הטכניון – מכון טכנולוגי לישראל, הפקולטה להנדסת מכונות

TECHNION—Israel Institute of Technology, Faculty of Mechanical Engineering

Introduction to Control (034040)

tutorial 3

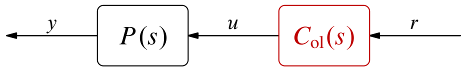


Fig. 1: Open-loop control system

Question 1. Consider the following plants controlled in open loop as illustrated in Fig. 1:

1. $P(s) = \frac{1}{s+1}$
2. $P(s) = \frac{s-2}{s+1}$
3. $P(s) = \frac{s+2}{s-1}$

Can these plants be controlled by the controller $C_{\text{ol}} = P^{-1}$?

Solution.

1. This process is stable ($P(s)$ is proper and has no poles in the RHP). The transfer function of the controller is
$$C_{\text{ol}}(s) = \frac{1}{P(s)} = s + 1.$$
This transfer function is non-proper (numerator order is 1 and denominator order is 0). In the time domain, it determines the control law $u(t) = \dot{r}(t) + r(t)$. Hence, it can be implemented only if we can measure both r and $\dot{r}$ and, in addition, if $|\dot{r}|$ is bounded.

2. This process is stable but has a RHP zero. The resulting controller has the transfer function
$$C_{\text{ol}}(s) = \frac{1}{P(s)} = \frac{s+1}{s-2}$$
This controller is proper but has a RHP pole and is therefore unstable, resulting in unbounded control signal u. For example, for a bounded input $r(t) = 2\mathbb{1}(t)$ we end up with $u(t) = (3e^{2t} - 1)\mathbb{1}(t)$. Thus, although we seem to get a perfect inversion $T_{yr} = PC_{\text{ol}} = 1$, it will demand an unrealistic control effort. Moreover, if the are any inaccuracies in the model, the unstable pole won't cancel and T_{yr} will be unstable. Thus, the system is *not internally stable* and cannot be used.

3. This process is unstable. The required controller for plant inversion has
$$C_{\text{ol}}(s) = \frac{1}{P(s)} = \frac{s-1}{s+2},$$
which is stable and non-minimum phase (has a RHP zero). Although we seem to have a perfect inversion, $T_{yr} = 1$, any external disturbance or inaccuracy in the model will result in an unbounded output. This is also a sign of *internal instability*.

   As a matter of fact, there is no open-loop controller that can fix this issue, i.e. unstable processes cannot be controlled in open loop.

That's all … ▽

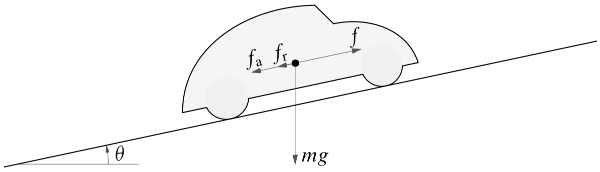


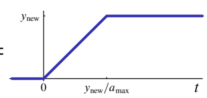
Fig. 2: System for Question 2

Question 2. Fig. 2 depicts a vehicle of mass $m = 1000$ [kg] driving uphill with the slope $\theta = 12°$. The driving force f generated by the engine is the control signal, whose goal is to maintain the car velocity v at a prespecified level. The resistance force has three major components: $f_g = mg \sin\theta$, the forces due to gravity; f_a, the aerodynamic drag; and f_r, the forces due to rolling friction. Assuming that the velocity of the car is always positive, the rolling resistance $f_r = mgc_r \cos\theta$, where the rolling resistance coefficient $c_r = 0.01$. The aerodynamic drag is proportional to the square of the speed, i.e. $f_a = \frac{1}{2}\alpha v^2$, where $\alpha \approx 1$ [kg/m] is a constant depending on the density of air, the frontal area of the car, and a shape-dependent aerodynamic drag coefficient.

1. Derive the equation of motion of the system and linearize it around the speed equilibrium point $v_{eq} = 80$ [km/h] $= \frac{80 \cdot 1000}{3600} = \frac{200}{9} \approx 22.22$ [m/sec] (denote the the control input and the regulated output in terms of deviation variables by u and y, respectively).

