, as k_p increases, the steady-state value of y decreases, cf. $T_d(0) = 1/k_p$. However, this decrease is limited by the stability condition, because unstable systems do not reach their steady state (see the response in Fig. 4(d) under $k_p = 7$, which diverges exponentially). Hence, we cannot expect the steady-state response to a step disturbance to be lower than or equal to $1/6$.

That's all . . . ∇

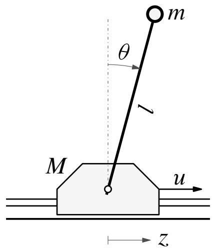


Fig. 5: Inverted pendulum on a cart

Question 3. Consider an inverted pendulum which consists of a point mass m on a mass-less rod of length l installed on a cart of mass M. An external force u is acting on the cart. The equations of motion of this system (see the Linear Systems M course notes) are

$$(M+m)\ddot{z}(t) + ml\ddot{\theta}(t)\cos\theta(t) - ml\dot{\theta}^2(t)\sin\theta(t) = u(t) \tag{1a}$$

$$\ddot{z}(t)\cos\theta(t) + l\ddot{\theta}(t) - g\sin\theta(t) = 0 \tag{1b}$$

where θ is the angle of the pendulum and z is the position of the cart. The parameters are $m = 5\,[\text{kg}]$, $M = 10\,[\text{kg}]$, $l = 1\,[\text{m}]$, and the standard gravity is taken $g = 9.8\,[\text{m/sec}^2]$

0. Derive the linearized state-space model of the system (in the "up" position) and the transfer function $P(s)$ with u as its input and the car acceleration $y = \ddot{z}$ as its output.

1. The system is controlled in a standard unity feedback closed-loop scheme, like that in Fig. 3. Can it be controlled (that is, stabilized) by the controller

$$C(s) = \frac{10(s^2 - 14.7)}{s^2 + 4s + 11.8} \tag{2}$$

   Check that both via the stability of the closed-loop transfer functions $T(s)$, $S(s)$, $T_{\text{d}}(s)$, and $T_{\text{c}}(s)$ and via the characteristic polynomial of the closed-loop system.

2. Analyze the step responses of the system $r \mapsto y$ under the controller above and no disturbances if the standard gravity is actually $g = 9.80665\,[\text{m/sec}^2]$ ($\approx 0.07\%$ deviation from the assumed g).

Solution.

0. Rewrite (1) as

$$\begin{bmatrix} M+m & ml\cos\theta(t) \\ \cos\theta(t) & l \end{bmatrix} \begin{bmatrix} \ddot{z}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} ml\dot{\theta}^2(t)\sin\theta(t) \\ g\sin\theta(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t).$$

   It appears to be natural to define the state vector of this system (mechanical system with no derivatives of its input) as

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} := \begin{bmatrix} z(t) \\ \theta(t) \\ \dot{z}(t) \\ \dot{\theta}(t) \end{bmatrix}.$$

The state vector then verifies the equation

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & M+m & ml\cos x_2(t) \\ 0 & 0 & \cos x_2(t) & l \end{bmatrix}}_{E(x)} \dot{x}(t) = \underbrace{\begin{bmatrix} x_3(t) \\ x_4(t) \\ mlx_4^2(t)\sin x_2(t) + u(t) \\ g\sin x_2(t) \end{bmatrix}}_{\phi(x,u)}.$$

This is not completely orthodox state equation, but it is convenient to keep it in that form. To see why, note that

$$\det E(x) = (M+m)l - ml\cos^2 x_2(t) = Ml + ml\sin^2 x_2(t) > 0,$$

so the matrix $E(x)$ is nonsingular. Hence, the equation above can be reduced to the standard state equation $\dot{x} = f(x,u)$ for

$$f(x,u) := E^{-1}(x)\phi(x,u).$$

We can calculate this $f(x,u)$ explicitly, but then determining its equilibrium and, especially, its derivatives (required in the linearization procedure) would be messy. Instead, note that as $E(x)$ is nonsingular, the equilibrium can be determined via the relation

$$\phi(x,u) = 0 \iff \begin{bmatrix} x_3(t) \\ x_4(t) \\ mlx_4^2(t)\sin x_2(t) + u(t) \\ g\sin x_2(t) \end{bmatrix} = 0 \iff \begin{cases} x_3(t) = 0 \\ x_4(t) = 0 \\ u(t) = 0 \\ x_2(t) = \pi k \text{ for some } k \in \mathbb{Z} \\ x_1(t) = z_0 \text{ is arbitrary constant} \end{cases}$$

