Introduction to Control (00340040)

tutorial 4

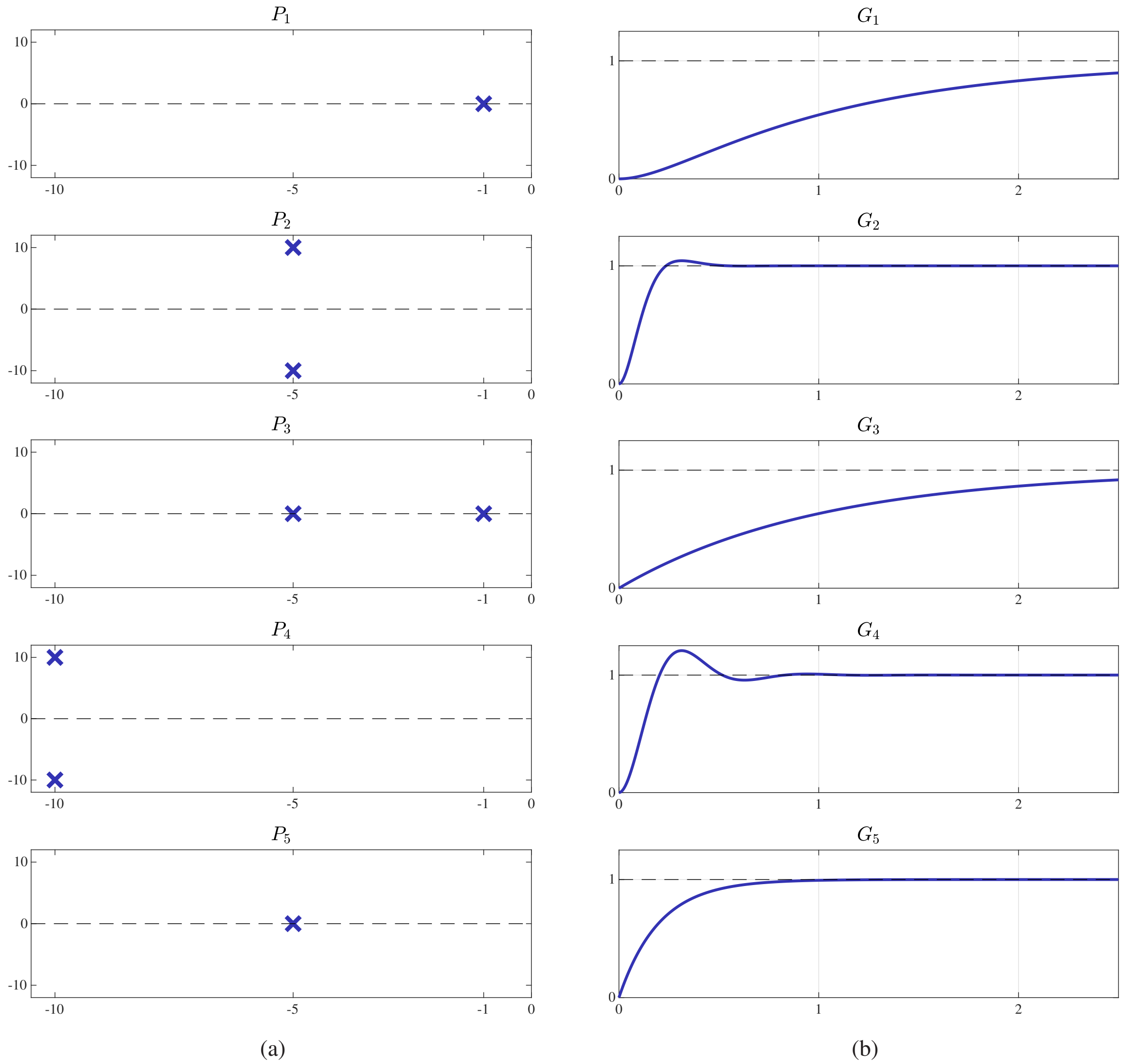


Fig. 1: Pole-zero maps and step responses for Question 1

Question 1. Match the pole-zero maps given in Fig. 1(a) to the step responses in Fig. 1(b).

Solution. It should be clear from the pole maps in Fig. 1(a) that we have either first- or second-order systems with no zeros. Identify then two step responses of second-order underdamped systems (G_2 and G_4) and two step responses of first-order systems (G_3 and G_5, they can be identified via non-zero slopes at $t = 0$). The response of G_1 might resemble that of a first-order system. But its slope at $t \to 0$ is zero, so we shall understand that it is actually an overdamped second-order system. Such a system should have two real (LHP) poles, hence

$$P_3 \leftrightarrow G_1.$$

A key to distinguish the first-order systems G_3 and G_5 is the fact that the latter is about 5 times faster than the former (say, in terms of their rise times). Because faster responses should correspond to poles further from the origin, we have that

$$P_5 \leftrightarrow G_5 \qquad \text{and} \qquad P_1 \leftrightarrow G_3.$$

Finally, comparing the pole maps of P_2 and P_4, we recognize that the poles of P_4 have smaller damping than those of P_4 (remember, the damping factor decreases as the ratio between the absolute values of the imaginary and real parts increases). Therefore, the step response of P_4 shall have a smaller overshoot and less oscillations[1]. This gives

$$P_4 \leftrightarrow G_2 \qquad \text{and} \qquad P_2 \leftrightarrow G_4.$$

We can also notice that the real term of the poles in a4 is larger (in its absolute value). Remembering that the real part of the complex pole pair is $-\zeta\omega_\mathrm{n}$ we understand that this response will converge to steady state faster.

Note also that the faster pole of P_3 is rather far away from its dominant pole at $s = -1$ (5 times further from the origin). We should therefore expect that the response of P_3 should resemble that of P_1. This indeed true, as can be seem via comparing the responses of G_1 and G_3. ∇

[1] To memorize that, consider the limits: poles which are on the real axis are overdamped and will have no oscillations, while poles on the imaginary axis are undamped and will oscillate without decay.

