Introduction to Control (034040)

tutorial 2

Question 1. Fig. 1 depicts a system for controlling the angle $\theta_a(t)$ of an antenna. The antenna, having

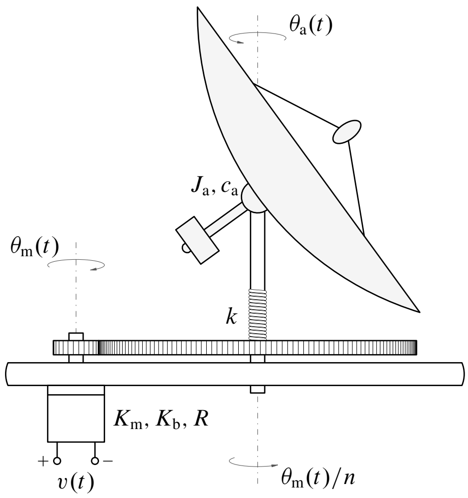

Fig. 1: Antenna actuated via a DC motor

moment of inertia J_a and balanced with respect to the vertical axis, is rotated by a DC motor with torque constant K_m, back emf constant K_b, armature resistance R and negligible inertia and armature inductance. The transmission system comprises two meshing gears, having a ratio of $1 : n$ and negligible inertia, a flexible rod with torsion coefficient (spring constant) k and negligible viscous friction, and a bearing system with viscous friction coefficient c_a. The control input is the voltage v applied to the DC motor.

1. Derive motion equations of the system.
2. Construct the block-diagram of the system and calculate the transfer function of the system $v \mapsto \theta_a$.

Solution.

1. First, consider the DC motor. Let θ_m denote the angle of its shaft. With neglected inductance L, the armature circuit satisfies

$$Ri(t) = v(t) - v_b(t),$$

where i is the armature current and v_b is the back emf voltage, which is proportional to the angular velocity $\omega_m = \dot\theta_m$ of the motor,

$$v_b(t) = K_b\dot\theta_m(t).$$

The torque generated by the motor is proportional to its armature current,

$$\tau_m(t) = K_m i(t).$$

Combining the three displayed equations above, we end up with the following torque equation:

$$\tau_m(t) = \frac{K_m}{R}\bigl(v(t) - K_b\,\dot\theta_m(t)\bigr) \quad \text{or, equivalently,} \quad T_m(s) = \frac{K_m}{R}\bigl(V(s) - K_b s\Theta_m(s)\bigr). \tag{1}$$

While v is the external input, θ_m is an internal signal in the system, which depends on the dynamics of the load and thus leads to an additional coupling between the motor and its load.

Consider now the load (i.e. the transmission and the antenna). We use Newtonian arguments to model it, by splitting to simpler free-body diagrams and formulating balances of momenta. To this end, note that there are two torques acting on the large gear: one is applied by the flexible rod, τ_rod, and another one—by the motor via the gear system (which merely amplifies it by a factor of n). Since the moment of inertia of the gear is assumed to be negligible, we have the following static torque equilibrium:

$$0 = \tau_\mathrm{rod}(t) + n\tau_\mathrm{m}(t). \tag{2}$$

The antenna then experiences the reaction torque applied by the spring and the torque of viscous friction in the bearing, τ_fric. The moment of inertia of the antenna is non-negligible, hence the balance of angular momenta at the antenna reads

$$J_\mathrm{a}\ddot{\theta}_\mathrm{a}(t) = -\tau_\mathrm{rod}(t) - \tau_\mathrm{fric}(t). \tag{3}$$

The torque applied by the flexible rod is proportional to the torsion of the spring, i.e. to the difference between the gear angle and the antenna angle θ_a. As the angle of the axis of the large gear is related to the motor angle via the transmission ratio, the rod torque verifies

$$\tau_\mathrm{rod}(t) = k\Bigl(\theta_\mathrm{a}(t) - \frac{1}{n}\theta_\mathrm{m}(t)\Bigr). \tag{4}$$

The torque generated by the viscous friction in the bearing system is then

$$\tau_\mathrm{fric}(t) = c_\mathrm{a}\dot{\theta}_\mathrm{a}(t). \tag{5}$$

Combining (2) and (4), we get the following equation for the motor shaft angle:

$$\theta_\mathrm{m}(t) = n\theta_\mathrm{a}(t) + \frac{n^2}{k}\tau_\mathrm{m}(t). \tag{6a}$$

