הטכניון – מכון טכנולוגי לישראל, הפקולטה להנדסת מכונות

TECHNION—Israel Institute of Technology, Faculty of Mechanical Engineering

Introduction to Control (034040)

tutorial 1

Question 1. Draw the asymptotic Bode magnitude plots of the transfer function

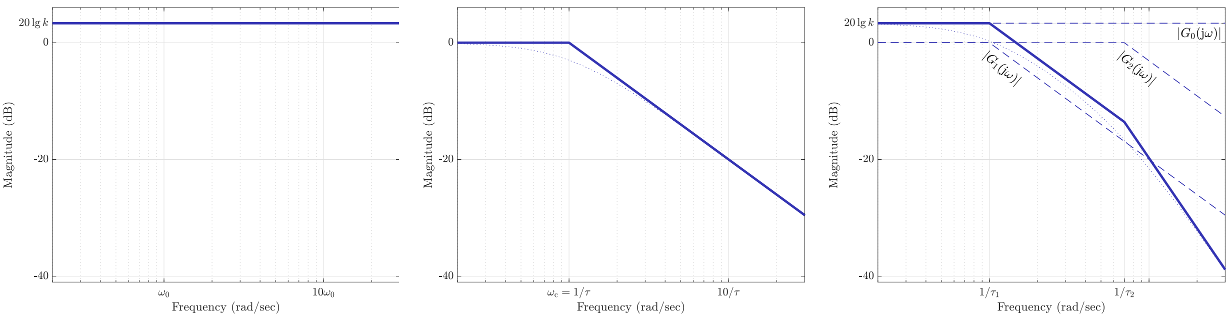
$$G(s) = \frac{k}{(\tau_1 s + 1)(\tau_2 s + 1)},$$

where $\tau_1 > 0$ and $\tau_2 > 0$.

Solution. Let us factor $G(s)$ as

$$G(s) = k \cdot \frac{1}{\tau_1 s + 1} \cdot \frac{1}{\tau_2 s + 1} =: G_0(s)G_1(s)G_2(s),$$

The transfer function $G_0(s) = k$ is static, whose magnitude bode diagram is the straight horizontal like at the level $20 \log k$ (remember, the Bode plot is in dB), see Fig. 1(a). The two other transfer functions are first-order transfer functions with the unit static gain of the form $1/(\tau s + 1)$. The asymptotic magnitude Bode plots of this kind of transfer functions comprises two straight lines: a horizontal one at 0 dB in the low-frequency range, up to the cutoff frequency $\omega_c = 1/\tau$, and a straight line starting at ω_c and decaying with the slope of -20 deg/dec (sometimes referred to as having a *rolloff* of 1), see Fig. 1(b). Now, we know (from the Linear Systems course) that the Bode magnitude plot of the cascade of systems is the superposition of their individual Bode magnitude plots. We then end up with the diagram presented by the solid line in Fig. 1(c). ∇



(a) $G(s) = k$ (b) $G(s) = 1/(\tau s + 1)$ (c) $G(s) = G_0(s)G_1(s)G_2(s)$

Fig. 1: Asymptotic Bode magnitude diagrams (here $k > 1$); dotted lines correspond to actual Bode plots

Question 2. Draw the Bode and polar plots for the following transfer functions:

1. $G_1(s) = \dfrac{1}{(\tau s+1)^2}$ for $\tau > 0$;
2. $G_2(s) = \dfrac{k}{s(\tau s+1)}$ for $\tau > 0$ and $k > 0$;
3. $G_3(s) = \dfrac{\tau_2 s+1}{\tau_1 s+1}$ for $\tau_1 = \frac{1}{3}$ and $\tau_2 = \frac{5}{3}$ and then for $\tau_1 = \frac{5}{3}$ and $\tau_2 = \frac{1}{3}$.