As $x_2 = \theta$ and we consider the pendulum in the "up" position, we may take $k = 0$ and end up with the equilibrium state and input

$$x_{\text{eq}} = \begin{bmatrix} z_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad u_{\text{eq}} = 0.$$

Then, to calculate the derivatives, use the relations

$$\frac{\mathrm{d}}{\mathrm{d}x_i}(E^{-1}\phi) = E^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}x_i}\phi - \frac{\mathrm{d}}{\mathrm{d}x_i}E \cdot E^{-1}\phi\right) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}u}(E^{-1}\phi) = E^{-1}\frac{\mathrm{d}}{\mathrm{d}u}\phi$$

(as E does not depend on u). Taking derivatives of E and ϕ separately is substantially simpler. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\phi = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x_2}\phi = \begin{bmatrix} 0 \\ 0 \\ mlx_4^2\cos x_2 \\ g\cos x_2 \end{bmatrix}, \quad \frac{\mathrm{d}}{\mathrm{d}x_3}\phi = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\mathrm{d}}{\mathrm{d}x_4}\phi = \begin{bmatrix} 0 \\ 1 \\ 2mlx_4\sin x_2 \\ 0 \end{bmatrix},$$

$$\frac{\mathrm{d}}{\mathrm{d}x_1}E = \frac{\mathrm{d}}{\mathrm{d}x_3}E = \frac{\mathrm{d}}{\mathrm{d}x_4}E = 0, \quad \frac{\mathrm{d}}{\mathrm{d}x_2}E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ml\sin x_2 \\ 0 & 0 & \sin x_2 & 0 \end{bmatrix},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}u}\phi = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}u}E = 0.$$

Moreover, at the equilibrium point $\frac{\mathrm{d}}{\mathrm{d}x_2}\phi\big|_{x=x_{\mathrm{eq}}} = \left[\begin{smallmatrix} 0 \\ 0 \\ 0 \\ g \end{smallmatrix}\right]$, $\frac{\mathrm{d}}{\mathrm{d}x_4}\phi\big|_{x=x_{\mathrm{eq}}} = \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \\ 0 \end{smallmatrix}\right]$, and $\frac{\mathrm{d}}{\mathrm{d}x_2}E\big|_{x=x_{\mathrm{eq}}} = 0$. Thus, the linearized model in terms of $\tilde{x} := x - x_{\mathrm{eq}}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & M+m & ml \\ 0 & 0 & 1 & l \end{bmatrix} \dot{\tilde{x}}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & g & 0 & 0 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t)$$

Then,

$$\tilde{X}(s) = \begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 0 & 0 & (M+m)s & mls \\ 0 & -g & s & ls \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} U(s) = \begin{bmatrix} (ls^2-g)/s^2 \\ -1 \\ (ls^2-g)/s \\ -s \end{bmatrix} \frac{1}{Mls^2-(M+m)g} U(s)$$

and the transfer function from $U(s)$ to $s^2 Z(s) = s\tilde{X}_3(s)$ is

$$P(s) = \frac{ls^2-g}{Mls^2-(M+m)g} = \frac{0.1(s^2-9.8)}{s^2-14.7}.$$

1. It is readily seen that

$$P(s)C(s) = \frac{0.1(s^2-9.8)}{s^2-14.7} \cdot \frac{10(s^2-14.7)}{s^2+4s+11.8} = \frac{s^2-9.8}{s^2+4s+11.8}.$$