2. Consider the open-loop control strategy in which a controller C_{ol} generates the control signal u from a reference velocity signal r_v. Draw the block-diagram of this system. Design C_{ol} by plant inversion. When can this controller be implemented? Simulate the response of the linearized system under

$$r_v(t) = \begin{cases} 0 & \text{if } t \le 0 \\ a_{max} t & \text{if } 0 \le t \le y_{new}/a_{max} \\ y_{new} & \text{if } t \ge y_{new}/a_{max} \end{cases} = \tag{1}$$



for the peak acceleration $a_{max} = 0.5$ [m/s^2] and $y_{new} = 10$ [km/h] $= \frac{25}{9} \approx 2.78$ [m/sec].

3. Assume that the road slope is actually different from the slope at which we modeled the car and designed our controller, viz. it equals $\bar{\theta} = 13°$. Write the new equation of motion in this case and and simulate the linearized response to the saturated ramp in (1). What is the settling time (to the settling level 5%) in this case?

4. To cope with slope uncertainty, a slope sensor can be installed in the vehicle. Suggest an open-loop control strategy, which exploits the information about the actual slope and guarantees that the regulated output $y = r_v$ for all t under any change in θ.

5. Assume now that the slope is as expected, but a passenger weighing $70g$ [N] entered the vehicle, so that its actual vehicle mass becomes $\bar{m} = 1070$ [kg]. Find the actual linearized dynamics and simulate the linearized response to the saturated ramp in (1). What is the settling time (to the settling level 5%) in this case?

6. Analyze the nonlinear system with the open-loop controller as in item 2. What is its steady-state response to the reference signal in (1)?

Solution.

1. The standard (Newtonian) force balance for the car reads

$$m\dot{v}(t) = f(t) - f_\mathrm{r}(t) - f_\mathrm{g}(t) - f_\mathrm{a}(t) = f(t) - \frac{1}{2}\alpha v^2(t) - mg\sin\theta - mgc_\mathrm{r}\cos\theta$$

This is a (nonlinear) first-order ODE, so the state can be chosen as $x(t) = v(t)$. The state equation reads then

$$\dot{x}(t) = \frac{1}{m}f(t) - \frac{1}{2m}\alpha x^2(t) - g(\sin\theta + c_\mathrm{r}\cos\theta) =: \phi(x, F).$$

Equilibria should then verify $\dot{x}_\mathrm{eq} = \phi(x_\mathrm{eq}, f_\mathrm{eq}) \equiv 0$, i.e.

$$f_\mathrm{eq} = \frac{1}{2}\alpha v_\mathrm{eq}^2 + mg(\sin\theta + c_\mathrm{r}\cos\theta). \qquad (\spadesuit)$$

There is a unique f_eq for any v_eq. In particular, with $v_\mathrm{eq} = 22.22$ [m/sec] we have $f_\mathrm{eq} = 2382.09$ [N]. Then, the parameters of the linearized system are

$$A = \frac{\partial\phi}{\partial v}\Big|_{v=v_\mathrm{eq}, f=f_\mathrm{eq}} = -\frac{\alpha v_\mathrm{eq}}{m} \approx -0.022 \quad \text{and} \quad B = \frac{\partial\phi}{\partial f}\Big|_{v=v_\mathrm{eq}, f=f_\mathrm{eq}} = \frac{1}{m} = 0.001,$$

giving the state equation in terms of deviation variables

$$\begin{cases} \dot{\tilde{x}}(t) = -\dfrac{\alpha v_\mathrm{eq}}{m}\tilde{x}(t) + \dfrac{1}{m}u(t) \\ y(t) = \tilde{x}(t) \end{cases} \iff \begin{cases} ms\tilde{X}(s) = -\alpha v_\mathrm{eq}\tilde{X}(s) + U(s) \\ Y(s) = \tilde{X}(s) \end{cases}$$

where $\tilde{x} := x - v_\mathrm{eq}$, $u := f - f_\mathrm{eq}$, and $y := v - v_\mathrm{eq}$. The transfer function of the system $u \mapsto y$ is

$$P(s) = \frac{1}{ms + \alpha v_\mathrm{eq}} = \frac{0.045}{45.07s + 1}. \qquad (\clubsuit)$$

This is a first-order system, whose time constant $\tau = m/(\alpha v_\mathrm{eq})$ depends on the equilibrium velocity. In fact, τ is a decreasing function of v_eq, meaning that the system responds faster at higher velocities.