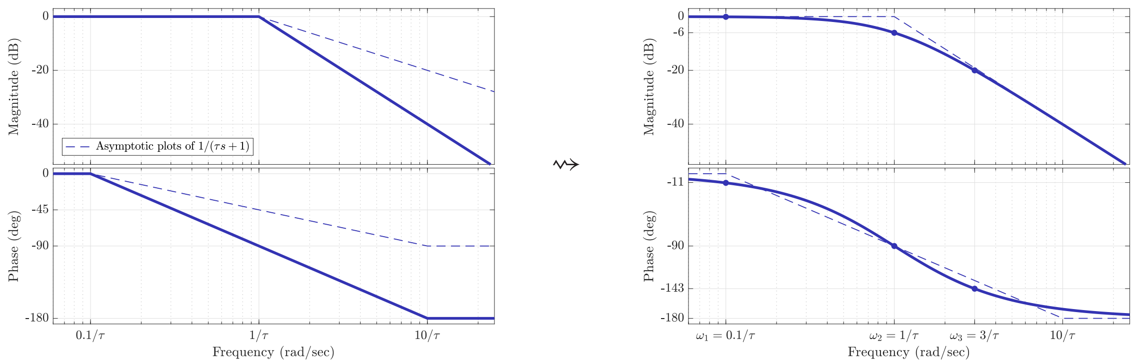
Solution. We shall follow the following procedure: first, draw asymptotic Bode plots, then actual Bode plots (via "rounding corners"), then present frequency responses at several frequency on the polar plot plane (from Bode), and then actual polar plots via connecting those points. The magnitude and the phase at the chosen points can be evaluated via the analytic expressions for the corresponding frequency responses.

1. This $G_1(s)$ can be presented as the cascade of two identical systems with the transfer function $1/(\tau s+1)$. The asymptotic plots for $G(\mathrm{j}\omega)$ can be presented, following the steps of Question 1, as in Fig. 2(a). The actual Bode plots are then shown by the solid lines in Fig. 2(b). Analytic expression for the frequency response is

$$G_1(\mathrm{j}\omega) = \frac{1}{(\mathrm{j}\omega\tau+1)^2} = \left(\frac{1}{\sqrt{\tau^2\omega^2+1}}\right)^2 \mathrm{e}^{\mathrm{j}2\arg(1/(\mathrm{j}\omega\tau+1))} = \frac{1}{\tau^2\omega^2+1}\mathrm{e}^{-\mathrm{j}2\arctan(\tau\omega)}.$$

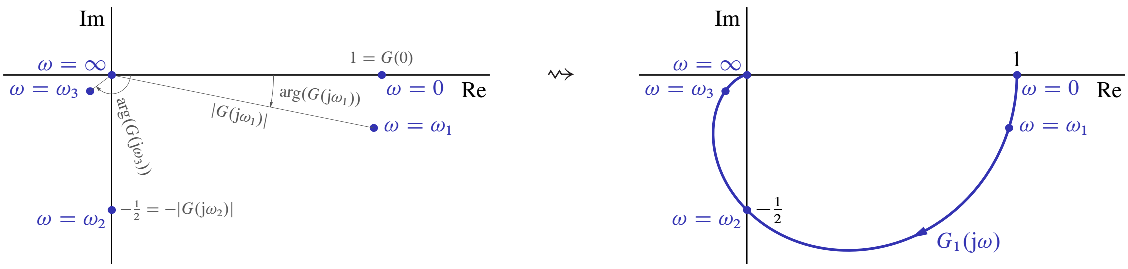
Let us pick the following frequency points:

ω	0	$\omega_1 = 0.1/\tau$	$\omega_2 = 1/\tau$	$\omega_3 = 3/\tau$	∞
$\lvert G_1(\mathrm{j}\omega)\rvert$	1	≈ 0.99	0.5	0.1	0
$\arg(G_1(\mathrm{j}\omega))$	$0°$	$\approx -11°$	$-90°$	$\approx -143°$	$-180°$



(a) Asymptotic Bode diagram

(b) Actual Bode diagram



(c) Several points of polar plot

(d) Actual polar plot

Fig. 2: Frequency response plots of $G_1(s)$

The frequency responses at the frequencies ω_i, $i = 1, 2, 3$, are marked by large dots in Fig. 2(b). The corresponding points at the complex plane, which is the plane of the polar plot, are presented in Fig. 2(c). Connecting these dots we end up with the polar plot in Fig. 2(d), where the arrow shows the direction of the plot as ω increases. Note that as $\omega \uparrow \infty$, the plot approaches the origin along the negative real axis, because the argument of $G_1(j\omega)$ approaches $-180°$ then.