The complementary sensitivity transfer function ($r \mapsto y$ and $d \mapsto -u$)

$$T(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{s^2-9.8}{2(s+1)^2}$$

is stable (proper and no RHP poles). The sensitivity function ($r \mapsto e$)

$$S(s) = \frac{1}{1+P(s)C(s)} = 1 - T(s) = \frac{s^2+4s+11.8}{2(s+1)^2}$$

is stable as well. The control sensitivity transfer function ($r \mapsto u$)

$$T_{\mathrm{c}}(s) = \frac{C(s)}{1+P(s)C(s)} = \frac{5(s^2-14.7)}{(s+1)^2}$$

is still stable. But the disturbance sensitivity transfer function ($d \mapsto y$)

$$T_{\mathrm{d}}(s) = \frac{P(s)}{1+P(s)C(s)} = \frac{0.05(s^2-9.8)(s^2+4s+11.8)}{(s+1)^2(s^2-14.7)}$$

is clearly unstable, which implies that the closed-loop system is not internally stable. Therefore, controller (2) cannot be used. The reason may be clearly seen in unstable cancellations between a pole of $P(s)$ at $s = \sqrt{14.7}$ and a zero of $C(s)$ at the same point.

The effect of unstable pole-zero cancellations may also be seen via the characteristic polynomial of the closed-loop system:

$$\begin{aligned}\chi_{\mathrm{cl}}(s) &= 0.1(s^2-9.8)10(s^2-14.7)+(s^2-14.7)(s^2+4s+11.8)\\ &= (s^2-14.7)(s^2-9.8+s^2+4s+11.8)\\ &= 2(s^2-14.7)(s+1)^2,\end{aligned}$$

which has three OLHP roots (two at $s=-1$ and one at $s=-\sqrt{14.7}\approx 3.834$) and one RHP root (at $s=\sqrt{14.7}$)). The conclusion is, again, that controller (2) cannot be used.

2. With the more accurate value of the standard gravity, the transfer function of the plant becomes

$$P(s)=\frac{ls^2-g}{Mls^2-(M+m)g}=\frac{0.1(s^2-9.80665)}{s^2-14.709975}.$$

The controller now no longer cancels the unstable pole of the plant at $s=\sqrt{14.709975}\approx 3.83536$. Then, the complementary sensitivity transfer function

$$\begin{aligned}T(s)=\frac{P(s)C(s)}{1+P(s)C(s)} &= \frac{(s^2-9.80665)(s^2-14.7)}{2(s^4+2s^3-13.7083s^2-29.42s-14.71)}\\ &= \frac{(s^2-9.80665)(s+3.83406)(s-\mathbf{3.83406})}{2(s+0.989117)(s+1.01114)(s+3.83496)(s-\mathbf{3.83522})}\\ &= \frac{1}{2}-\frac{200.283}{s+0.989117}+\frac{199.284}{s+1.01114}-\frac{0.000275578}{s+3.83496}+\frac{\mathbf{0.000122043}}{s-3.83522}.\end{aligned}$$

Although the unstable pole and zero in this $T(s)$ are very close, they do not cancel each other. Hence, the transfer matrix is *unstable*.

That's all … ▽

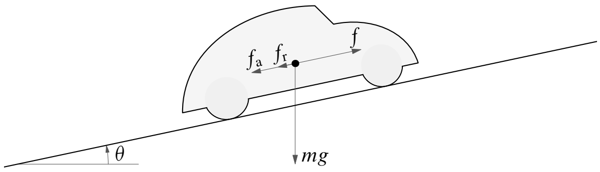


Fig. 6: System for Question 4

Question 4 (self study)**.** Fig. 6 depicts a vehicle of mass $m = 1000$ [kg] driving uphill with the slope $\theta = 12°$. The driving force f generated by the engine is the control signal, whose goal is to maintain the car velocity v at a pre-specified level. The resistance force has three major components: $f_g = mg \sin\theta$, the forces due to gravity; f_a, the aerodynamic drag; and f_r, the forces due to rolling friction. Assuming that the velocity of the car is always positive, the rolling resistance $f_r = mgc_r \cos\theta$, where the rolling resistance coefficient $c_r = 0.01$. The aerodynamic drag is proportional to the square of the speed, i.e. $f_a = \frac{1}{2}\alpha v^2$, where $\alpha \approx 1$ [kg/m] is a constant depending on the density of air, the frontal area of the car, and a shape-dependent aerodynamic drag coefficient. As shown in Tutorial 3, the nonlinear motion equation of this system is

$$\dot{v}(t) = \frac{1}{m}F(t) - \frac{1}{2m}\alpha v^2(t) - g(\sin\theta + c_r \cos\theta)$$

and linearized motion equation around the equilibrium velocity $v_{eq} = 80$ [km/h] $= 200/9 \approx 22.22$ [m/sec] is

