Introduction to Control (034040)
lecture no. 11

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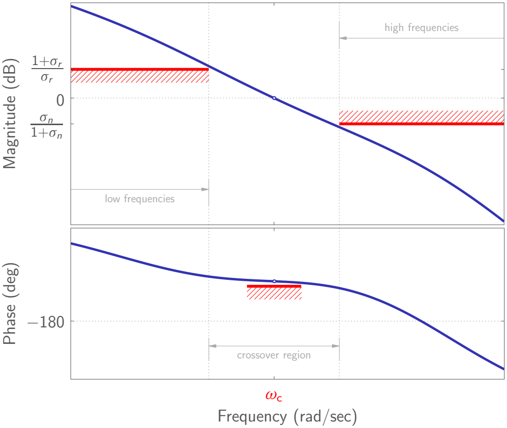
Typical operations with $L(j\omega)$



Typical course of action:

- set required crossover, ω_c
- shape high-frequency gain
- shape phase around ω_c
- shape low-frequency gain

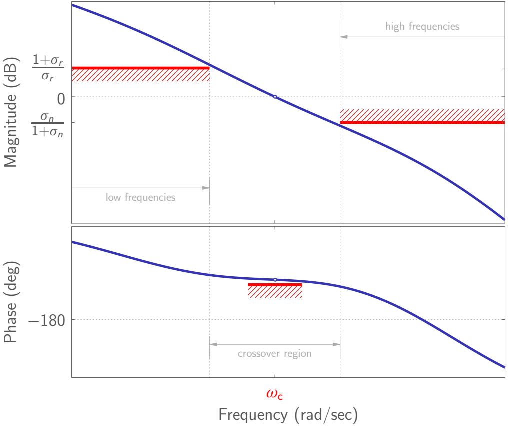
by cascade adjustments of $C(s)$.

The goal of today's lecture is to

- give a flavor of basic tools of loop shaping

for frequency responses with monotonically decreasing magnitude and phase.

Typical operations with $L(j\omega)$



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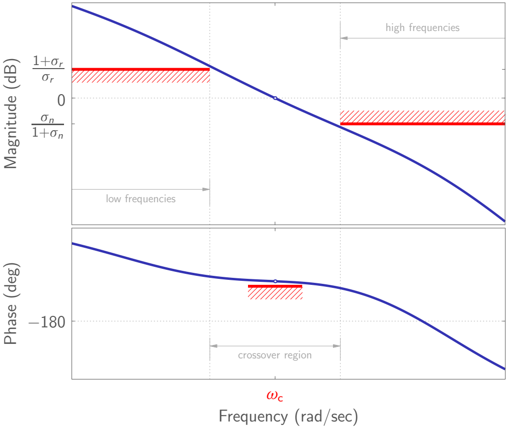

Typical course of action:

- set required crossover, ω_c (means: proportional controller)
- shape high-frequency gain (means: low-pass filter)
- shape phase around ω_c (means: lead controller)
- shape low-frequency gain (means: lag controller)

by cascade adjustments of $C(s)$.

The goal of today's lecture is to

- give a flavor of basic tools of loop shaping

for frequency responses with monotonically decreasing magnitude and phase.

Outline

Assigning required crossover

Phase shaping around crossover: lead controller

Shaping low-frequency gain: lag controller

Design examples

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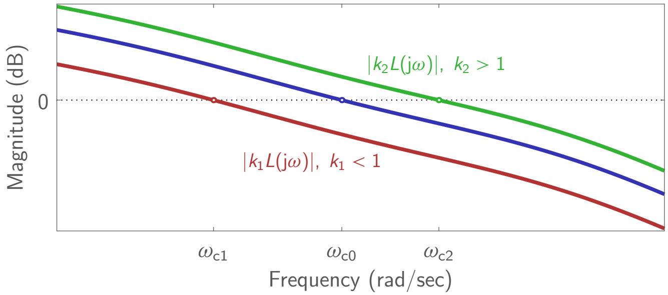
Gain effects



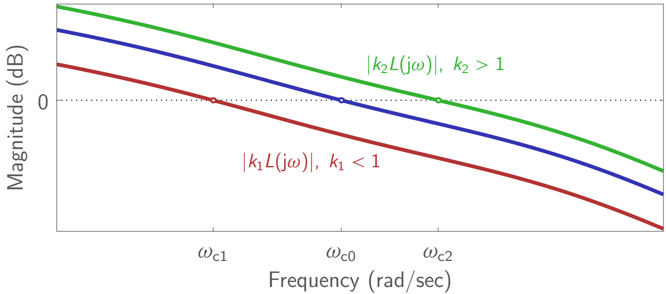
If $|L(j\omega)|$ is monotonically decreasing and continuous,

– increase / decrease of the loop gain $\implies$ increase / decrease of ω_c

If we aim at $\omega_c = \omega_c^*$, then we may just use the following P controller:

$$C(s) = k_p = \frac{1}{|P(j\omega_c^*)|}.$$

Gain effects



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Phase shaping around crossover: lead controller

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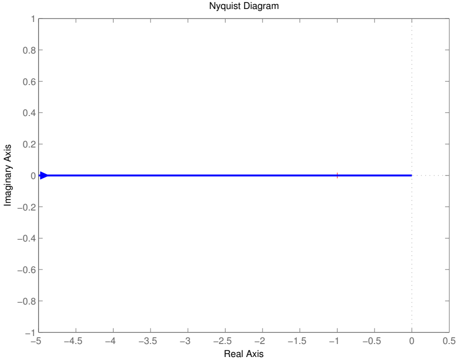
Design examples

Motivating example

Consider control of $P(s) = 1/s^2$. Its polar plot



shows clearly that this plant cannot be stabilized by a P controller. Some

— phase lead

must be added in the crossover region.