2. The steps here are similar to those taken in the previous system. The asymptotic and actual Bode diagrams are then presented in Figs. 3(a) and 3(c), respectively. The frequency response

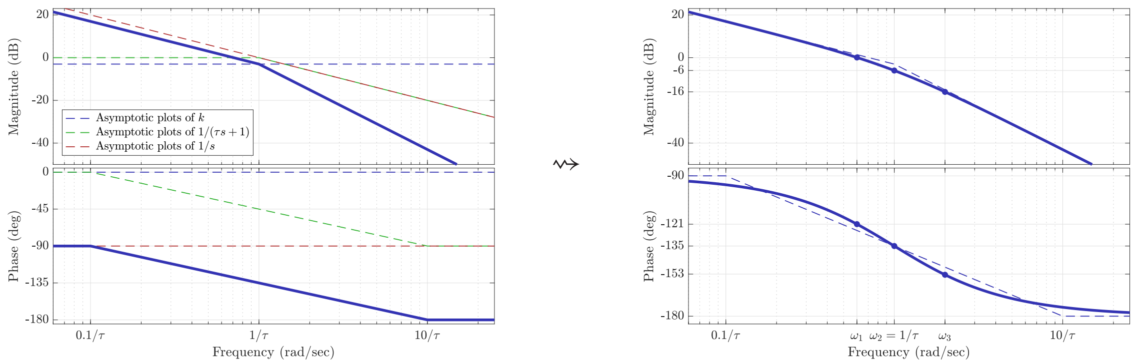
$$G_2(j\omega) = \frac{k}{j\omega(j\tau\omega + 1)} = \frac{k}{\omega\sqrt{\tau^2\omega^2 + 1}} e^{-j(\pi/2 + \arctan(\tau\omega))},$$

from which the frequency responses at the chosen points are

ω	0	$\omega_1 = 0.6/\tau$	$\omega_2 = 1/\tau$	$\omega_3 = 3/\tau$	∞
$\lvert G_2(j\omega)\rvert$	∞	$\approx 1.43k\tau$	$k\tau/\sqrt{2}$	$\approx 0.11k\tau$	0
$\arg(G_2(j\omega))$	$-90°$	$\approx -121°$	$-135°$	$\approx -153°$	$-180°$

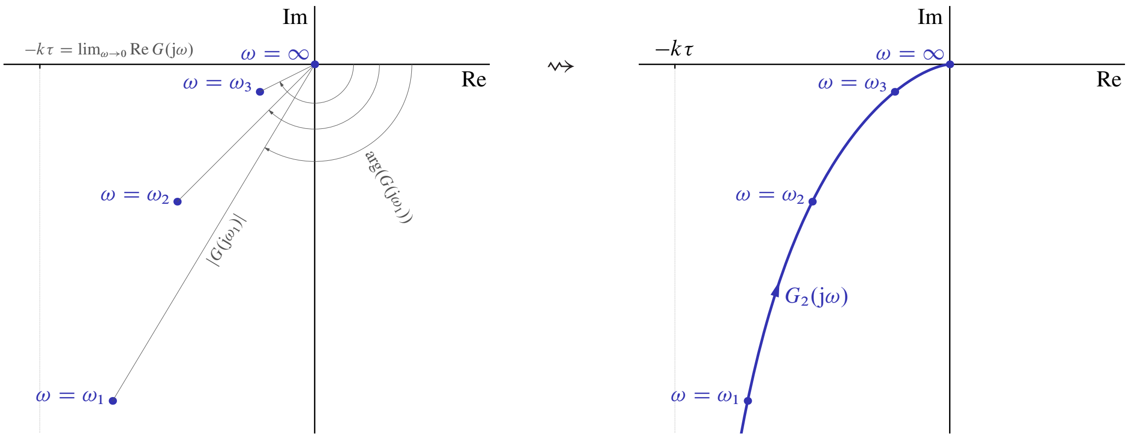
The only nontrivial difference is that due to the presence of an integrator in $G_2(s)$, $|G_2(0)| = \infty$. To understand the behavior of the hodograph at small frequencies, rewrite

$$G_2(j\omega) = \frac{k}{j\omega(j\tau\omega + 1)} = -\frac{k\tau}{\tau^2\omega^2 + 1} - j\frac{k}{\omega(\tau^2\omega^2 + 1)}.$$