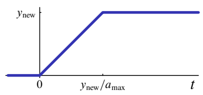
$$\dot{y}(t) = -\frac{\alpha v_{eq}}{m} y(t) + \frac{1}{m} u(t),$$

where the deviation variables $y := v - v_{eq}$ and $u := f - 0.5\alpha v_{eq}^2 - mg(\sin\theta + c_r \cos\theta)$.

1. Consider the unity feedback closed-loop control strategy in which a *proportional* controller $C(s) = k_p$ generates the control signal $u(t)$ from the mismatch between the reference velocity signal $r_v(t)$ and the measured deviation from the equilibrium velocity $y(t)$. Draw the block-diagram of this system. Under what values of k_p the closed-loop system is stable?

2. Consider the reference signal r_v such that

$$r_v(t) = \begin{cases} 0 & \text{if } t \le 0 \\ a_{max} t & \text{if } 0 \le t \le y_{new}/a_{max} \\ y_{new} & \text{if } t \ge y_{new}/a_{max} \end{cases} = \tag{3}$$



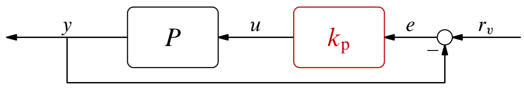
   for the peak acceleration $a_{max} = 0.5$ [m/s^2] and $y_{new} = 10$ [km/h] $= 25/9 \approx 2.78$ [m/sec]. How the choice of k_p affects the steady-state error in general? Simulate the response of the linearized system under k_p's for which the steady-state error is $e_{ss} = |\lim_{t\to\infty} r_v(t) - y(t)| \in \{2, 1, 0.1\}$ [km/h].

3. How does the steady-state error of the previous item change if the road slope changes? Simulate with the change from Tutorial 3, $\bar{\theta} = 13°$, under the controller gains obtained in the previous item. How does it differ from the open-loop results of Tutorial 3?

4. Analyze the nonlinear system with the unity feedback closed-loop controller as in item 1. What is its steady-state response to the reference signal in (3)?

Solution. We should always remember that we design our controllers in terms of deviation variables. The actual responses require adding v_{eq} to the output y and adding f_{eq} to the input u.

1. The block-diagram of the unity feedback control system is presented in the figure below:



where, as we know from Tutorial 3, the transfer function of the linearized plant

$$P(s) = \frac{1}{ms + \alpha v_{\rm eq}} = \frac{0.045}{45.07s + 1}.$$

To analyze the (internal) stability of this closed-loop system, we need its characteristic polynomial

$$\chi_{\rm cl}(s) = ms + \alpha v_{\rm eq} + k_{\rm p}.$$

This is a first-order polynomial, whose root is in the OLHP iff all its coefficients have the same sign. Because $m > 0$, the system is stable iff

$$\alpha v_{\rm eq} + k_{\rm p} > 0 \iff k_{\rm p} > -\alpha v_{\rm eq} \approx -22.19.$$

2. The closed-loop transfer function $r_v \mapsto e$ is the sensitivity transfer function

$$S(s) = \frac{1}{1 + P(s)k_{\rm p}} = \frac{ms + \alpha v_{\rm eq}}{ms + (\alpha v_{\rm eq} + k_{\rm p})}.$$

Its static gain

$$S(0) = \frac{\alpha v_{\rm eq}}{\alpha v_{\rm eq} + k_{\rm p}} = \frac{1}{1 + k_{\rm p}/(\alpha v_{\rm eq})} \approx \frac{1}{1 + k_{\rm p}/22.19}.$$

This is a decreasing function of $k_{\rm p}$ for all admissible $k_{\rm p} > -\alpha v_{\rm eq}$. Furthermore, it is positive for all admissible controller gains, implying that the actual steady-state velocity always "undershoots." The steady-state error is then

$$e_{\rm ss} = S(0)y_{\rm new} = \frac{y_{\rm new}}{1 + k_{\rm p}/(\alpha v_{\rm eq})} \approx \frac{2.78}{1 + k_{\rm p}/22.19}.$$