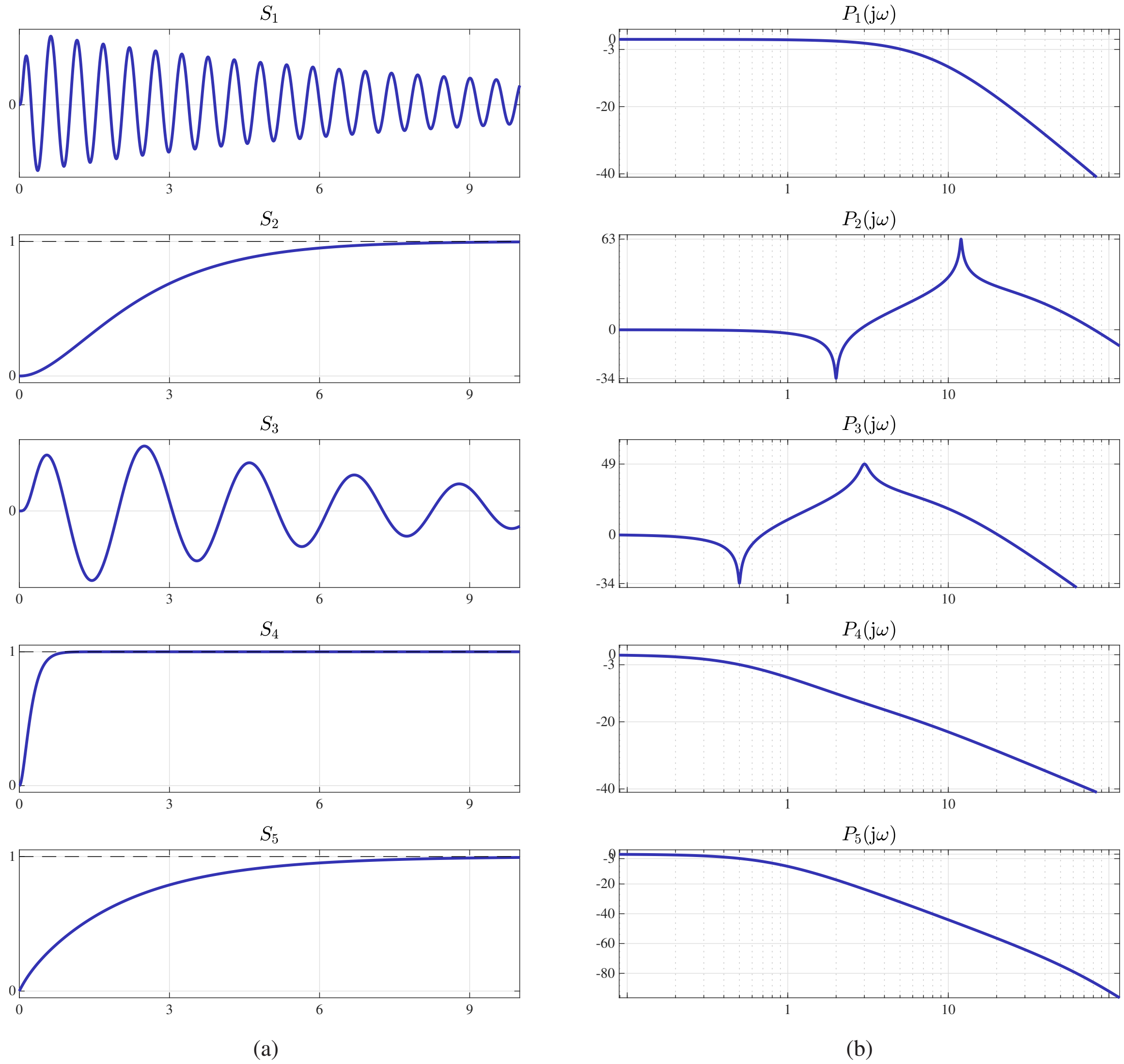


Fig. 2: Step responses and Bode magnitude plots for Question 2

Question 2. Match the step responses in Fig. 2(a) to the Bode magnitude plots in Fig. 2(b).

Solution. We know that systems with monotonically decreasing frequency-response magnitudes tend to have non-oscillatory step responses. Moreover, the wider is the bandwidth of such systems, the faster is their step response. There are three monotonic frequency responses in Fig. 2(b), those of P_1, P_4, and P_5, and three non-oscillatory step responses in Fig. 2(a), those of S_2, S_4, and S_5. Of those, S_4 is substantially faster than the others and the bandwidth of P_1, which is about 5 rad/sec, is about a decade wider than those of P_4 and P_5 (both are about 0.5 rad/sec). Hence,

$$P_1 \leftrightarrow S_4$$

This line of reasoning can no longer be used to tell P_4 from P_5, they have similar bandwidths and the step responses of S_2 and S_5 have similar raise times. Thus, a different differentiation shall be sought. What differentiates frequency responses of P_4 from P_5 is their high-frequency slopes. While for the former it is about -20 dB/dec, the latter has it smaller than -40 dB/dec (although the slope of the magnitude plot of $P_5(j\omega)$ is not constant at $\omega \approx 100$, it is clearly smaller than -40). This implies that the pole excess of $P_4(s)$ is 1, whereas the pole excess of $P_5(s)$ is at least 3 (actually, it is even 4). Consequently, the step response y_4 of P_4 must have

$$\lim_{t\to 0} \dot{y}_4(t) = \lim_{s\to\infty} s \cdot sP_4(s)\frac{1}{s} = \lim_{s\to\infty} sP_4(s) \neq 0$$

(by the Initial Value Theorem and because $sP_4(s)$ is bi-proper), whereas the step response y_5 of P_5 must have

$$\lim_{t\to 0} \dot{y}_5(t) = \lim_{s\to\infty} s \cdot sP_5(s)\frac{1}{s} = \lim_{s\to\infty} sP_5(s) = 0$$