Motivating example

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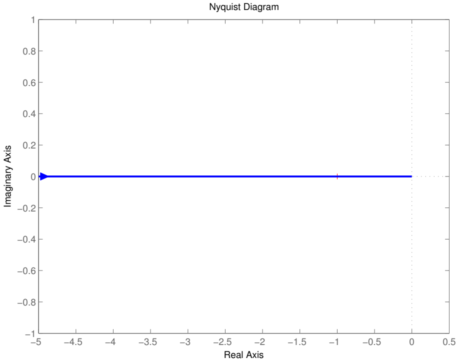

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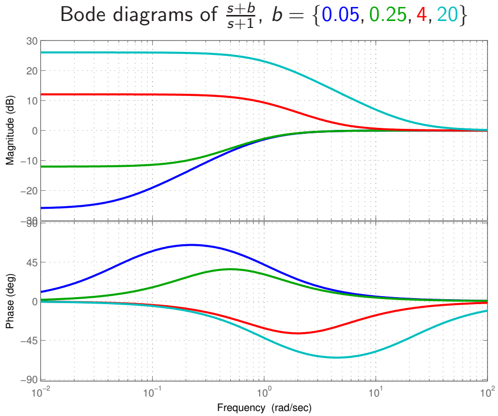
Motivating example (contd)

Consider 1-order controllers of the form $C_1(s) = \frac{s+b}{s+a}$. For $a = 1$ we have:



Any $C_1(s)$ with $a > b$ adds a phase lead $\forall\omega$ and thus stabilizes the double integrator. Such a C_1, however, would alter ω_c, which might be undesirable.

Motivating example (contd)

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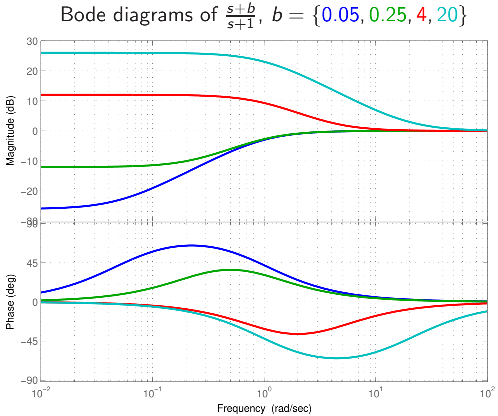

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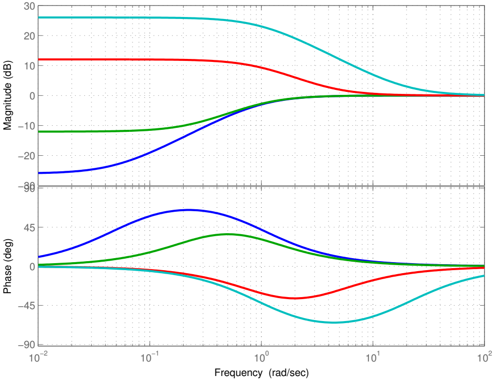
Motivating example (contd)

Consider 1-order controllers of the form $C_1(s) = \frac{s+b}{s+a}$. For $a = 1$ we have:

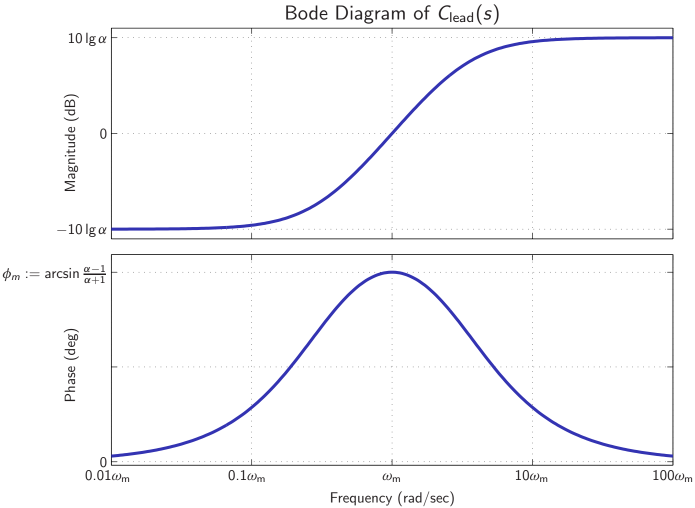
Bode diagrams of $\frac{s+b}{s+1}$, $b = \{0.05, 0.25, 4, 20\}$



Any $C_1(s)$ with $a > b$ adds a phase lead $\forall\omega$ and thus stabilizes the double integrator. Such a C_1, however, would alter ω_c, which might be undesirable.

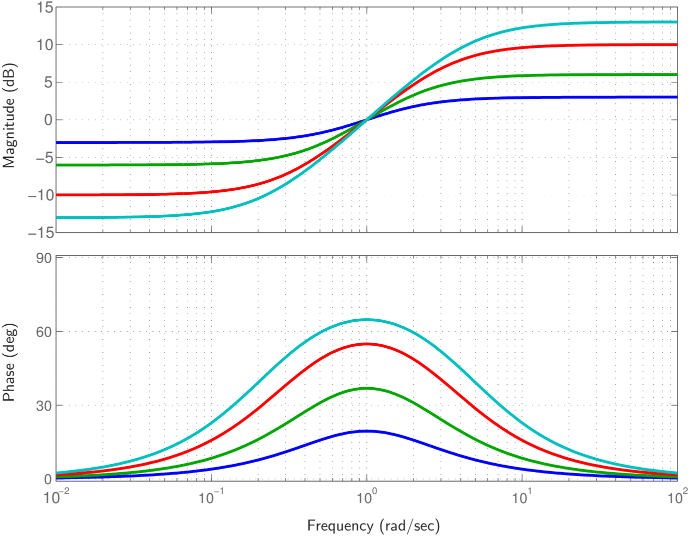
Lead controller

General form: $C_{\text{lead}}(s) = \dfrac{\sqrt{\alpha}\, s + \omega_m}{s + \sqrt{\alpha}\, \omega_m}$, where $\alpha > 1$ is a parameter.