Hence,

$$e_{\rm ss} = x \iff k_{\rm p} = \alpha v_{\rm eq}\Big(\frac{y_{\rm new}}{x} - 1\Big),$$

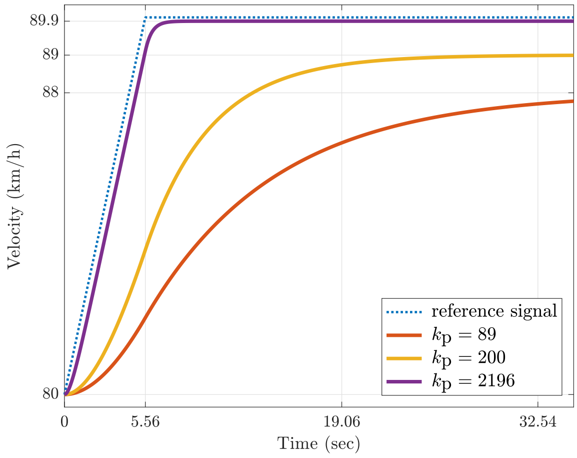
which results in $k_{\rm p} \approx \{89, 200, 2196\}$ for $x = 3.6 \cdot \{2, 1, 0.1\}$ [m/sec], respectively. The vehicle velocities for each of these choices are presented in Fig. 7(a). Note that the convergence to their respective steady-state values becomes faster as $k_{\rm p}$ increases. This can be explained by the fact that the time constant of the complementary sensitivity transfer function,

$$T(s) = \frac{P(s)k_{\rm p}}{1 + P(s)k_{\rm p}} = \frac{k_{\rm p}}{ms + (\alpha v_{\rm eq} + k_{\rm p})} = \frac{k_{\rm p}/(\alpha v_{\rm eq} + k_{\rm p})}{m/(\alpha v_{\rm eq} + k_{\rm p})s + 1},$$

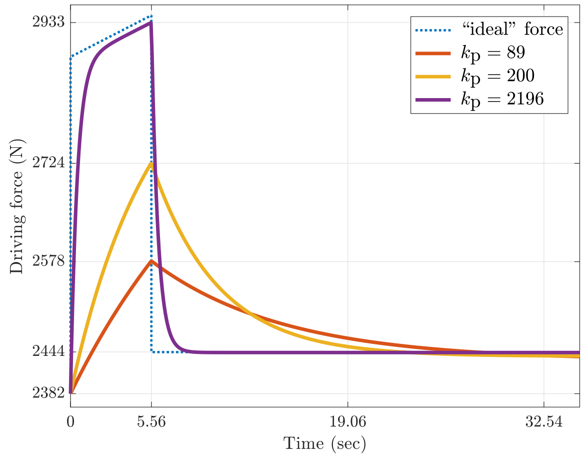
decreases as $k_{\rm p}$ increases. Fig. 7(b) presents corresponding driving forces, i.e. control signals, which is

$$F = f_{\rm eq} + T_{\rm c}(s)r_v = f_{\rm eq} + \frac{k_{\rm p}(ms + \alpha v_{\rm eq})}{ms + (\alpha v_{\rm eq} + k_{\rm p})} r_v.$$

For the sake of comparison, the dotted line represents the "ideal" force $f_{\rm idl} = P^{-1}r_v + f_{\rm eq}$, under which we have $y \equiv r_v$. This force was implemented in open loop in Tutorial 3, but was shown to be very sensitive to changes in the vehicle mass and the road slope. The high-gain feedback could reproduce this force quite well. But what about sensitivity to uncertainties?