($sP_5(s)$ is still strictly proper). Comparing the responses of S_2 and S_5 we can see that the former has zero derivative at $t = 0$, whereas the latter has it at some positive value. Hence,

$$P_4 \leftrightarrow S_5 \qquad \text{and} \qquad P_5 \leftrightarrow S_2.$$

We are also supposed to know that systems with narrow resonance peaks tend to have oscillatory step responses, with oscillation frequencies close to the resonance frequencies. Two frequency responses in Fig. 2(b), those of P_2 and P_3, have one resonance peak each. The resonance of P_2 is at a higher frequency ($\omega \approx 12$ rad/sec) than that of P_3 ($\omega \approx 3$ rad/sec), so the step response of P_2 may be expected to have faster oscillations than that of P_3. Oscillatory step responses in Fig. 2(a) are S_1 (faster) and S_3 (slower). Hence,

$$P_2 \leftrightarrow S_1. \qquad \text{and} \qquad P_3 \leftrightarrow S_3.$$

Note that in many situations the simplest way to associate frequency and step responses is via frequency-response gains at the zero frequency and steady-state values of step responses. However, all five magnitude frequency responses in Fig. 2(b) start at 0 dB. Hence, the static gains $P_i(0) = 1$ for all $i = 1, \dots, 5$. We therefore cannot distinguish the step responses of these systems by their steady-state values. Indeed, all step responses in Fig. 2(a) converge to 1 (this might not be evident from the responses of S_1 and S_3, yet it is still true). ∇

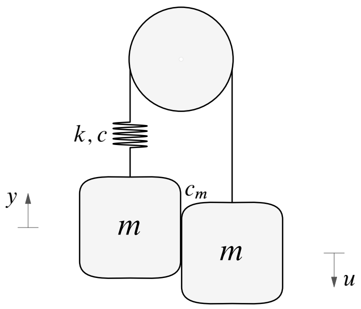


Fig. 3: System for Question 3

Question 3. Consider the system presented in Fig. 3, which consists of two masses $m = 1$ [kg] connected via a massless pulley and a spring having the spring constant $k = 1$ [N/m] and the damping coefficient $c = 0.1$ [N sec/m]. The contact between the masses causes a friction force which is proportional to the velocity, with the coefficient c_m and opposes the motion. The displacement of the right mass is the system input u and the displacement of the left mass is the output y. The system is controlled in a standard open-loop scheme with a controller $C_{\mathrm{ol}} : r \mapsto u$ and controlled response $T_{yr} : r \mapsto y$ for a reference signal r.

1. If the masses do not touch each other (i.e. no friction force is acting between them, with $c_m = 0$), then the plant $P_0 : u \mapsto y$ has the transfer function (cf. Lecture 2)
$$P_0(s) = \frac{cs + k}{ms^2 + cs + k}.$$
Design the controller C_{ol} for which the controlled responds to a reference signal r has the transfer function
$$T_{yr1}(s) = \frac{1}{\tau s + 1}.$$
Under what conditions on τ the resulting controller is admissible (i.e. internally stabilizing)?

2. Plot the step responses of u and y for $\tau \in \{0.1, 0.5, 1, 10\}$ [sec]. Explain trends under decreasing τ using frequency-domain arguments.

3. Now assume that we need to ensure the zero steady-state error between r and y for both $r(t) = \mathbb{1}(t)$ and $r(t) = \sin(\omega_r t + \phi_r)\mathbb{1}(t)$ for a given $\omega_r > 0$ and every $\phi_r \in \mathbb{R}$. Consider the family of controlled transfer functions of the form
$$T_{yr2}(s) = \frac{\alpha_2 s^2 + \alpha_1 \omega_{\mathrm{n}} s + \alpha_0 \omega_{\mathrm{n}}^2}{((\beta/\omega_{\mathrm{n}})s + 1)(s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2)}$$
for $\omega_{\mathrm{n}} = 1$, $\zeta = 3$, $\beta = 2$, and some $\alpha_i \in \mathbb{R}$, $i \in \{0, 1, 2\}$, to be chosen to satisfy the steady-state requirements. Is the resulting controller C_{ol} stabilizing? It it is, calculate α_i for $\omega_r = 1$. Plot the Bode diagram of its frequency response.

4. Now, suppose that the masses touch each other, producing a friction force with friction coefficient $c_m = 0.5$ [N sec/m]. Derive the motion equation and the transfer function of the system $P_{c_m} : u \mapsto y$ in that case.

5. Draw (schematically) the systems step response.

6. Is the controller designed for P_{c_m} under the desired controlled system $T_{\mathrm{ref},1}$ as in item 1 admissible? What requirements must be satisfied by the controlled transfer function in this case to result in an admissible (internally stabilizing) controller?

7. Consider now the design of C_{ol} for P_{c_m} so that the controlled dynamics have

$$T_{yr3}(s) = \frac{\omega_{\mathrm{n}}^2 (b_1 s + 1)}{s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2}.$$

   What constraints do we have on b_1? Design the controller C_{ol} such that the resulting system responds to reference signals r exactly as T_{yr3} for the chosen b_1. Plot the step responses of u and y for $\omega_{\mathrm{n}} \in \{0.1, 1, 2, 5\}$ and $\zeta = \sqrt{0.5}$. Explain trends under increasing ω_{n} using frequency-domain and modal arguments.

Solution.

1. With the given plant P_0 and required controlled system T_{yr1}, we have that

$$T_{yr1} = P_0 C_{\mathrm{ol}} \iff C_{\mathrm{ol}} = P_0^{-1} T_{yr1}.$$

   Hence,

$$C_{\mathrm{ol}}(s) = \frac{T_{yr1}(s)}{P_0(s)} = \frac{ms^2 + cs + k}{(cs + k)(\tau s + 1)} = \frac{s^2 + 0.1s + 1}{(0.1s + 1)(\tau s + 1)}.$$

   Internal stability requires that both P_0 and C_{ol} are stable (in that case $P_0 C_{\mathrm{ol}} = T_{yr1}$ is always stable as well).