Lead controller for different α's

Bode diagrams of $C_{\text{lead}}(s)$ for $\omega_{\text{m}} = 1$ and $\alpha = \{2, 4, 10, 20\}$



C_{lead}: properties

Pros: if we choose $\omega_m = \omega_c$, the use of $C_{\text{lead}}(s)$

- does not alter ω_c
- increases μ_{ph}

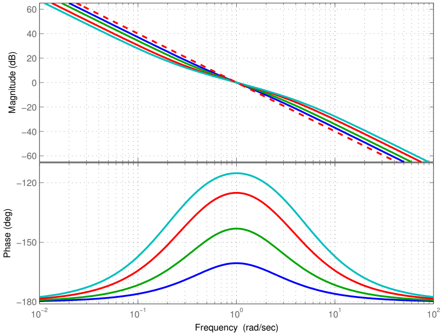
Cons:

- decreases the low-frequency gain
- increases the high-frequency gain
- decreases μ_g
  (because, typically, $\omega_c < \omega_\phi$ and $|C_{\text{lead}}(j\omega)| > 1$ for $\omega > \omega_m$)

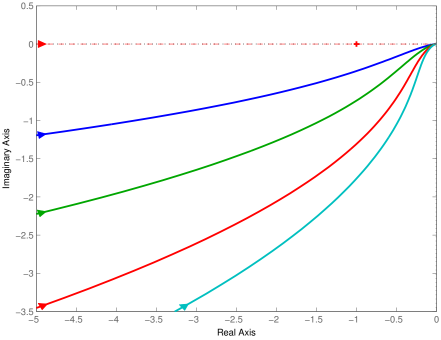
Motivating example (contd)

Assume that the required crossover frequency is $\omega_c = 1\,\text{rad/sec}$. Applying $C_{\text{lead}}(s)$ for $\omega_m = 1$ and $\alpha = \{2, 4, 10, 20\}$ to $P(s) = 1/s^2$ we get:

Bode diagrams of $P(s)C_{\text{lead}}(s)$

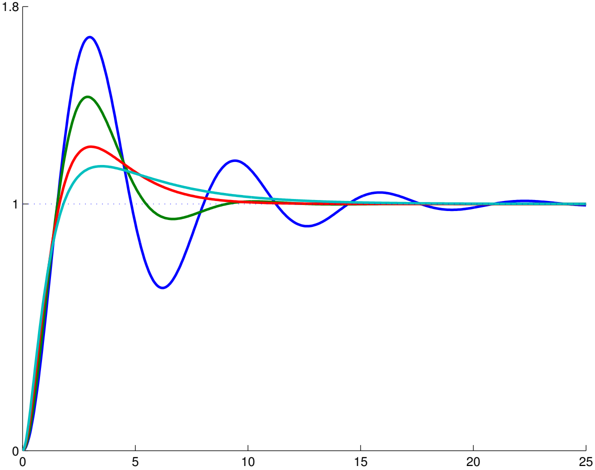


Nyquist diagrams of $P(s)C_{\text{lead}}(s)$



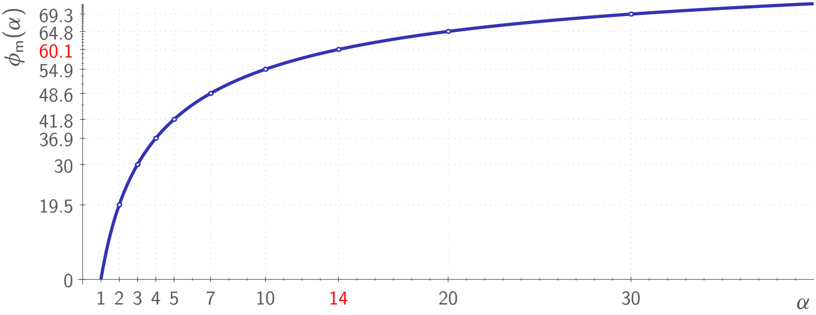
Motivating example (contd)

Closed-loop command responses with C_{lead} for $\omega_{\text{m}} = 1$ and $\alpha = \{2, 4, 10, 20\}$



Phase lead of lead controllers

Maximal phase lead, $\phi_m = \arcsin \frac{\alpha-1}{\alpha+1}$, is achieved at $\omega = \omega_m$:



Practically, as the slope of the curve above for large α is low, the values
$\implies \alpha \geq 14$ are undesirable.

For larger α we do not gain much (in terms of the phase lead), while pay a steep price (in terms of low- and high-frequency controller gains).