Substituting (4) and (5) into (3), we get $J_\mathrm{a}\ddot{\theta}_\mathrm{a}(t) = -k\bigl(\theta_\mathrm{a}(t) - \frac{1}{n}\theta_\mathrm{m}(t)\bigr) - c_\mathrm{a}\dot{\theta}_\mathrm{a}(t)$. The first term in the right-hand side of this equation equals, via (2), $n\tau_\mathrm{m}(t)$, so we end up with the following antenna dynamics:

$$J_\mathrm{a}\ddot{\theta}_\mathrm{a}(t) + c_\mathrm{a}\dot{\theta}_\mathrm{a}(t) = n\tau_\mathrm{m}(t). \tag{6b}$$

Equations (6) define the dynamics of the load with the motor torque τ_m as its input. In the Laplace domain these equation read

$$\Theta_\mathrm{m}(s) = n\Theta_\mathrm{a}(s) + \frac{n^2}{k}T_\mathrm{m}(s) \quad \text{and} \quad \Theta_\mathrm{a}(s) = \frac{n}{s(J_\mathrm{a}s + c_\mathrm{a})}T_\mathrm{m}(s). \tag{7}$$

2. The block diagram corresponding to equations (1) and (7) is presented in Fig. 2, where the "black" blocks represent (1) and the "blue" blocks—(7).

   The derivation of the transfer function of the system $v \mapsto \theta_\mathrm{a}$ is easier to do by tracing signals in this case. Namely, it is readily seen (cf. (7)) that

$$\Theta_\mathrm{m}(s) = \biggl(\frac{n^2}{k} + \frac{n^2}{s(J_\mathrm{a}s + c_\mathrm{a})}\biggr)T_\mathrm{m}(s) = \frac{n^2(J_\mathrm{a}s^2 + c_\mathrm{a}s + k)}{ks(J_\mathrm{a}s + c_\mathrm{a})}T_\mathrm{m}(s)$$

   and then

$$T_\mathrm{m}(s) = \frac{K_\mathrm{m}}{R}\biggl(V(s) - \frac{n^2K_\mathrm{b}s(J_\mathrm{a}s^2 + c_\mathrm{a}s + k)}{ks(J_\mathrm{a}s + c_\mathrm{a})}T_\mathrm{m}(s)\biggr)$$

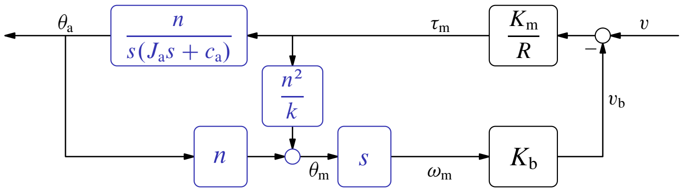

Fig. 2: Block-diagram of the system in Question 1

or, equivalently,

$$\left(1 + \frac{n^2 K_{\mathrm{m}} K_{\mathrm{b}}\, s(J_{\mathrm{a}}s^2 + c_{\mathrm{a}}s + k)}{kRs(J_{\mathrm{a}}s + c_{\mathrm{a}})}\right) T_{\mathrm{m}}(s) = \frac{K_{\mathrm{m}}}{R} V(s).$$

Thus,

$$T_{\mathrm{m}}(s) = \frac{K_{\mathrm{m}} k s(J_{\mathrm{a}}s + c_{\mathrm{a}})}{kRs(J_{\mathrm{a}}s + c_{\mathrm{a}}) + n^2 K_{\mathrm{m}} K_{\mathrm{b}}\, s(J_{\mathrm{a}}s^2 + c_{\mathrm{a}}s + k)} V(s)$$

and then

$$\Theta_{\mathrm{a}}(s) = \frac{n K_{\mathrm{m}} k}{kRs(J_{\mathrm{a}}s + c_{\mathrm{a}}) + n^2 K_{\mathrm{m}} K_{\mathrm{b}}\, s(J_{\mathrm{a}}s^2 + c_{\mathrm{a}}s + k)} V(s).$$