(a) Asymptotic Bode diagram

(b) Actual Bode diagram



(c) Several points of polar plot

(d) Actual polar plot

Fig. 3: Frequency response plots of $G_2(s)$ (here $k = 1/\sqrt{2}$ and $\tau = 1$)

It is now seen that while the imaginary part goes to $-\infty$, the real part approaches a finite value, $-k\tau$ (in fact, the real part belongs to $(-k\tau, 0)$ for all ω). This yields the polar plot in Figs. 3(d).

3. This transfer function can be presented as

$$G_3(s) = \frac{\tau_2 s + 1}{\tau_1 s + 1} = (\tau_2 s + 1) \cdot \frac{1}{\tau_1 s + 1},$$

which is the cascade of a first-order system and the inverse of another first-order system. The asymptotic plots of the former are as the dashed lines in Fig. 2(a) and those of the latter—are the same plots modulo the opposite signs (inversion of a transfer function means sign inversion on Bode). The form of the convolution of such plots depends on the relation between τ_1 and τ_2.

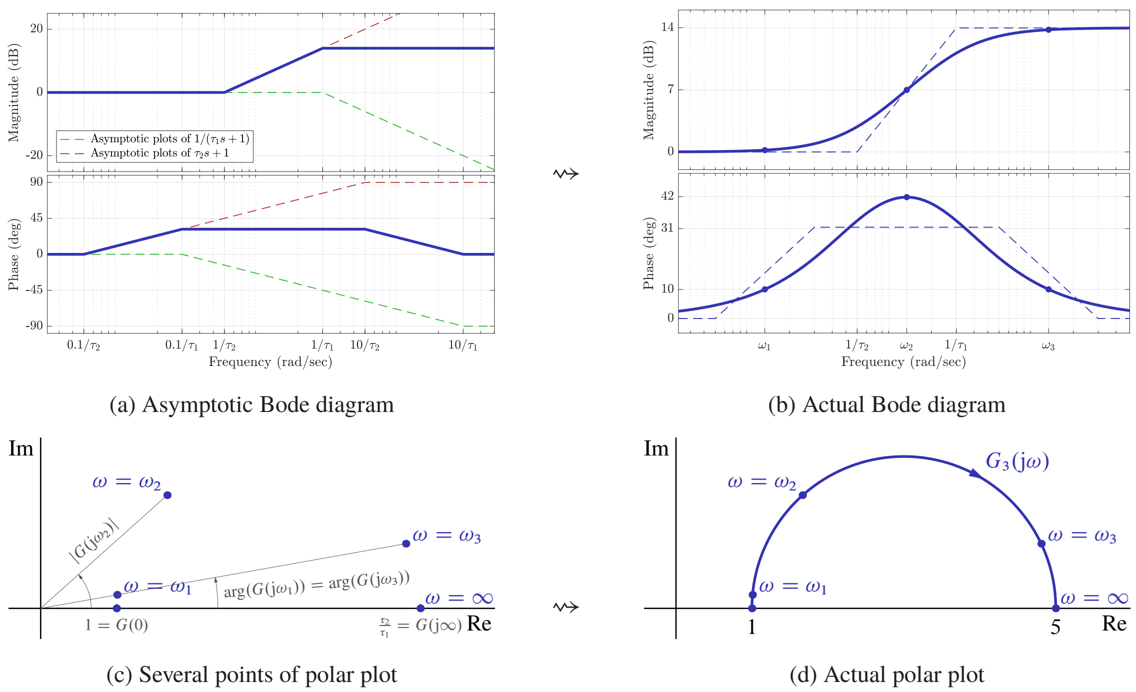
- If $\frac{1}{3} = \tau_1 < \tau_2 = \frac{5}{3}$, the effect of the zero precedes that of the pole (as ω increases). Hence, the magnitude starts at 0 bB (this is the static gain), then gets up at $\omega_{c2} := 1/\tau_2 = 0.6$ and then becomes flat again at $\omega_{c1} := 1/\tau_1 = 3$. This is what we can see in Fig. 4(a). The actual Bode diagram is presented in Fig. 4(b). To construct the polar plot, pick the following frequencies:

ω	0	$\omega_1 = 0.1342$	$\omega_2 = 1.342$	$\omega_3 = 13.42$	∞
$\lvert G_3(\mathrm{j}\omega)\rvert$	1	≈ 1.02	≈ 2.2361	≈ 4.88	5
$\arg(G_3(\mathrm{j}\omega))$	$0°$	$\approx 10.04°$	≈ 41.81	≈ 10.04	$0°$

where the values can be obtained from the frequency response

$$G_3(\mathrm{j}\omega) = \frac{\mathrm{j}\tau_2\omega + 1}{\mathrm{j}\tau_1\omega + 1} = \sqrt{\frac{\tau_2^2\omega + 1}{\tau_1^2\omega + 1}}\, \mathrm{e}^{\mathrm{j}(\arctan(\tau_2\omega) - \arctan(\tau_1\omega))}$$

(with $\frac{1}{3} = \tau_1 > \tau_2 = \frac{5}{3}$, although it is also true for $\tau_1 > \tau_2$). The polar plot is then as shown in Fig. 4(d). Note that $\arg(G_3(\mathrm{j}\omega_1)) = \arg(G_3(\mathrm{j}\omega_3))$, so the corresponding points lie on the same radial line in the complex plane.



(a) Asymptotic Bode diagram

(b) Actual Bode diagram

(c) Several points of polar plot

(d) Actual polar plot

Fig. 4: Frequency response plots of $G_3(s)$ for $\frac{1}{3} = \tau_1 < \tau_2 = \frac{5}{3}$

- If $\frac{5}{3} = \tau_1 > \tau_2 = \frac{1}{3}$, the effect of the pole precedes that of the zero (as ω increases). Hence, the magnitude starts at 0 bB (this is the static gain), then gets down at $\omega_{c1} := 1/\tau_1 = 0.6$ and then becomes flat again at $\omega_{c2} := 1/\tau_2 = 3$. This is what we can see in Fig. 5(a). The actual Bode diagram is presented in Fig. 5(b). To construct the polar plot, pick the same frequencies as in the previous case. We then have:

ω	0	$\omega_1 = 0.1342$	$\omega_2 = 1.342$	$\omega_3 = 13.42$	∞
$\lvert G_3(\mathrm{j}\omega)\rvert$	1	≈ 0.98	≈ 0.45	≈ 0.2	0.2
$\arg(G_3(\mathrm{j}\omega))$	$0°$	$\approx -10.04°$	≈ -41.81	≈ -10.04	$0°$

  The polar plot is then as shown in Fig. 5(d).

As a matter of fact, it can be verified that for all τ_1 and τ_2, the real and imaginary parts of $G_3(\mathrm{j}\omega)$,