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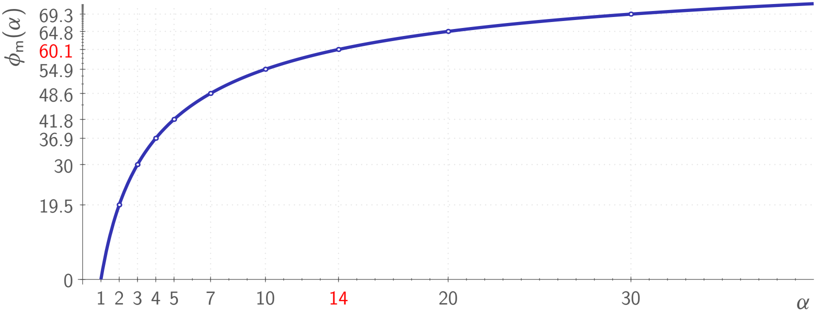

Practically, as the slope of the curve above for large α is low, the values

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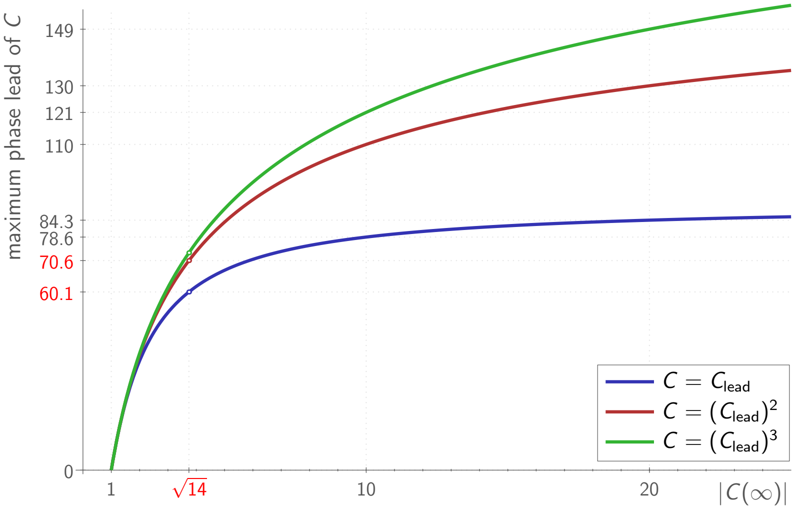
For larger α we do not gain much (in terms of the phase lead), while pay a steep price (in terms of low- and high-frequency controller gains).

More phase lead

If $\phi_m > 60^\circ$ is required, use 2 (seldom, even more) lead controllers:



Outline

Assigning required crossover

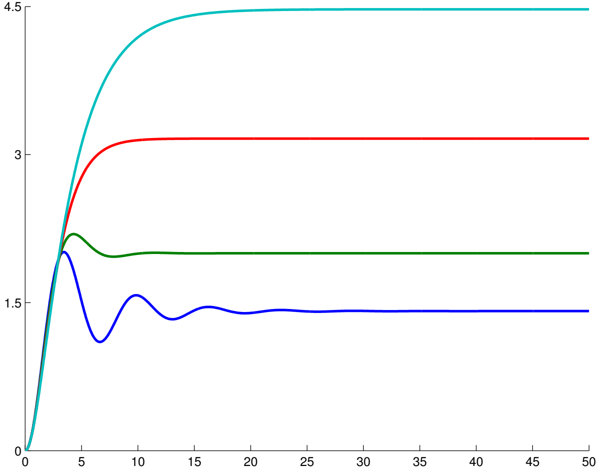
Phase shaping around crossover: lead controller

Shaping low-frequency gain: lag controller

Design examples

Motivating example (contd)

Closed-loop disturbance responses with C_{lead} for $\omega_{\text{m}} = 1$ and $\alpha = \{2, 4, 10, 20\}$



Motivating example (contd)

To explain this, note that

$$T_{\mathrm{d}}(s) = \frac{P(s)}{1 + P(s)C_{\mathrm{lead}}(s)} = \frac{1}{1/P(s) + C_{\mathrm{lead}}(s)} = \frac{1}{s^2 + C_{\mathrm{lead}}(s)},$$

so that

$$\lim_{t\to\infty} y_d(t) = T_{\mathrm{d}}(0) = \frac{1}{C_{\mathrm{lead}}(0)} = \sqrt{\alpha}.$$

Therefore, with the use of C_{lead}

- the vulnerability to low-frequency disturbances increases as α grows.

Motivating example (contd)

Possible remedy is to

- increase the static gain of C_{lead},

by the controller $C(s) = kC_{\text{lead}}(s)$ for some $k > 1$. Yet this would

- increase high-frequency gain as well;
- increase the system bandwidth beyond required.

We would like to

- increase low-frequency gain without altering gain / phase at $\omega \approx \omega_c$.

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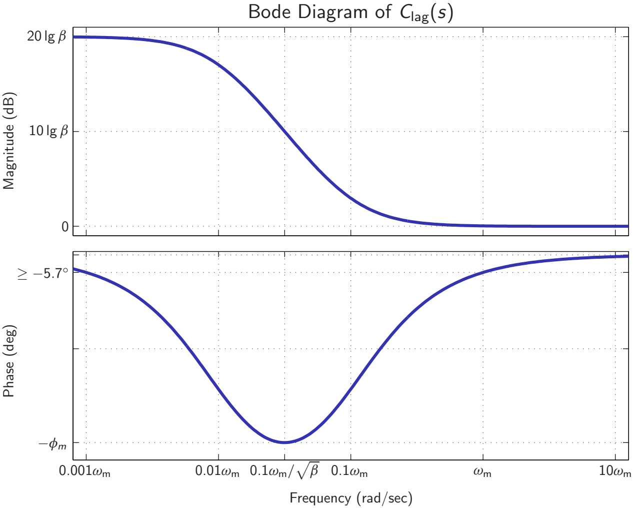
We would like to

- increase low-frequency gain without altering gain / phase at $\omega \approx \omega_{\text{c}}$.