Thus, we end up with the third-order transfer function of the system $v \mapsto \theta_{\mathrm{m}}$

$$\frac{n K_{\mathrm{m}} k}{s(n^2 K_{\mathrm{m}} K_{\mathrm{b}}\, J_{\mathrm{a}} s^2 + (kRJ_{\mathrm{a}} + n^2 K_{\mathrm{m}} K_{\mathrm{b}}\, c_{\mathrm{a}})s + (Rc_{\mathrm{a}} + n^2 K_{\mathrm{m}} K_{\mathrm{b}})k)}.$$

As a matter of fact, this transfer function has one pole at the origin ($s = 0$) and two poles in the open left-half place (because this is the case with any second-order polynomial with positive coefficients).

That's all … ∇

Question 2. Consider the magnetic levitation system described in Fig. 3. The electric current i running

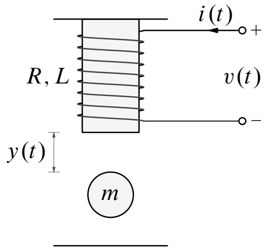

Fig. 3: Magnetic levitation system

through a coil, having resistance R and inductance L, creates a magnetic field, which attracts an iron ball of mass m. The objective is to control the ball position y via the input voltage v. The electromagnetic force applied by the magnetic field to the ball is

$$f_{\mathrm{em}}(t) = \alpha \frac{i^2(t)}{y^2(t)}, \tag{8}$$

where $\alpha > 0$ is constant.

1. Write dynamic equations of the systems and its state-space realization.

2. For a given equilibrium position of the ball, $y(t) \equiv y_0$, find the state and input at the equilibrium, linearize the system dynamics around that point, and derive the relation between the deviation variables $\tilde{v}$ and $\tilde{y}$ in the Laplace transform domain (the s-domain).

3. Is the linearized system BIBO stable?

4. Assume that the model and the deviation variables are derived for some m, but the actual mass of the ball is $\bar{m} \neq m$. How does this affect the linearized relation between $\tilde{v}$ and $\tilde{y}$?

Solution.

1. The force balance equation on the ball is $m\ddot{y}(t) = -f_{\mathrm{em}}(t) + mg$. Substituting (8) into this equation, we obtain

$$m\ddot{y}(t) = -\alpha \frac{i^2(t)}{y^2(t)} + mg.$$

The dynamics of the electric RL circuit are

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big(Li(t)\big) + Ri(t) = v(t).$$

The dynamics of the whole system is then a combination of the two differential equations above.

It is convenient to present these equation in the state-space form. As our system comprises the second- and first-order dynamics, we may expect to have a 3-dimensional state vector. The following choice for it might appear natural:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} := \begin{bmatrix} y(t) \\ \dot{y}(t) \\ i(t) \end{bmatrix}.$$

Then the equations above can be rewritten as

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ g - \frac{\alpha}{m} x_3^2(t)/x_1^2(t) \\ \frac{1}{L} v(t) - \frac{R}{L} x_3(t) \end{bmatrix} &&=: f(x, v), \\ y(t) &= x_1(t) &&=: h(x, v). \end{aligned} \tag{9}$$

which is indeed a first-order differential equation, as any state equation is supposed to be.

2. As we probably know (from "Linear Systems M") the linearization procedure for the state equation $\dot{x} = f(x, v)$ and the output equation $y = h(x, u)$ follows the following steps:
   (a) find equilibria $(x_{\mathrm{eq}}, v_{\mathrm{eq}})$, which are the all points satisfying $\dot{x} \equiv 0$, i.e. $f(x_{\mathrm{eq}}, u_{\mathrm{eq}}) = 0$;
   (b) at a chosen equilibrium point $(x_{\mathrm{eq}}, u_{\mathrm{eq}})$, calculate the matrices

$$A := \frac{\partial f(x, u)}{\partial x}\bigg|_{(x,u)=(x_{\mathrm{eq}},u_{\mathrm{eq}})} \in \mathbb{R}^{n\times n}, \qquad B := \frac{\partial f(x, u)}{\partial u}\bigg|_{(x,u)=(x_{\mathrm{eq}},u_{\mathrm{eq}})} \in \mathbb{R}^{n\times 1},$$
$$C := \frac{\partial h(x, u)}{\partial x}\bigg|_{(x,u)=(x_{\mathrm{eq}},u_{\mathrm{eq}})} \in \mathbb{R}^{1\times n}, \qquad D := \frac{\partial h(x, u)}{\partial u}\bigg|_{(x,u)=(x_{\mathrm{eq}},u_{\mathrm{eq}})} \in \mathbb{R}^{1\times 1},$$

   where the derivative of a vector $\phi \in \mathbb{R}^n$ with respect to a vector $\eta \in \mathbb{R}^m$ is defined as the $n \times m$ matrix M, whose (i, j) entry equals $\partial\phi_i/\partial\eta_j$.