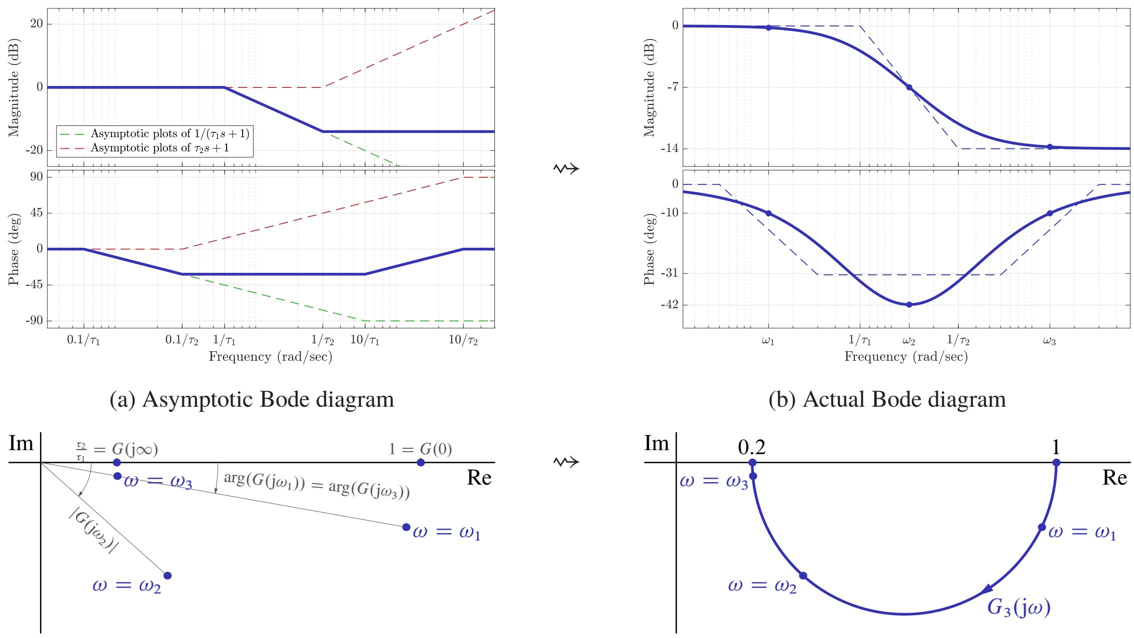
$$\mathrm{Re}\big(G_3(\mathrm{j}\omega)\big) = \frac{\tau_1\tau_2\omega^2 + 1}{\tau_1^2\omega^2 + 1} \quad \text{and} \quad \mathrm{Im}\big(G_3(\mathrm{j}\omega)\big) = \frac{(\tau_2 - \tau_1)\omega}{\tau_1^2\omega^2 + 1},$$

verify

$$\left(\mathrm{Re}\big(G_3(\mathrm{j}\omega)\big) - \frac{\tau_1 + \tau_2}{2\tau_1}\right)^2 + \Big(\mathrm{Im}\big(G_3(\mathrm{j}\omega)\big)\Big)^2 = \left(\frac{\tau_1 - \tau_2}{2\tau_1}\right)^2.$$

This implies that whenever $\tau_1 \neq \tau_2$, the polar plot of $G_3(\mathrm{j}\omega)$ is a semi-circle centered at $\frac{1}{2}(1 + \frac{\tau_2}{\tau_1})$ and having the radius $\frac{1}{2}|1 - \frac{\tau_2}{\tau_1}|$. This is also true for the particular case when $\tau_2 = 0$, which is the standard first-order transfer function.

That's all … ∇



(a) Asymptotic Bode diagram

(b) Actual Bode diagram

(c) Several points of polar plot

(d) Actual polar plot

Fig. 5: Frequency response plots of $G_3(s)$ for $\frac{5}{3} = \tau_1 > \tau_2 = \frac{1}{3}$

Question 3. Simplify the block-diagram in Fig. 6 and find the transfer function $T_y(s)$ from r to y and $T_u(s)$ from r to u.

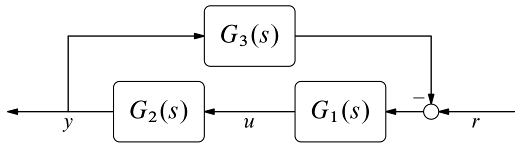


Fig. 6: Block-diagram for Question 3

Solution. Finding $T_y(s)$ is a straightforward application of the cascade (for $G_2(s) \cdot G_1(s)$) and feedback rules:

$$T_y(s) = \frac{G_2(s)G_1(s)}{1 + G_3(s)G_2(s)G_1(s)}.$$

To derive $T_u(s)$, rearrange the system as shown in Fig. 7 (all signal names remain unchanged). The resulting