   (a) The plant has a strictly proper transfer function (2-order denominator and 1-order numerator) and the coefficients of its 2-order denominator are all positive. Hence, P_0 is stable.

   (b) The transfer function $C_{\mathrm{ol}}(s)$ is proper iff $\tau \neq 0$ (2-order denominator and 2-order numerator) and has its poles at $s = -10$ and $s = -1/\tau$, which are in the open LHP $\mathbb{C} \setminus \bar{\mathbb{C}}_0$ iff $\tau \geq 0$.

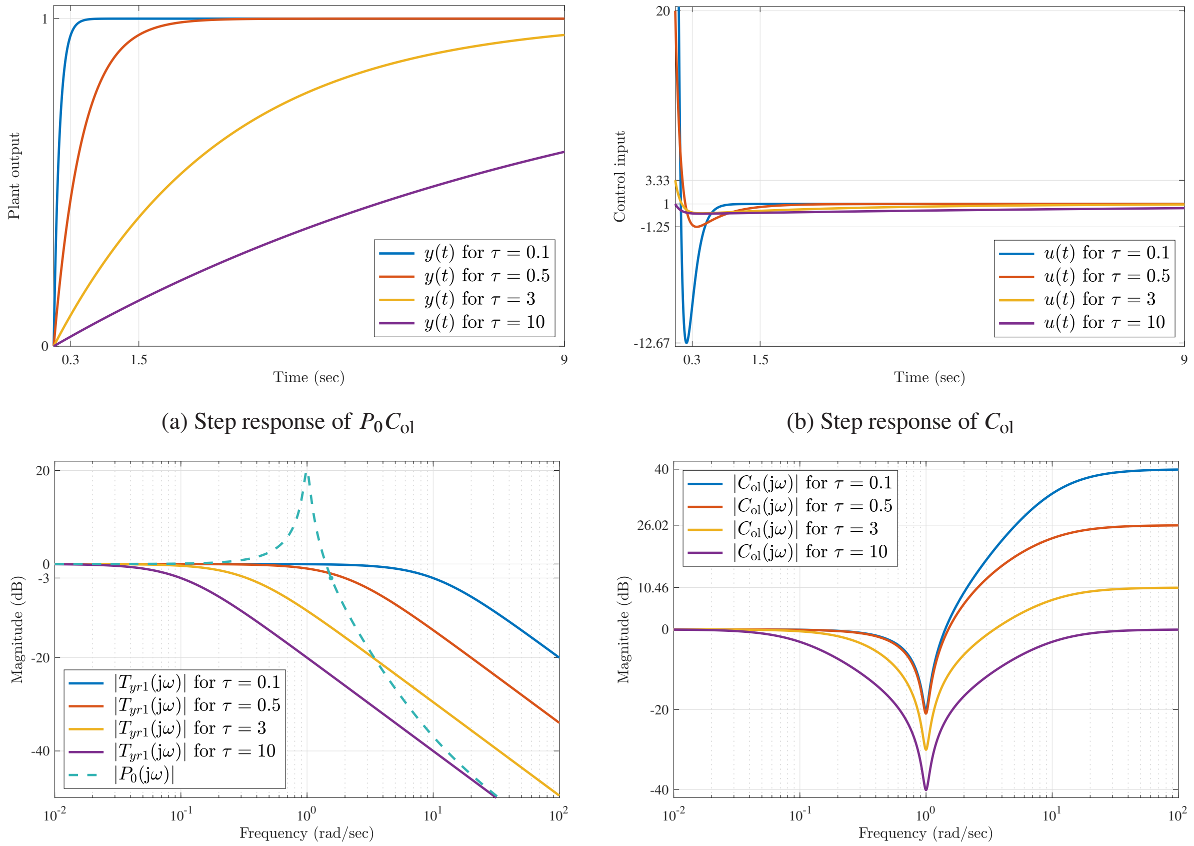
   Thus, the controller is admissible iff $\tau > 0$.

2. The responses to a step r of the plant output y and the control input u are presented in Figs. 4(a) and 4(b), respectively. We can see that as the time constant of the controlled system, τ, decreases, the response of y becomes faster, which is what we expect from the 1-order T_{yr1}. This can also be explained via frequency-domain reasoning. Indeed, as τ decreases, the *bandwidth* of T_{yr1} increases, see Fig. 4(c). There are no free lunches though. A faster response requires a higher control effort (the peak value of u under $\tau = 0.1$ is 100, far beyond the boundary of the plot). This can also be seen via the magnitude Bode diagrams of the controller in Fig. 4(d). The large gain at high frequencies there is responsible for the high spikes in the control signal at the beginning, i.e. at the moment the step (quick input change) is applied. The increase of the high-frequency part of $|C_{\mathrm{ol}}(\mathrm{j}\omega)|$ can be explained by comparing the high-frequency behavior of $|T_{yr1}(\mathrm{j}\omega)|$ and $|P_0(\mathrm{j}\omega)|$ (dashed cyan line in Fig. 4(c)). We can see that at small τ the former decays substantially slower than the latter. Then their ratio, which is $|C_{\mathrm{ol}}(\mathrm{j}\omega)|$ is, grows.

3. To ensure that the controlled system is T_{yr2}, we need again the controller $C_{\mathrm{ol}} = P_0^{-1} T_{yr2}$, i.e.

$$C_{\mathrm{ol}}(s) = \frac{T_{yr2}(s)}{P_0(s)} = \frac{(\alpha_2 s^2 + \alpha_1 \omega_{\mathrm{n}} s + \alpha_0 \omega_{\mathrm{n}}^2)(ms^2 + cs + k)}{((\beta/\omega_{\mathrm{n}})s + 1)(s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2)(cs + k)}.$$

   This transfer function is proper (4-order denominator and 4-order numerator) and all its poles, $s = -\omega_{\mathrm{n}}/\beta$, $s = (\zeta \pm \sqrt{\zeta^2 - 1})\omega_{\mathrm{n}}$, and $s = -k/c$, in the open LHP. Hence, C_{ol} is stable and, because of the stability of P_0, the control system is internally stable.