2. The block-diagram of the open-loop control system is exactly as that in Fig. 1 modulo the replacement of r with r_v. The plant inversion controller has the transfer function

$$C_\mathrm{ol}(s) = \frac{1}{P(s)} = ms + \alpha v_\mathrm{eq}.$$

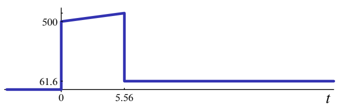
and generates

$$u(t) = m\dot{r}_v(t) + \alpha v_\mathrm{eq} r_v(t).$$

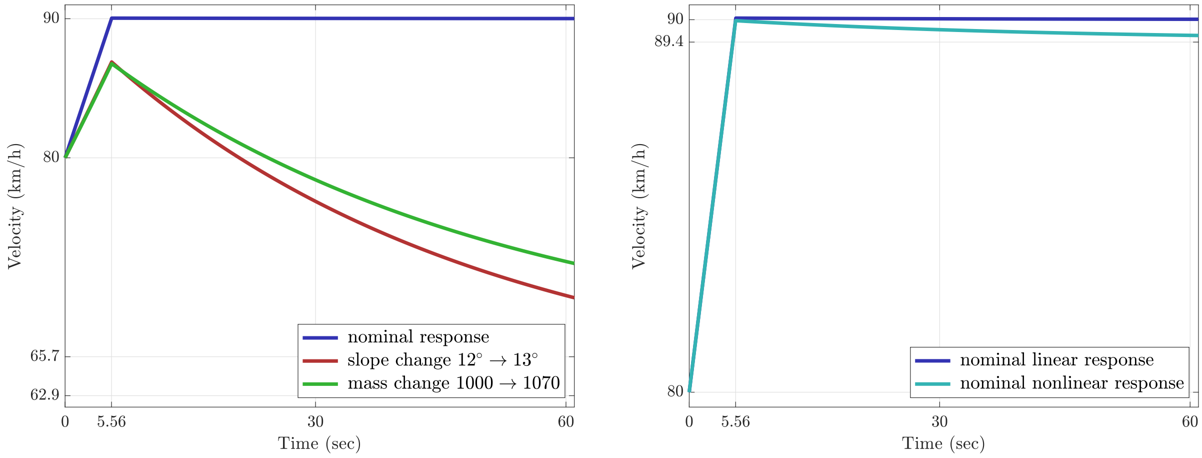
Although this C_ol is unstable ($C_\mathrm{ol}(s)$ is non-proper), it generates an unbounded u only if r_v or its derivative $\dot{r}_v$ is unbounded (we assume that they are measurable). For the reference signal r_v in (1),

$$u(t) = ma_\mathrm{max}\big(\mathbb{1}(t) - \mathbb{1}(t - y_\mathrm{new}/a_\mathrm{max})\big) + \alpha v_\mathrm{eq} r_v(t)$$

$$\approx 500\big(\mathbb{1}(t) - \mathbb{1}(t - 5.56)\big) + 22.19\, r_v(t) = \qquad (\diamondsuit)$$



is actually bounded (r_v is obviously bounded). Hence, we can safely implement C_ol above for this class of reference signals (that might change for other choices of r_v). Note that the pulse at the



(a) Linearized model

(b) Nonlinear model

Fig. 3: Velocity responses to r_v from (1)

beginning has a substantially higher amplitude (500) than the saturated ramp (≤ 61.63). In other words, the acceleration mode requires substantially higher force than cruise.

It is worth emphasizing that the actual driving force under this control law is $u + f_{\text{eq}}$, i.e.

$$f(t) = m\big(\dot{r}_v(t) + (\sin\theta + c_{\text{r}}\cos\theta)g\big) + \alpha v_{\text{eq}}(r_v(t) + 0.5v_{\text{eq}}),$$

and the actual velocity is $y + v_{\text{eq}}$.

The response of the vehicle to this input is presented in Fig. 3(a) by the blue line. As expected, it perfectly matches the (shifted by v_{eq}) reference velocity $r_v(t)$, raising linearly from 80 [km/h] to 90 [km/h] with the slope $a_{\max} = 0.5$ [m/sec^2] in $\frac{y_{\text{new}}}{a_{\max}} \approx 5.56$ sec and then staying at 90 [km/h].