(a) Vehicle velocities



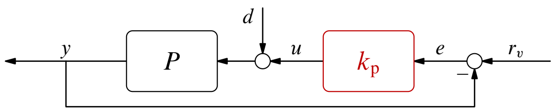
(b) Driving forces

Fig. 7: Velocity responses to r_v from (3) ("ideal" force is that under which $v \equiv r_v$)

3. As we may remember from Tutorial 3, a change in the road slope does not alter the linearized plant model. It only causes the force equilibrium changes to $\frac{1}{2}\alpha v_{\text{eq}}^2 + mg(\sin\bar{\theta} + c_r\cos\bar{\theta})$. Because we keep correcting u by the nominal equilibrium force, the step disturbance $d = d_\theta \mathbb{1}$, where

$$d_\theta := -2mg\Big(\cos\frac{\bar{\theta}+\theta}{2} - c_r\sin\frac{\bar{\theta}+\theta}{2}\Big)\sin\frac{\bar{\theta}-\theta}{2} \approx -166.79$$

should be added to account for the mismatch (d_θ is a decreasing function of $\bar{\theta}$). The resulting system is then presented by the block-diagram below:



The error signal in this case comprises the effect of r_v, via the sensitivity function $S : r_v \mapsto e$ and the effect of d via the disturbance sensitivity function, $d \mapsto e$,

$$-T_d(s) = -\frac{P(s)}{1+P(s)k_p} = \frac{1}{ms + (\alpha v_{\text{eq}} + k_p)}.$$

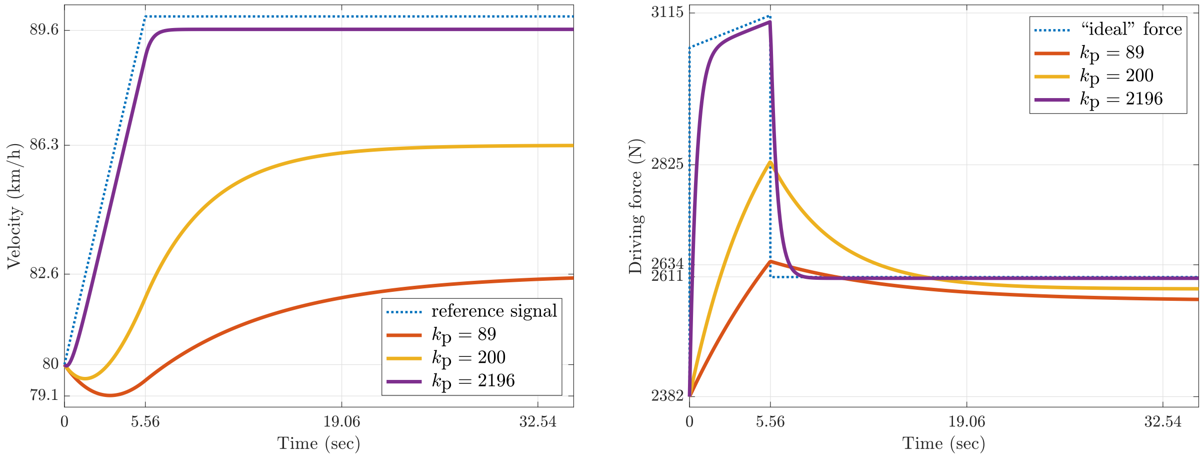
The resulting steady-state error,

$$e_{\text{ss}} = |S(0)y_{\text{new}} - T_d(0)d_\theta| = \left|\frac{y_{\text{new}} - d_\theta/(\alpha v_{\text{eq}})}{1 + k_p/(\alpha v_{\text{eq}})}\right| > \frac{y_{\text{new}}}{1 + k_p/(\alpha v_{\text{eq}})},$$

increases with $\bar{\theta}$ (because $d_\theta < 0$ for $\bar{\theta} > 0$ and it decreases as $\bar{\theta}$ grows). Still, the increment of the steady-state error with respect to the nominal case is smaller than that under open-loop control (calculated in Tutorial 3):

$$\frac{|d_\theta|}{\alpha v_{\text{eq}} + k_p} < \frac{|d_\theta|}{\alpha v_{\text{eq}}} \approx 27.06\,[\text{km/h}], \quad \forall k_p > 0$$

and as k_p increases, the error decreases. The simulated results presented in Fig. 8(a) confirm the conclusions above. Fig. 8(b) presents then corresponding driving forces. The dotted line again rep-



(a) Vehicle velocities

(b) Driving forces

Fig. 8: Velocity responses to r_v from (3) under changed slope ("ideal" force is that under which $v \equiv r_v$)

resents the "ideal" force under which $y \equiv r_v$, which is now $f_{\text{idl}} = P^{-1} r_v - d + f_{\text{eq}}$. This force cannot be implemented in open loop because we do not measure d. As we can see from Fig. 8(b), high-gain feedback can reproduce f_{idl} reasonably well without explicitly measuring that force.