(a) Step response of $P_0 C_{\rm ol}$

(b) Step response of $C_{\rm ol}$

(c) Bode magnitude diagrams of P_0 and T_{yr1}

(d) Bode magnitude diagrams of $C_{\rm ol} = P_0^{-1} T_{yr1}$

Fig. 4: Controlled behaviors for different time constants τ for the design in item 1

To ensure zero steady-state error for a harmonic reference signal with a frequency ω, the controlled dynamics must satisfy $|1 - T_{yr2}(\mathrm{j}\omega)| = 0$ (plus $|1 - T_{yr2}(-\mathrm{j}\omega)| = 0$ if $\omega \neq 0$, to ensure that the resulting controller has real parameters). In our case this condition reads

$$T_{yr2}(0) = 1, \quad T_{yr2}(\mathrm{j}\omega_r) = 1, \quad \text{and} \quad T_{yr2}(-\mathrm{j}\omega_r) = 1. \qquad (\spadesuit)$$

Because the static gain of T_{yr2}, $T_{yr2}(0) = \alpha_0$, the first condition of ($\spadesuit$) requires

$$\boxed{\alpha_0 = 1}.$$

With this choice, the condition $T_{yr2}(s) = 1$ for any $s \in \mathbb{C}$ reads

$$\alpha_2 s^2 + \alpha_1 \omega_{\rm n} s + \omega_{\rm n}^2 = ((\beta/\omega_{\rm n})s + 1)(s^2 + 2\zeta\omega_{\rm n} s + \omega_{\rm n}^2)$$

or, equivalently,

$$\alpha_2 s^2 + \alpha_1 \omega_{\rm n} s = \frac{\beta}{\omega_{\rm n}} s^3 + (1 + 2\beta\zeta)s^2 + (\beta + 2\zeta)\omega_{\rm n} s.$$

Dividing both sides by $\omega_{\rm n} s$ and regrouping the terms, we end up with

$$\frac{s}{\omega_{\rm n}}\alpha_2 + \alpha_1 = \left(1 + \frac{s^2}{\omega_{\rm n}^2}\right)\beta + 2\zeta + \frac{s}{\omega_{\rm n}}(1 + 2\beta\zeta).$$

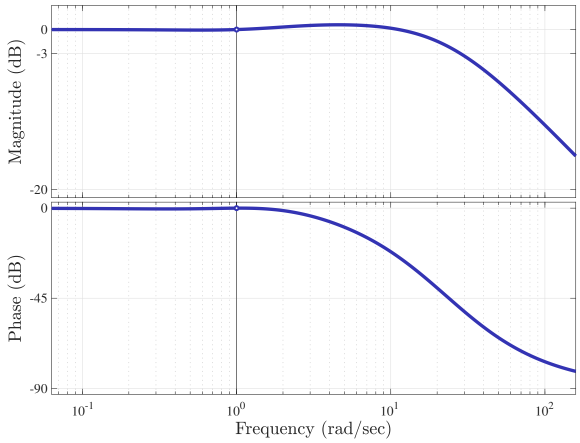


Fig. 5: Bode diagram of T_{yr2}

If $s = \mathrm{j}\omega_r$, then the real part in the left-hand side above depends only on α_1 and the imaginary part depends only on α_2. This immediately yields

$$\boxed{\alpha_1 = \left(1 - \frac{\omega_r^2}{\omega_\mathrm{n}^2}\right)\beta + 2\zeta} \quad \text{and} \quad \boxed{\alpha_2 = 1 + 2\beta\zeta}.$$

Note that $\lim_{\beta\to 0}\alpha_1 = 2\zeta$ and $\lim_{\beta\to 0}\alpha_2 = 1$. Substituting the given numerical values, we end up with

$$T_{yr2}(s) = \frac{26(s + 0.7247)(s + 1.698)}{(s + 0.6863)(s + 2)(s + 23.31)}.$$

Its Bode diagram is presented in Fig. 5. We can see that in $\omega \in [0, 1]$ not only the magnitude plot is very close to 0 dB, but also the phase is close to 0 deg. At $\omega = 1$ we have that $|T_{yr2}(\mathrm{j})| = 1$ and $\arg T_{yr2}(\mathrm{j}) = 0$, as required.

Remark 1. There is a simpler solution, which only needs to assume that the controlled transfer functions $T_{yr2}(s)$ has relative degree 1. Namely, note that a stable T_{yr2} guarantees $T_{yr2}(0) = 1$ and $T_{yr2}(\pm\mathrm{j}\omega_r) = 1$ iff $T_{er2} := 1 - T_{yr2}$ is stable and has zeros at $s = 0$ and $s = \pm\mathrm{j}\omega_r$. Moreover, $T_{yr2}(s) = 1 - T_{er2}(s)$ is strictly proper iff $T_{er2}(\infty) = 1$ (convince yourselves in that). We may thus start with

$$T_{er2}(s) = \frac{s(s^2 + \omega^2)}{\phi(s)}$$

for any *monic* and *Hurwitz* polynomial $\phi(s)$ of degree 3, i.e. $\phi(s) = s^3 + \phi_2 s^2 + \phi_1 s + \phi_0$ with positive coefficients such that $\phi_2\phi_1 > \phi_0$, and calculate $T_{yr2} = 1 - T_{er2}$. Treating situations where the relative degree of the controlled transfer function is higher than 1 is more delicate. ∇

4. We study the free body diagram of the left mass, assuming that the system is in its equilibrium with $y = u = 0$. There are two forces acting on it:

$$f_{\mathrm{spring}}(t) = k(u(t) - y(t)) + c(\dot{u}(t) - \dot{y}(t)) \quad \text{and} \quad f_{\mathrm{massf}}(t) = -c_m(\dot{u}(t) + \dot{y}(t))$$