   This procedure results in the linearized state and output equations $\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$ and $\tilde{y} = C\tilde{x} + D\tilde{u}$, respectively, in terms of the deviation variables

$$\tilde{x}(t) := x(t) - x_{\mathrm{eq}}, \quad \tilde{u}(t) := u(t) - u_{\mathrm{eq}}, \quad \text{and} \quad \tilde{y}(t) := y(t) - h(x_{\mathrm{eq}}, u_{\mathrm{eq}}).$$

   Applying this technique to (9), we have:
   (a) An equilibrium point must satisfy (assuming that y and i are always positive)

$$\begin{bmatrix} x_{\mathrm{eq}2} \\ g - \frac{\alpha}{m} x_{\mathrm{eq}3}^2/x_{\mathrm{eq}1}^2 \\ \frac{1}{L} v_{\mathrm{eq}} - \frac{R}{L} x_{\mathrm{eq}3} \end{bmatrix} = 0 \quad \iff \quad \begin{cases} x_{\mathrm{eq}2} = 0 \\ \sqrt{mg}\, x_{\mathrm{eq}1} = \sqrt{\alpha}\, x_{\mathrm{eq}3} \\ R x_{\mathrm{eq}3} = v_{\mathrm{eq}} \end{cases}$$

   Thus, for any given ball position y_0, we have $x_{\mathrm{eq}1} = y_0$ and the equilibrium equalities above have a unique solution,

$$(x_{\mathrm{eq}}, v_{\mathrm{eq}}) = \left(\begin{bmatrix} y_0 \\ 0 \\ \sqrt{mg/\alpha}\, y_0 \end{bmatrix}, R\sqrt{\frac{mg}{\alpha}}\, y_0 \right) \quad \text{and then } y_{\mathrm{eq}} = y_0. \tag{10}$$

   The corresponding deviation input and output variables satisfy then

$$\tilde{u}(t) := u(t) - R\sqrt{\frac{mg}{\alpha}}\, y_0 \quad \text{and} \quad \tilde{y}(t) := y(t) - y_0. \tag{11}$$

   (b) Now,

$$\frac{\partial f(x, v)}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ 2\alpha x_3^2/(m x_1^3) & 0 & -2\alpha x_3/(m x_1^2) \\ 0 & 0 & -R/L \end{bmatrix}, \qquad \frac{\partial f(x, v)}{\partial v} = \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix},$$
$$\frac{\partial h(x, v)}{\partial x} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad \frac{\partial h(x, v)}{\partial v} = 0.$$

Substituting the equilibrium point (10) into these expressions, we end up with

$$\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right] = \left[\begin{array}{ccc:c} 0 & 1 & 0 & 0 \\ 2g/y_0 & 0 & -2\sqrt{g\alpha/m}/y_0 & 0 \\ 0 & 0 & -R/L & 1/L \\ \hdashline 1 & 0 & 0 & 0 \end{array}\right].$$

Note that the eigenvalues of the block-diagonal A are $\lambda_1 = -R/L$ and $\lambda_{2,3} = \pm\sqrt{2g/y_0}$.

To derive the relation between v and y in the s-domain, note that the equation $\dot{\tilde{x}} = A\tilde{x} + B\tilde{v}$ reads there as $s\tilde{X}(s) = A\tilde{X}(s) + B\tilde{V}(s)$ or, equivalently,

$$\tilde{X}(s) = (sI - A)^{-1}B\tilde{V}(s).$$

Then,

$$\begin{aligned}\tilde{Y}(s) &= C\tilde{X}(s) + D\tilde{V}(s) = \bigl(D + C(sI - A)^{-1}B\bigr)\tilde{V}(s) \\ &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} s & -1 & 0 \\ -2g/y_0 & s & 2\sqrt{g\alpha/m}/y_0 \\ 0 & 0 & s + R/L \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix}\tilde{V}(s) \\ &= -\frac{2\sqrt{g\alpha/m}}{(Ls + R)(y_0s^2 - 2g)}\tilde{V}(s),\end{aligned}$$

which is what we need.