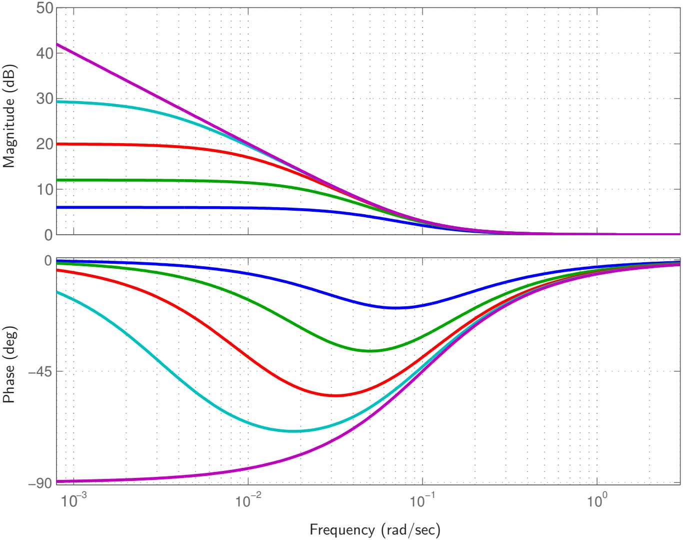
Lag controller

General form: $C_{\text{lag}}(s) = \dfrac{10s + \omega_m}{10s + \omega_m/\beta}$, where $\beta > 1$ is a parameter.



Lag controller for different β's

Bode diagrams of $C_{\text{lag}}(s)$ for $\omega_{\text{m}} = 1$ and $\beta = \{2, 4, 10, 30, \infty\}$



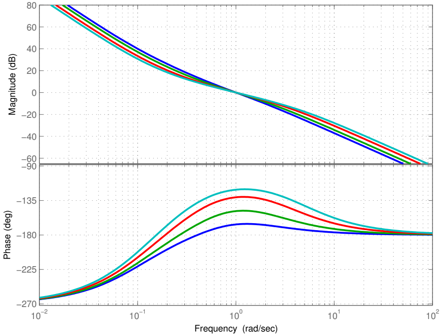
Motivating example (contd)

If the zero steady-state error under $d = \mathbb{1}$ is required, then we shall choose $\beta = \infty$. Let's also choose $\omega_m = 1$ for C_{lag} and consider

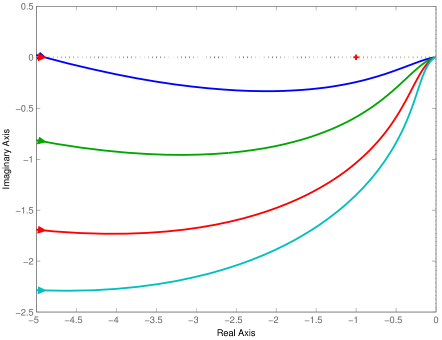
$$C(s) = C_{\text{lead}}(s)C_{\text{lag}}(s).$$

We have (again, in C_{lead} $\alpha = \{2, 4, 10, 20\}$):

Bode diagrams of $P(s)C_{\text{lead}}(s)C_{\text{lag}}(s)$

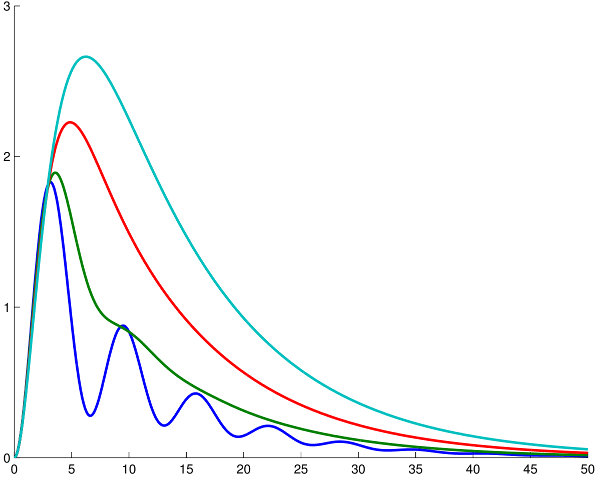


Nyquist diagrams of $P(s)C_{\text{lead}}(s)C_{\text{lag}}(s)$



Motivating example (contd)

Closed-loop disturbance responses with $C_{\text{lead}} C_{\text{lag}}$ for $\omega_{\text{m}} = 1$, $\alpha = \{2, 4, 10, 20\}$, and $\beta = \infty$



C_{lag}: properties

Pros: if we choose $\omega_m = \omega_{\mathrm{c}}$, the use of $C_{\mathrm{lag}}(s)$

- increases the low-frequency gain of $L(\mathrm{j}\omega)$
- adds at most 5.7° phase lag at ω_{c}
  (this may be compensated by a bit larger phase lead of $C_{\mathrm{lead}}(s)$)

Cons:

- adds phase lag
- gain increase within a decade of ω_{c} is modest (at most by $\sqrt{2} \approx 1.414$)

Outline

Assigning required crossover

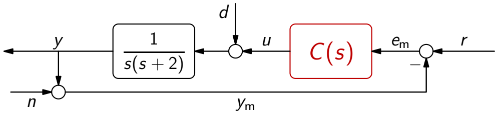
Phase shaping around crossover: lead controller

Shaping low-frequency gain: lag controller

Design examples

System

A DC motor controlled in closed loop:



Requirements:

- closed-loop stability (of course)
- zero steady-state error for a step in r
- zero steady-state error for a step in d
- $\mu_{\text{ph}} \geq 45^\circ$
- ω_c is treated as a tuning parameter

Remark: We implicitly assume that the plant is normalized, in a sense that the control amplitude $|u(t)| < 1$ is "small" and $|u(t)| > 1$ is "large".