4. The actual control force generated by the feedback controller satisfies

$$f(t) = u(t) + f_{\text{eq}} = k_{\text{p}}(r_v(t) - y(t)) + f_{\text{eq}} = k_{\text{p}}(r_v(t) - v(t) + v_{\text{eq}}) + f_{\text{eq}}.$$

Substituting this expression into the system dynamics, we get

$$\begin{aligned}\dot{v}(t) &= \frac{1}{m}(k_{\text{p}}(r_v(t) - v(t) + v_{\text{eq}}) + f_{\text{eq}}) - \frac{1}{2m}\alpha v^2(t) - g(\sin\theta + c_{\text{r}}\cos\theta) \\ &= -\frac{\alpha}{2m}v^2(t) - \frac{k_{\text{p}}}{m}v(t) + \frac{\alpha}{2m}v_{\text{eq}}^2 + \frac{k_{\text{p}}}{m}v_{\text{eq}} + \frac{k_{\text{p}}}{m}r_v(t)\end{aligned}$$

We do not have tools to analyze the stability of this (nonlinear) system. So consider it as a fact that it is stable. Assuming stability (important), the steady-state velocity v_{ss} can be computed by setting $r_v = y_{\text{new}} = v_{\text{new}} - v_{\text{eq}}$ and $\dot{v} = 0$ (here v_{new} is the required steady-state level of v). These conditions yield the algebraic quadratic equation

$$\frac{\alpha}{2}v_{\text{ss}}^2 + k_{\text{p}}v_{\text{ss}} - \frac{\alpha}{2}v_{\text{eq}}^2 - k_{\text{p}}v_{\text{new}} = 0 \iff v_{\text{ss}}^2 + 2\varkappa v_{\text{ss}} - v_{\text{eq}}^2 - 2\varkappa v_{\text{new}} = 0.$$

where $\varkappa := k_{\text{p}}/\alpha$. The only positive solution to this equation is

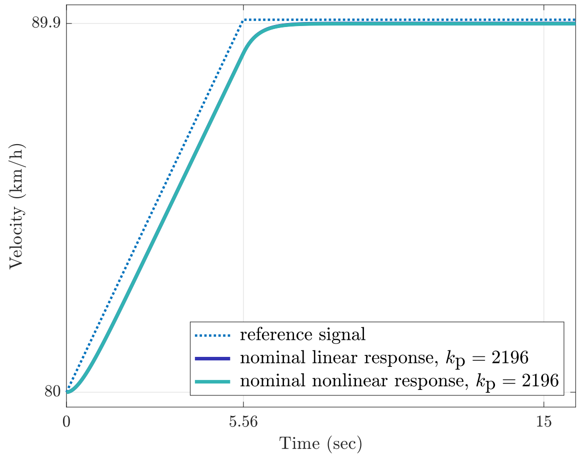
$$v_{\text{ss}} = \sqrt{\varkappa^2 + 2\varkappa v_{\text{new}} + v_{\text{eq}}^2} - \varkappa = \sqrt{(\varkappa + v_{\text{new}})^2 - v_{\text{new}}^2 + v_{\text{eq}}^2} - \varkappa.$$