3. If the road slope is $\bar{\theta} \neq \theta$, the linearized model reads $y = P\bar{u}$, where $\bar{u}$ is the deviation of the input force F from its equilibrium f_{eq} given by (♠) modulo the replacement of θ with $\bar{\theta}$ (note that $P(s)$ in (♣) does not depend on the slope angle θ). Hence,

$$\begin{aligned}\bar{u}(t) &= f(t) - 0.5\alpha v_{\text{eq}}^2 - mg(\sin\bar{\theta} + c_{\text{r}}\cos\bar{\theta}) = u(t) + mg(\sin\theta - \sin\bar{\theta} + c_{\text{r}}(\cos\theta - \cos\bar{\theta}))\\ &= u(t) + 2mg\Big(\cos\frac{\theta+\bar{\theta}}{2} - c_{\text{r}}\sin\frac{\theta+\bar{\theta}}{2}\Big)\sin\frac{\theta-\bar{\theta}}{2}.\end{aligned}$$

This situation corresponds to the block-diagram in Fig. 4, where the "disturbance"

$$d(t) = d_{\bar{\theta}} := 2mg\Big(\cos\frac{\theta+\bar{\theta}}{2} - c_{\text{r}}\sin\frac{\theta+\bar{\theta}}{2}\Big)\sin\frac{\theta-\bar{\theta}}{2} \approx -166.79 \qquad (♡)$$

(not $-166.79\mathbb{1}(t)$, a constant for all t). For this system

$$y = Pd + r_v = r_v + \frac{1}{ms + \alpha v_{\text{eq}}}d_{\bar{\theta}}.$$

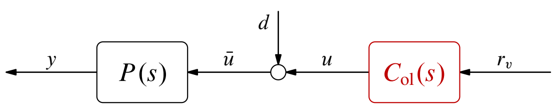


Fig. 4: Open-loop cruise control under uncertain road slope

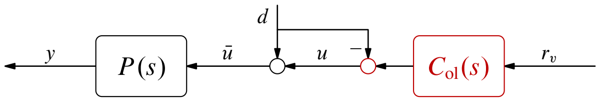


Fig. 5: Open-loop cruise control under uncertain road slope with slope measurements

In this case

$$\lim_{t\to\infty} r_v(t) - y(t) = -\frac{1}{\alpha v_{\text{eq}}} d_{\bar\theta} \approx 7.52\,[\text{m/sec}] \approx 27.06\,[\text{km/h}],$$

which is about 34% of v_{eq} (the deviation of the slope is now more that 8% of its nominal value)..

The response of the vehicle is presented in Fig. 3(a) by the red line. We can see that the resulted velocity is quite far from the expected one. After a fast raise in the right direction at the beginning (when the driving force is rather large), the velocity starts to drop gradually to its steady state value $62.94 < 80$. The dynamics of this drop is slow, because they are driven by the time constant $\tau \approx 45.07$ of the plant model. Hence, the settling time to the settling level 5% is

$$t_{\text{s}} = \frac{y_{\text{new}}}{a_{\max}} + 3\tau \approx 140.77\,[\text{sec}].$$

4. To compensate the effect of slope variations, we need to compensate the effect of the constant disturbance d in Fig. 4. In fact, we can calculate this disturbance if we measure the actual slope angle, $\bar\theta$, via the formula in (♡). Having d measured, the compensation scheme is presented in Fig. 5. Indeed, the control signal generated by the controller is now

$$u = C_{\text{ol}} r_v - d = C_{\text{ol}} r_v - d_{\bar\theta},$$

which results in

$$\bar u = u + d = C_{\text{ol}} r_v \implies y = P\bar u = r_v,$$

as required.

5. The mass of the vehicle appears in our model in two places. It's a part of the transfer function in (♣), affecting its time constant, and a part of the equilibrium driving force in (♠). In other words, if we knew the mass, the model would be

$$\bar P(s) = \frac{1}{\bar m s + \alpha v_{\text{eq}}} = \frac{0.045}{48.23s + 1}$$

under the input

$$\bar u(t) = f(t) - 0.5\alpha v_{\text{eq}}^2 - \bar m g(\sin\theta + c_{\text{r}}\cos\theta) = u(t) + (m - \bar m) g(\sin\theta + c_{\text{r}}\cos\theta).$$

Because the controller is "unaware" of this, it still generates u chosen on the basis of $P(s)$. This situation can be described via the block diagram depicted in Fig. 6, where the "disturbance"

$$d(t) = d_{\bar m} := (m - \bar m) g(\sin\theta + c_{\text{r}}\cos\theta) \approx -149.49$$