Remark 1. A way to invert the 3×3 matrix $sI - A$ is to split it into blocks as

$$sI - A = \left[\begin{array}{cc:c} s & -1 & 0 \\ -2g/y_0 & s & 2\sqrt{g\alpha/m}/y_0 \\ \hdashline 0 & 0 & s + R/L \end{array}\right]$$

and note that it is block lower triangular. We may then use the formula

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix},$$

which requires inverting only 2×2 and 1×1 matrices,

$$\begin{bmatrix} s & -1 \\ -2g/y_0 & s \end{bmatrix}^{-1} = \frac{y_0}{y_0s^2 - 2g}\begin{bmatrix} s & 1 \\ 2g/y_0 & s \end{bmatrix} \quad \text{and} \quad (s + R/L)^{-1} = \frac{L}{Ls + R}$$

which is simpler than the direct inversion of a 3×3 matrix. A yet simpler way is to use some symbolic software, like Wolfram Mathematica. $\triangledown$

3. The transfer function of the linearized system $\tilde{v} \mapsto \tilde{y}$ is obviously

$$P(s) = -\frac{2\sqrt{g\alpha/m}}{(Ls + R)(y_0s^2 - 2g)}.$$

It is proper, but one of its poles is in $\bar{\mathbb{C}}_0$, viz. at $\sqrt{2g/y_0}$. Therefore, the linearized system is unstable.

As a matter of fact, the fact that the "A" matrix of the linearized system has an eigenvalue in $\mathbb{C}_0$, at $\sqrt{2g/y_0}$, implies that the nonlinear model (9) is not Lyapunov stable either (by Lyapunov's indirect method).

4. The parameters of the linearized model under mass $\bar{m}$ are

$$\left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 2g/y_0 & 0 & -2\sqrt{g\alpha/\bar{m}}/y_0 & 0 \\ 0 & 0 & -R/L & 1/L \\ \hline 1 & 0 & 0 & 0 \end{array}\right],$$

which only changes one element of $\bar{A}$, and the corresponding equilibria are

$$(\bar{x}_{\text{eq}}, \bar{v}_{\text{eq}}, \bar{y}_{\text{eq}}) = \left(\begin{bmatrix} y_0 \\ 0 \\ \sqrt{\bar{m}g/\alpha}\, y_0 \end{bmatrix}, R\sqrt{\frac{\bar{m}g}{\alpha}} y_0, y_0 \right). \tag{12}$$

The linearized model in the time domain can then be written as

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{B}(v(t) - \bar{v}_{\text{eq}}) = \bar{A}\bar{x}(t) + \bar{B}(v(t) - v_{\text{eq}}) + \bar{B}(v_{\text{eq}} - \bar{v}_{\text{eq}}) \\ y(t) &= \bar{C}\bar{x}(t) + y_0, \end{aligned}$$

where $\bar{x}(t) := x(t) - \bar{x}_{\text{eq}}$. Note that because we do not see the state vector in the input/output relation, there is no need to translate the deviation of the state from $\bar{x}_{\text{eq}}$ to that from x_{eq}. Because $\tilde{v}(t) = v(t) - v_{\text{eq}}$ and $\tilde{y}(t) = y(t) - y_{\text{eq}} = y(t) - y_0$, the relation between these "old" deviation signals reads

$$\tilde{Y}(s) = -\frac{2\sqrt{g\alpha/\bar{m}}}{(Ls+R)(y_0 s^2 - 2g)}\big(\tilde{V}(s) + D(s)\big),$$

where the disturbance

$$d(t) := v_{\text{eq}} - \bar{v}_{\text{eq}} = R\sqrt{\frac{g}{\alpha}}(\sqrt{m} - \sqrt{\bar{m}})y_0.$$

In other words, a miscalculation of the equilibrium points results not only in different parameters of the plant transfer function (the gain, in this case), but also in a constant load disturbance.

That's all … $\triangledown$