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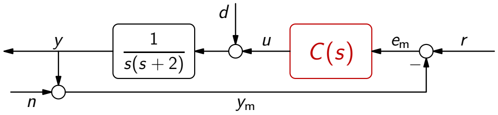

Requirements:

- closed-loop stability (of course)
- zero steady-state error for a step in r always holds
- zero steady-state error for a step in d
- $\mu_{\rm ph} \geq 45^\circ$
- $\omega_{\rm c}$ is treated as a tuning parameter

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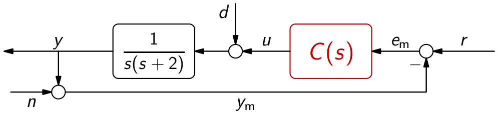

Requirements:

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- zero steady-state error for a step in r — always holds
- zero steady-state error for a step in d — integrator in $C(s)$
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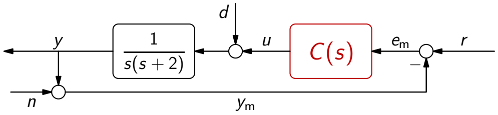


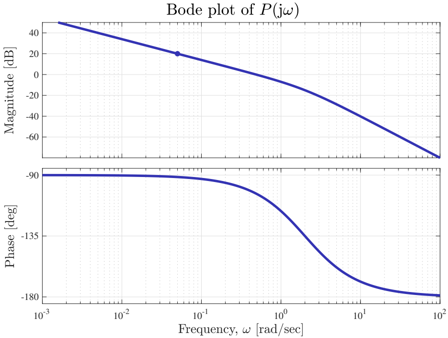
Requirements:

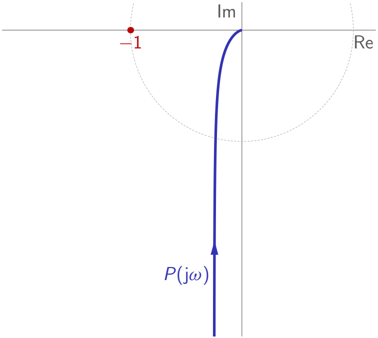
- closed-loop stability (of course)
- zero steady-state error for a step in r — always holds
- zero steady-state error for a step in d — integrator in $C(s)$
- $\mu_{ph} \geq 45^\circ$ — *might* require phase lead
- ω_c is treated as a tuning parameter

Remark: We implicitly assume that the plant is normalized, in a sense that the control amplitude $|u(t)| < 1$ is "small" and $|u(t)| > 1$ is "large".

Example 1: the plant

Let $\omega_c = 0.05$:

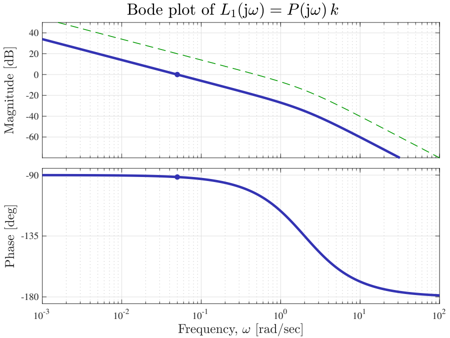


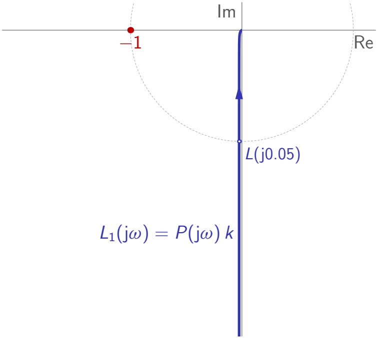


This crossover can be attained by the gain $k \approx 0.1$.

Example 1: adjusting crossover

We get:





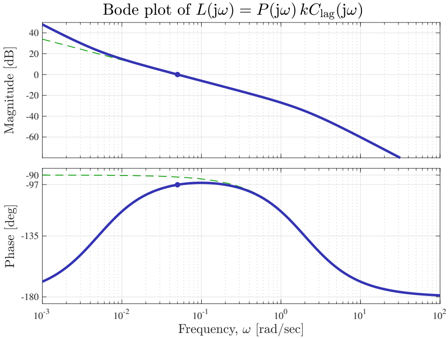
Here $\mu_{ph} \approx 90^\circ$ and

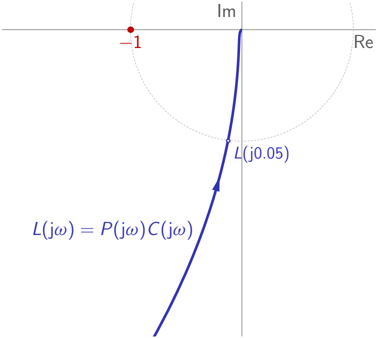
– we don't need a lead,

even after the lag controller will add its 5.7°.

Example 1: adjusting low-frequency gain

Use the lag controller with $\omega_m = 0.05$ and $\beta = \infty$:

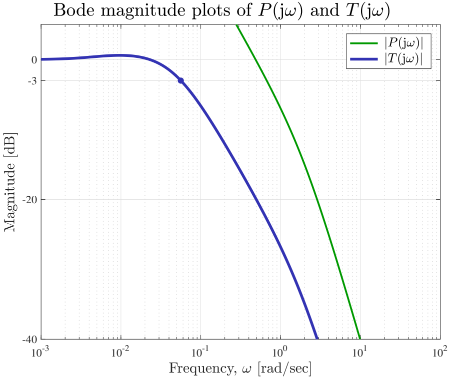


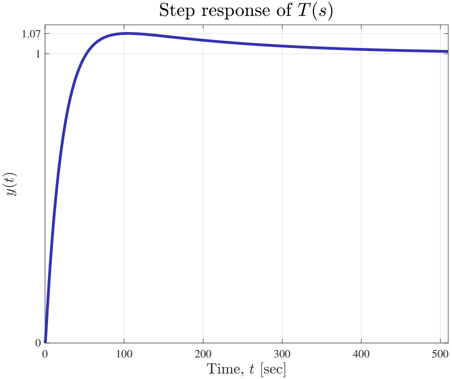


It yields $\mu_{ph} \approx 83°$, which is more than enough. Resulting controller:

$$C(s) = kC_{lag}(s) = \frac{0.10003(s + 0.005)}{s}.$$

Example 1: closed-loop command response

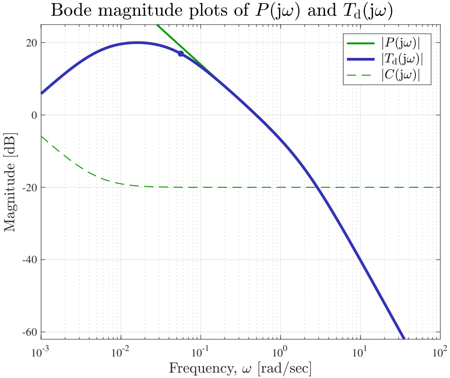


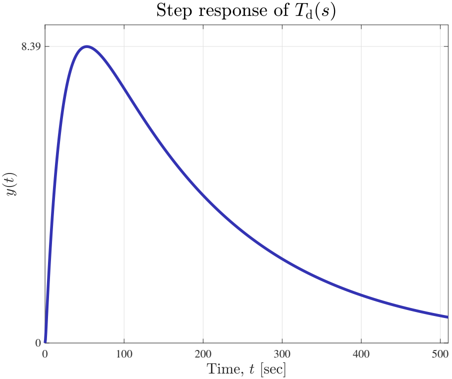


To note:

- resonance peak is very small $\implies$ small OS
- closed-loop bandwidth $\omega_b \approx 0.0562$, which is a bit above the designed $\omega_c = 0.05$ and smaller than the open-loop bandwidth

Example 1: closed-loop disturbance response

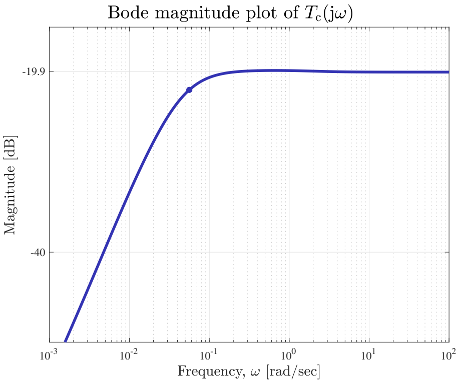


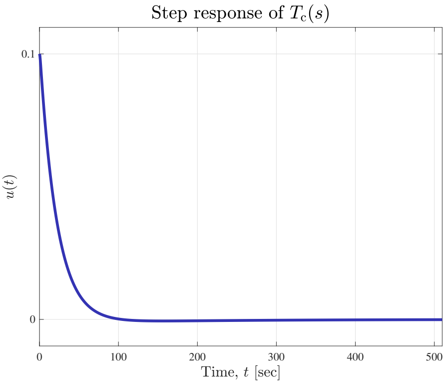


To note:

– not enough controller gain $\implies$ high disturbance sensitivity

Example 1: closed-loop control signal



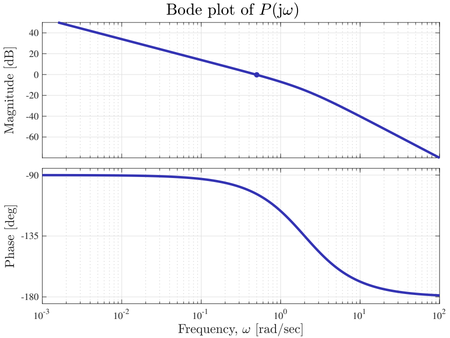


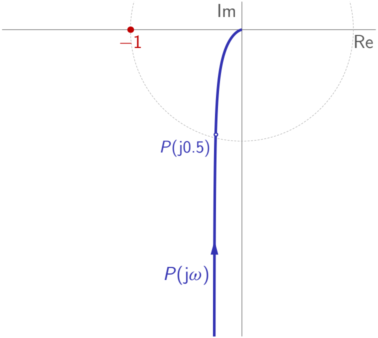
To note:

– closed-loop bandwidth $\ll$ open-loop bandwidth $\implies$ low control effort

Example 2: the plant

Let $\omega_c = 0.5$ now:

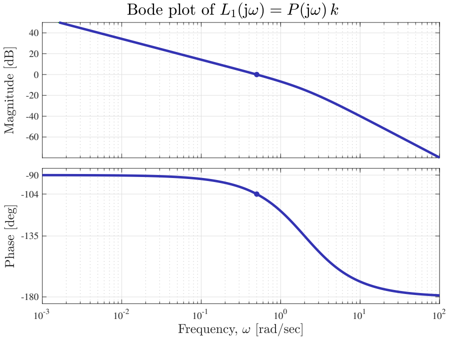


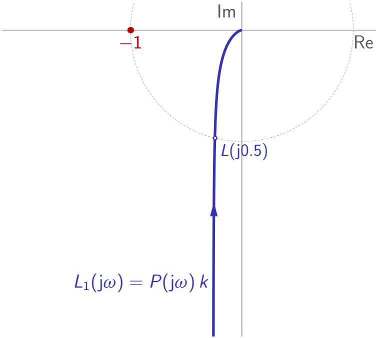


This is about the actual crossover, so can be attained by the gain $k \approx 1.04$.