(the friction between the masses is resisting the relative change in y and u; since their motion directions are opposite, a positive change in either of the coordinates results in a resistance force on the right mass). Therefore, the equation of motion for the left mass is

$$m\ddot{y}(t) = f_{\mathrm{spring}}(t) - f_{\mathrm{massf}}(t) = k(u(t) - y(t)) + c(\dot{u}(t) - \dot{y}(t)) - c_m(\dot{u}(t) + \dot{y}(t))$$

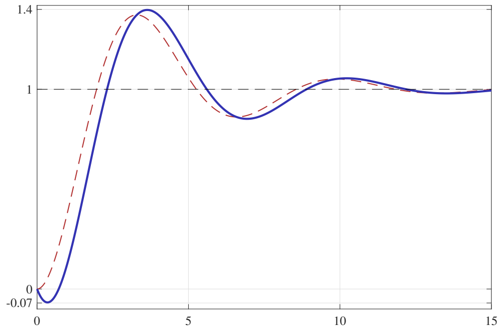


Fig. 6: Step response of P_{c_m} in the case when it is nonminimum-phase

or, equivalently,

$$m\ddot{y}(t) + (c + c_m)\dot{y}(t) + ky(t) = (c - c_m)\dot{u}(t) + ku(t).$$

Hence, the transfer function of the system $P_{c_m} : u \mapsto y$ is

$$P_{c_m}(s) = \frac{(c - c_m)s + k}{ms^2 + (c + c_m)s + k}.$$

Note that this plant is nonminimum-phase iff $c_m > c$. If the damping coefficient of the spring exceeds the friction coefficient of the masses, the plant is minimum-phase, exactly as in the case of $c_m = 0$. If $c_m = c$, then the pole excess of $P_{c_m}(s)$ is 2. In all other cases it is 1. For the given numerical values we end up with

$$P_{c_m}(s) = \frac{-0.4s + 1}{s^2 + 0.6s + 1},$$

which is nonminimum-phase, it has one zero in the RHP $\bar{\mathbb{C}}_0$ (at $s = 2.5$).

5. Consider a system P_a having the transfer function

$$P_\mathrm{a}(s) = \frac{1}{s^2 + 0.6s + 1},$$

which is the zero-free version of $P_{c_m}(s)$. This is an underdamped second-order transfer function with the natural frequency $\omega_\mathrm{n} = 1$, the damping factor $\zeta = 0.3$, and the static gain $k_\mathrm{st} = 1$. By the formulae provided in the last section of Lecture 3, the step response of this P_a should have the overshoot $\mathrm{OS} = \mathrm{e}^{-\pi\zeta/\sqrt{1-\zeta^2}} \cdot 100\% \approx 37\%$, the peak time $t_\mathrm{p} = \pi/(\omega_\mathrm{n}\sqrt{1-\zeta^2}) \approx 3.29$ [sec], and the steady-state level 1. It is presented by the red dashed line in Fig. 6. The addition of a zero at $s = 2.5$ should give rise to some undershoot and to a higher overshoot. Moreover, the step response of P_{c_m} must intersect that of P_a at every peak point of the latter, i.e. at every point where the step response of P_a has zero derivative. The blue solid line in Fig. 6 represents the step response of P_{c_m}. It has a slightly higher overshoot ($\approx 40\%$) and a slight undershoot ($\approx 7\%$).

6. The controller now has the transfer function

$$C_\mathrm{ol}(s) = \frac{T_{yr1}(s)}{P_{c_m}(s)} = \frac{ms^2 + (c + c_m)s + k}{((c - c_m)s + k)(\tau s + 1)} = \frac{s^2 + 0.6s + 1}{(-0.4s + 1)(\tau s + 1)}$$

This controller is inadmissible because it is unstable (has a RHP pole at $s = 2.5$).

The RHP zero of the plant becomes a pole of $C_\mathrm{ol}(s)$, unless it is canceled by a zero of $T_{yr}(s)$. This implies that the system is internally stable only if the controlled transfer function has a zero at $s = 2.5$.

Moreover, because $1/P_{c_m}(s)$ has 1-order denominator and 2-order numerator, $T_{yr}(s)$ must be strictly proper to result in a proper $C_{\mathrm{ol}}(s)$. Thus, the controller is admissible (internally stabilizing) iff T_{yr} is stable and its transfer function is strictly proper and has a zero at $s = 2.5$.

7. Like in all designs discussed above, the controller rendering the controlled response to be equal T_{yr3} is

$$C_{\mathrm{ol}}(s) = \frac{T_{yr3}(s)}{P_{c_m}(s)} = \frac{\omega_{\mathrm{n}}^2(b_1 s + 1)}{s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2} \frac{ms^2 + (c + c_m)s + k}{(c - c_m)s + k}$$

Three cases are possible:

(a) If $c_m < c$, then all three poles of this $C_{\mathrm{ol}}(s)$ are stable and the pole excess is 0, both regardless of b. Hence, b may be arbitrary.

(b) If $c_m = c$, then two poles of this $C_{\mathrm{ol}}(s)$ are still stable, but we only have 2-rder denominator. Hence, $C_{\mathrm{ol}}(s)$ is proper iff $b = 0$, which is the choice to render C_{ol} stabilizing.

(c) If $c_m > c$, then one pole of $C_{\mathrm{ol}}(s)$ is in the RHP, at $s = k/(c_m - c)$, unless it is canceled by the zero at $s = -1/b_1$. Hence, we must have $b_1 = (c - c_m)/k < 0$ (the controller is proper then).

For our parameters the plant is nonminimum-phase, so the third choice is ours. The controller is then calculated as

$$C_{\mathrm{ol}}(s) = \frac{\omega_{\mathrm{n}}^2(ms^2 + (c + c_m)s + k)}{k(s^2 + 2\zeta\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2)} = \frac{\omega_{\mathrm{n}}^2(s^2 + 0.6s + 1)}{s^2 + \sqrt{2}\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2}$$

and it is stable and proper, as expected.