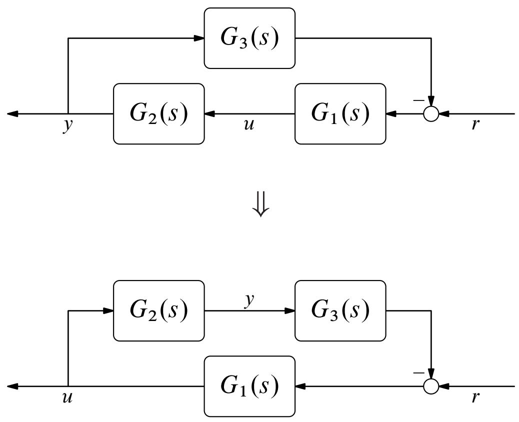


Fig. 7: Transformation of the block-diagram in Fig. 6

configuration is also standard, resulting in

$$T_u(s) = \frac{G_1(s)}{1 + G_3(s)G_2(s)G_1(s)}.$$

That's all … ∇

Question 4. Simplify the block-diagram in Fig. 8 and find the transfer function $P(s)$ from v_a to y.

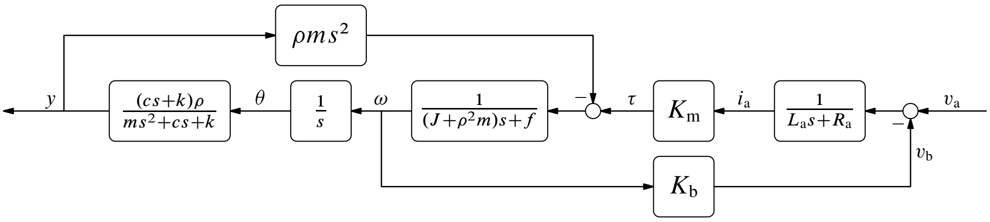


Fig. 8: Block-diagram for Question 4

Solution. A possible sequence of transformations is presented in Fig. 9. The first transformation is the

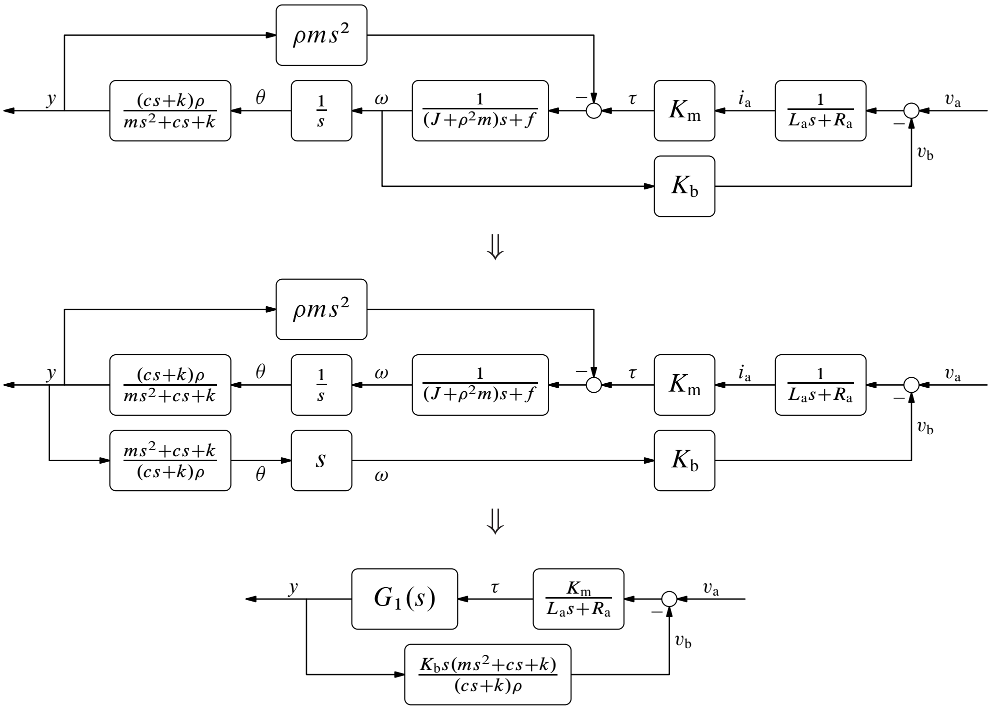


Fig. 9: Simplifications of the block-diagram in Fig. 8

movement of the node of ω to the output y. The second one just gathers the blocks, with