(again, not $-149.49\mathbb{1}(t)$, a constant for all t). Thus, the output (which itself is a deviation from v_{eq}) is

$$y = \bar P(s) d + \frac{\bar P(s)}{P(s)} r_v = \frac{1}{\bar m s + \alpha v_{\text{eq}}} d_{\bar m} + \frac{ms + \alpha v_{\text{eq}}}{\bar m s + \alpha v_{\text{eq}}} r_v = r_v + \frac{1}{\bar m s + \alpha v_{\text{eq}}} d_{\bar m} + \frac{(m - \bar m)s}{\bar m s + \alpha v_{\text{eq}}} r_v$$

$$= r_v + \frac{1}{\bar m s + \alpha v_{\text{eq}}} \big((m - \bar m) s r_v + d_{\bar m} \big).$$

Thus, the response deviated from the reference trajectory and the deviation is proportional to the mismatch between the assumed and actual vehicle masses. This deviation even does not vanish in steady state. Indeed, in the limit we have (substituting $s \to 0$) that

$$\lim_{t\to\infty} r_v(t) - y(t) = \frac{1}{\alpha v_{\rm eq}} d_{\bar m} \approx 6.74\,[\text{m/sec}] \approx 24.26\,[\text{km/h}],$$

which is almost a third of $v_{\rm eq}$ (30.3%, to be precise). This is a lot, taking into account that the mass change is only 7% of the nominal mass.

The response of the vehicle is presented in Fig. 3(a) by the green line. Similarly to the slope change, we can see that the resulted velocity is quite far from the expected one. After a fast raise in the right direction at the beginning (when the driving force is rather large), the velocity starts to drop gradually to its steady state value $65.74 < 80$. The dynamics of this drop is slow, because they are driven by the time constant $\tau \approx 48.23$ of the plant model. Hence, the settling time to the settling level 5% is

$$t_{\rm s} = \frac{y_{\rm new}}{a_{\max}} + 3\tau \approx 150.25\,[\text{sec}].$$

6. The response of the nonlinear car dynamics to $u + f_{\rm eq}$, where u is as in ($\diamondsuit$), will of course deviate from the linearized response. The analysis of transients in this case is quite complicated. Still, we can simulate the nonlinear system, obtaining the response presented by the cyan line in Fig. 3(b). It deviates from the linear response, especially in its steady state.

   Steady-state analysis is simpler, so we carry it out below. First, because the initial pulse does not affect the steady-state level of the control signal in ($\diamondsuit$), we have that

$$u_{\rm ss} = \alpha v_{\rm eq} y_{\rm new}.$$

   Because $f = u + f_{\rm eq}$, we have the following steady-state level of the driving force:

$$f_{\rm ss} = \alpha v_{\rm eq} y_{\rm new} + f_{\rm eq} = \alpha v_{\rm eq} y_{\rm new} + \frac{1}{2}\alpha v_{\rm eq}^2 + mg(\sin\theta + c_{\rm r}\cos\theta)$$

   (taking into account (♠)). But it follows from (♠) that the actual steady-state value of the vehicle velocity (assuming its positivity) for a given steady-state level of F is

$$\begin{aligned} v_{\rm ss} &= \sqrt{\frac{2(f_{\rm ss} - mg(\sin\theta + c_{\rm r}\cos\theta))}{\alpha}} = \sqrt{\frac{2\alpha v_{\rm eq} y_{\rm new} + \alpha v_{\rm eq}^2}{\alpha}} = \sqrt{2v_{\rm eq} y_{\rm new} + v_{\rm eq}^2} \\ &= \sqrt{2v_{\rm eq} v_{\rm new} - v_{\rm eq}^2} = \sqrt{v_{\rm new}^2 - (v_{\rm new} - v_{\rm eq})^2}, \end{aligned}$$

   where $v_{\rm new} := v_{\rm eq} + y_{\rm new}$ is the required new steady-state velocity level. Thus, we always end up at a lower steady-state velocity level with respect to the required $v_{\rm new}$. This is consistent with what we see in Fig. 3(b).

That's all … ▽

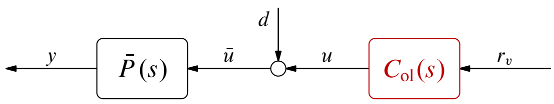


Fig. 6: Open-loop cruise control under uncertain vehicle mass