Thus, the steady-state error

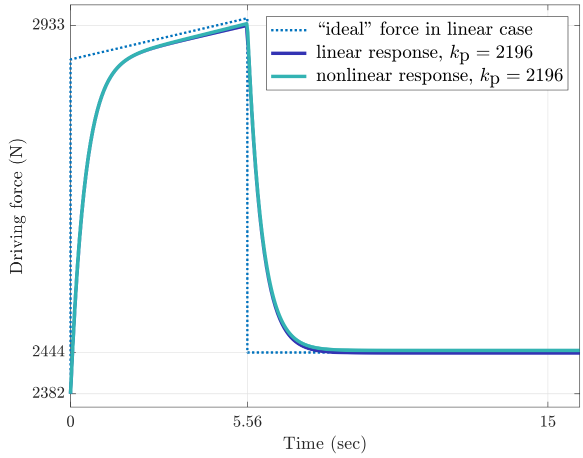
$$e_{\text{ss}} = v_{\text{new}} - v_{\text{ss}} = \varkappa + v_{\text{new}} - \sqrt{(\varkappa + v_{\text{new}})^2 - v_{\text{new}}^2 + v_{\text{eq}}^2},$$

which is a decreasing function of $\varkappa$, vanishing as $\kappa \to \infty$. With $k_{\text{p}} = 2196$ (the gain yielding a steady-state error of 0.1 [km/h] in the linear case), this formula yields

$$e_{\text{ss}} \approx 0.106\,\text{km/h},$$



(a) Vehicle velocities



(b) Driving forces

Fig. 9: Closed-loop control under nonlinear model

which is about the same as that in the linear case. Fig. 9 demonstrates that the transient behavior of the nonlinear system is also close to that of the linearized system. This demonstrates that feedback can cope with modeling inaccuracies, unlike the open-loop control studied in Tutorial 3.

That's all . . . ▽

Question 5 (self study)**.** Consider the unity feedback closed-loop system in Fig. 3. Let

$$P(s) = \frac{s+1}{s(s^2+s+1)} \quad \text{and} \quad C(s) = \frac{k(\tau s+1)}{s(s+1)}.$$

Determine and draw the closed-loop stability area in the (τ, k)-plane.

Solution. The characteristic polynomial of the closed-loop system is

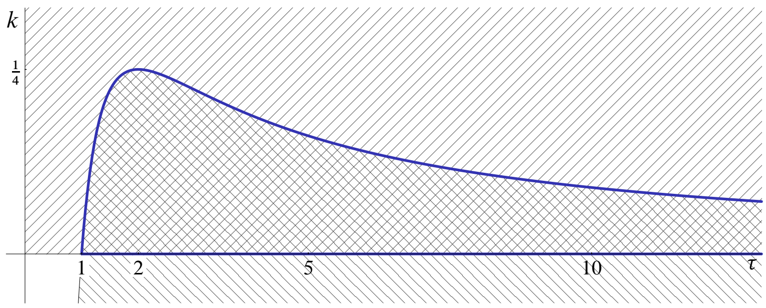
$$\chi_{\rm cl}(s) = (s+1)\cdot k(\tau s+1) + s(s^2+s+1)\cdot s(s+1) = (s+1)(s^4+s^3+s^2+k\tau s+k).$$

One stable root at $s = -1$ is obvious. To check the stability of the others, construct the Routh table for $s^4+s^3+s^2+k\tau s+k$:

s^4	1	1	k
s^3	1	$k\tau$	0
s^2	$1-k\tau$	k	0
s^1	$k\tau - \frac{k}{1-k\tau}$	0	
s^0	k		

Thus, stability conditions are $1-k\tau > 0$, $k\big(\tau - \frac{1}{1-k\tau}\big) > 0$, and $k > 0$. The third condition implies that the second one reads $\tau > \frac{1}{1-k\tau}$, which, in turn, implies that τ must be positive (taking into account that $1-k\tau > 0$). Thus, we can rewrite the stability conditions above as $k > 0$, $k < \frac{1}{\tau}$, and $k < \frac{1}{\tau} - \frac{1}{\tau^2}$. Because the last condition is stronger than the second one, the latter is redundant and we end up with

$$\left.\begin{array}{c} 0 < k < \dfrac{1}{\tau} - \dfrac{1}{\tau^2} \\ \tau > 0 \end{array}\right\} \iff$$



where the area $\{k > 0, \tau > 0\}$ is hatched by "NE-SW" lines, whereas the area $k < \frac{1}{\tau} - \frac{1}{\tau^2}$ corresponds to the region hatched by "NW-SE" lines. The stability region is obviously the intersection of these regions (the crosshatched area).

Note that $k = 0.25$ is the tightest upper bound for admissible k's and the largest k range is attained with $\tau = 2$. ▽