$$G_1(s) = \frac{\frac{(cs+k)\rho}{ms^2+cs+k} \cdot \frac{1}{s} \cdot \frac{1}{(J+\rho^2 m)s+f}}{1 + \frac{(cs+k)\rho}{ms^2+cs+k} \cdot \frac{1}{s} \cdot \frac{1}{(J+\rho^2 m)s+f} \cdot \rho m s^2} = \frac{(cs+k)\rho}{(ms^2+cs+k)s((J+\rho^2 m)s+f) + \rho m s^2 (cs+k)\rho}$$
$$= \frac{1}{s} \frac{(cs+k)\rho}{m(J+\rho^2 m)s^3 + (cJ + fm + 2c\rho^2 m)s^2 + (cf + Jk + (1+k)\rho^2 m)s + kf}.$$

Thus, we end up with

$$P(s) = \frac{G_1(s) \cdot \frac{K_m}{L_a s + R_a}}{1 + G_1(s) \cdot \frac{K_m}{L_a s+R_a} \cdot \frac{K_b s(ms^2+cs+k)}{(cs+k)\rho}} = \frac{G_1(s) K_m (cs+k)\rho}{(L_a s + R_a)(cs+k)\rho + G_1(s) K_m K_b s(ms^2+cs+k)},$$

which can be further simplified after substituting the formula for $G_1(s)$, but such a simplification goes beyond the scope of this question.

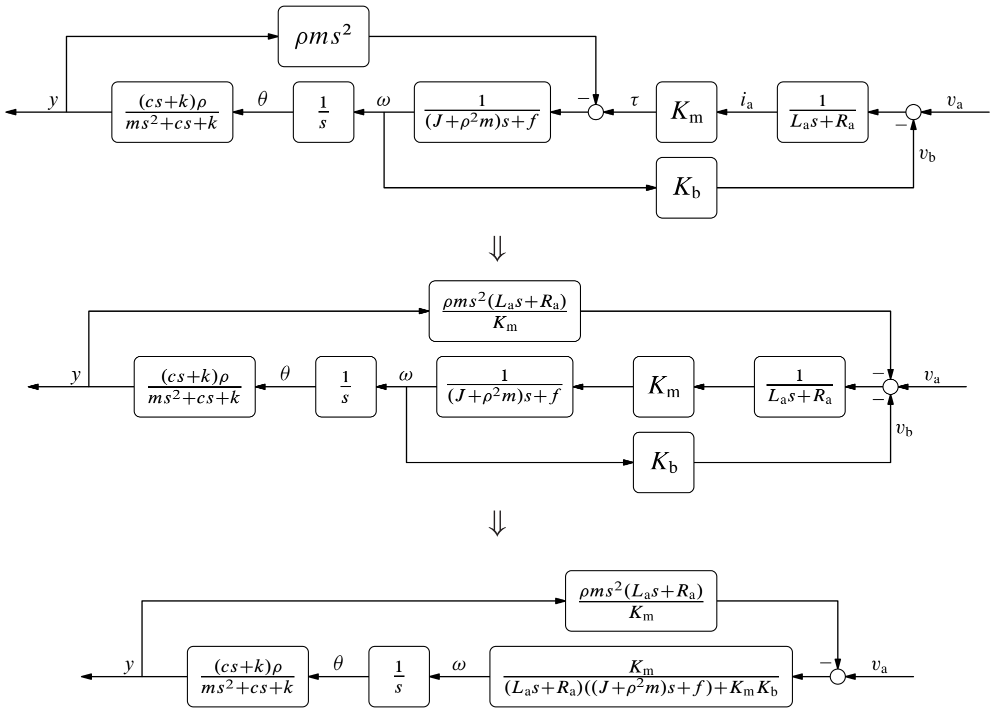


Fig. 10: Simplifications of the block-diagram in Fig. 8, another take

This sequence of transformations is not unique. Another approach is presented in Fig. 10. Needless to say, the resulted system P from v_a to y does not depend on the sequence of block-diagram manipulations (provided we do it right, of course). $\triangledown$