The controlled system with

$$T_{yr3}(s) = \frac{\omega_{\mathrm{n}}^2(-0.4s + 1)}{s^2 + \sqrt{2}\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2}$$

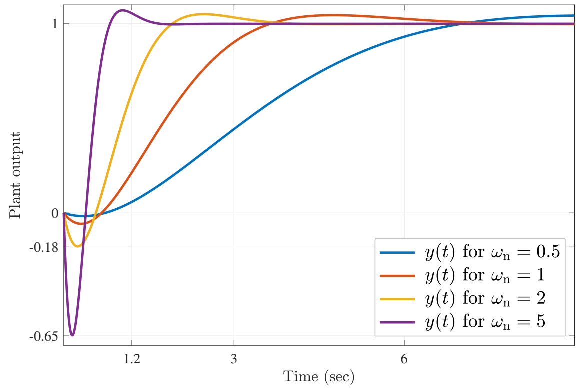
has a pair of complex conjugate poles at $s = (-1 \pm \mathrm{j})\sqrt{0.5}\,\omega_{\mathrm{n}}$. These poles yield a faster response as ω_{n} increases. In other words, the bandwidth of T_{yr3} grows with ω_{n}, see Fig. 7(c). But the increase of ω_{n} has two unwelcome consequences.

(a) Similarly to the discussion in item 2, the increase of the bandwidth of T_{yr3} requires higher control effort. This can be expected from the Bode magnitude plots of $|C_{\mathrm{ol}}(\mathrm{j}\omega)|$ in Fig. 7(d) and shows clearly in the step responses of u in Fig. 7(b) (the peak value of u under $\omega_{\mathrm{n}} = 5$ is 25, which is located beyond the boundary of the plot). The underlying reason for this behavior is again a faster decay of $|P_{c_m}(\mathrm{j}\omega)|$ (dashed cyan line in Fig. 7(c)) than $|T_{yr3}(\mathrm{j}\omega)|$ at high frequencies for large ω_{n}. Then $|C_{\mathrm{ol}}(\mathrm{j}\omega)| = |T_{\mathrm{r}2}(\mathrm{j}\omega)|/|P_{c_m}(\mathrm{j}\omega)|$ grows.

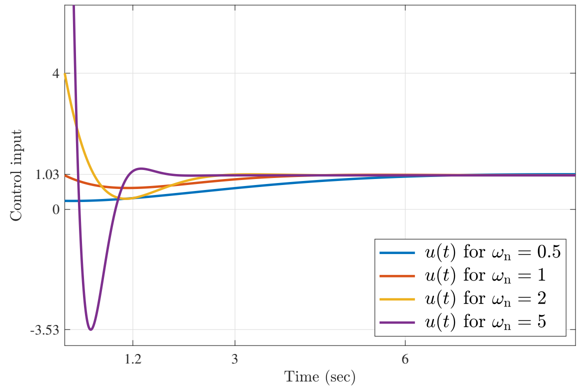
(b) As the response of y to $r = \mathbb{1}$ becomes faster, it exhibits higher and higher overshoot and, especially, undershoot. This behavior can be explained from the pole-zero maps of $T_{yr3}(s)$, see Fig. 7(e). As ω_{n} increases, the zero at $s = 2.5$ becomes more dominant. The undershoot is the very result of this dominancy. Note also that the the frequency response of T_{yr3} for $\omega_{\mathrm{n}} = 5$ has a visible resonance peak (at $\omega \approx 4.42$), which is indeed an indicator of high overshoot / undershoot. As this peak is rather wide, no oscillatory behavior should be expected (and the response in Fig. 4(a) is indeed not oscillatory).

At the same time, it may be seen from Fig. 7(e) that the poles of $T_{yr3}(s)$ are dominant at small natural frequencies. For that reason the step response for $\omega_{\mathrm{n}} = 0.5$ is quite cose to the step responses of the zero-free system $\omega_{\mathrm{n}}^2/(s^2 + \sqrt{2}\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2)$.

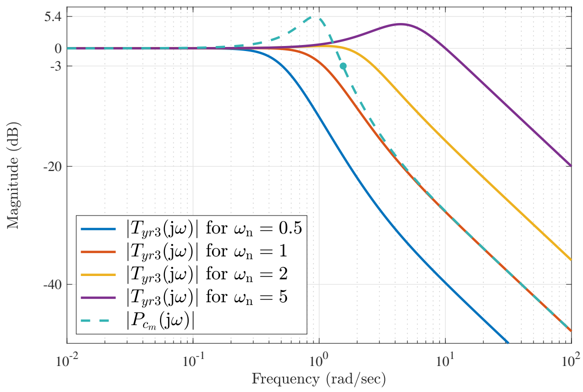
That's all … $\triangledown$



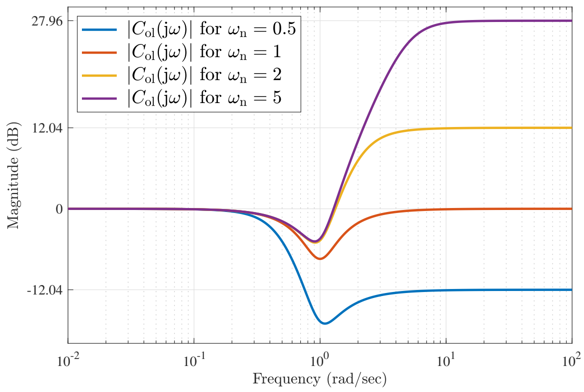
(a) Step response of $P_{c_m} C_{\mathrm{ol}}$



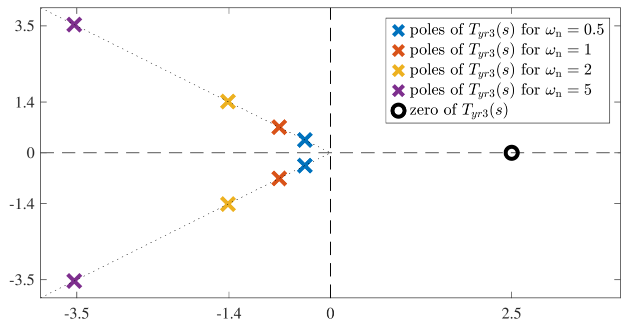
(b) Step response of C_{ol}



(c) Bode magnitude diagrams of P_{c_m} and T_{yr3}



(d) Bode magnitude diagrams of $C_{\mathrm{ol}} = T_{yr3}/P_{c_m}$



(e) Pole-zero map of T_{yr3}

Fig. 7: Controlled behaviors under T_{yr3} for different natural frequencies ω_n

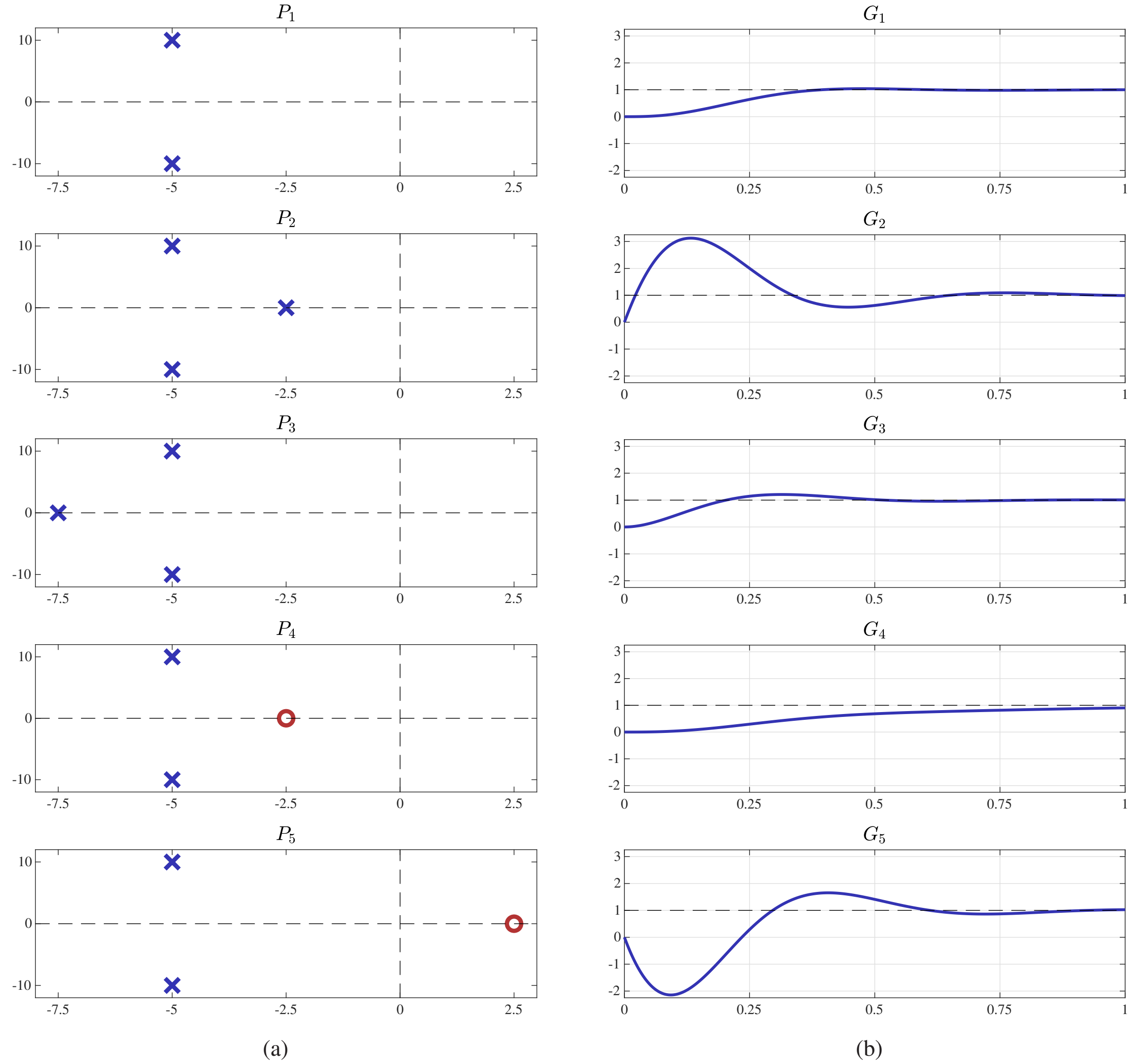


Fig. 8: Pole-zero maps and step responses for Question 4

Question 4 (self study)**.** Consider now a second-order underdamped system, like that given by P_2 in Fig. 1(a) with various modifications. Match the pole-maps given in Fig. 8(a) to the step time-responses in Fig. 8(b).

Solution. First, note that all systems have a pair of poles at $s = -5 \pm \mathrm{j}10$. This pair should produce an underdamped response with overshoot $\approx 20\%$ and peak time ≈ 0.313, like the response of G_4 in Fig. 1(b). Of the responses in Fig. 8(b), only that of G_3 corresponds to these parameters, so we have

$$P_1 \leftrightarrow G_3.$$

Of the other maps in Fig. 8(a), two (P_2 and P_3) have an additional pole added and two (P_4 and P_5)—an additional zero. We know that the addition of a pole slows down the response (longer rise times) and reduces its overshoot. This description suits the responses of G_1 and G_4. Moreover, the close this additional pole to the origin is, the more visible its effect on the step response is. Hence,

$$P_2 \leftrightarrow G_4 \qquad \text{and} \qquad P_3 \leftrightarrow G_1$$

(the response of G_4 is slower and its overshoot is lower). The addition of a zero speeds up the response (shorter rise times) and increases the overshoot. Moreover, if this additional zero is in the RHP, the response has the undershoot. This description should make it clear that

$$P_4 \leftrightarrow G_2 \qquad \text{and} \qquad P_5 \leftrightarrow G_5,$$

just because G_5 undershoots, whereas G_2 does not. ∇