Example 2: adjusting crossover

We get:





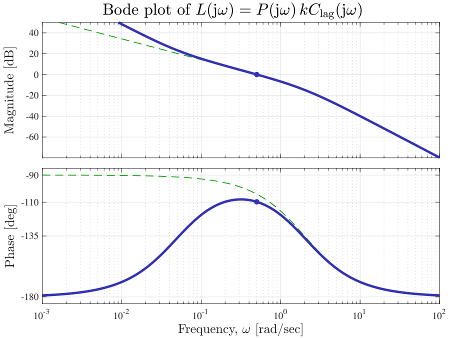
Here $\mu_{\mathrm{ph}} \approx 76^\circ$ and

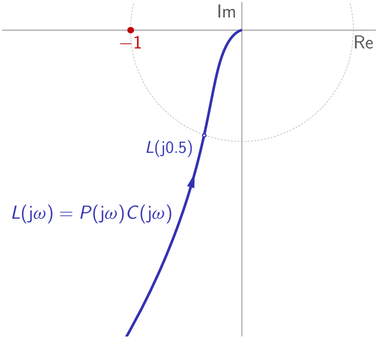
– we still don't need a lead,

even after the lag controller will add its 5.7°.

Example 2: adjusting low-frequency gain

Use the lag controller with $\omega_m = 0.5$ and $\beta = \infty$:

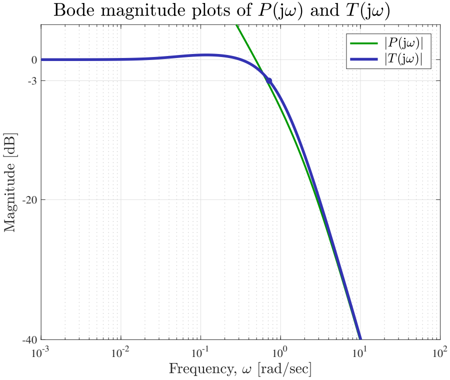


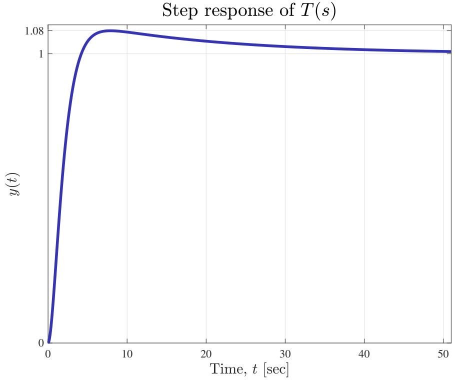


It yields $\mu_{ph} \approx 70^\circ$, which is also more than enough. Resulting controller:

$$C(s) = kC_{lag}(s) = \frac{1.0308(s + 0.05)}{s}.$$

Example 2: closed-loop command response

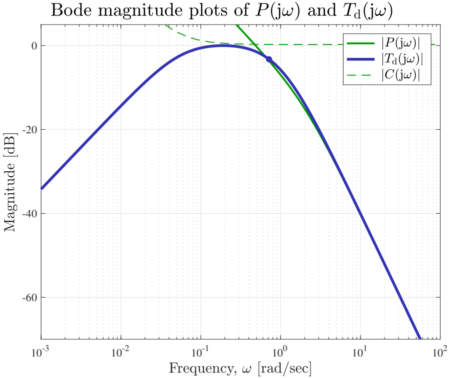


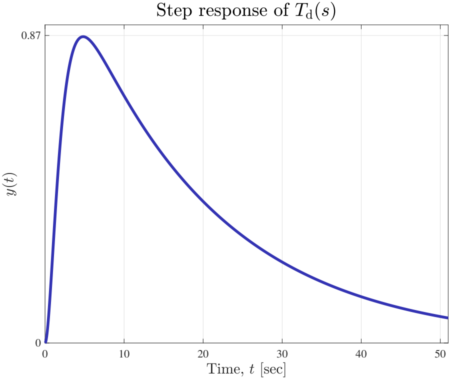


To note:

- resonance peak is very small $\implies$ small OS
- closed-loop bandwidth $\omega_b \approx 0.7161$, which is a bit above the designed $\omega_c = 0.5$ and about the same as the open-loop bandwidth

Example 2: closed-loop disturbance response

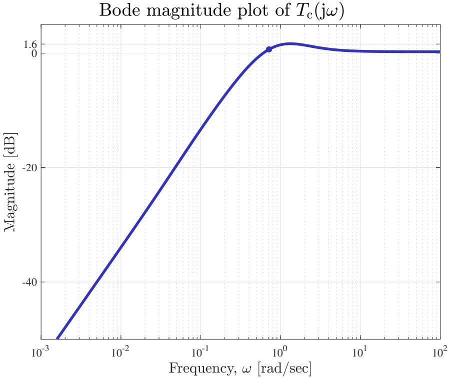


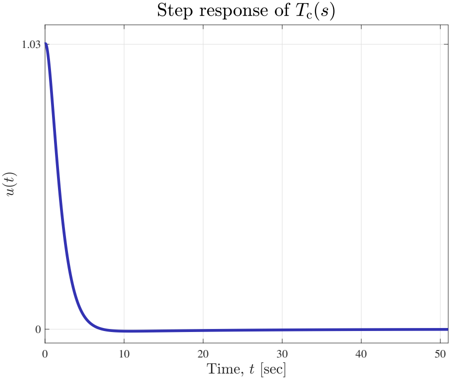


To note:

– unit controller gain $\implies$ neutral disturbance sensitivity

Example 2: closed-loop control signal



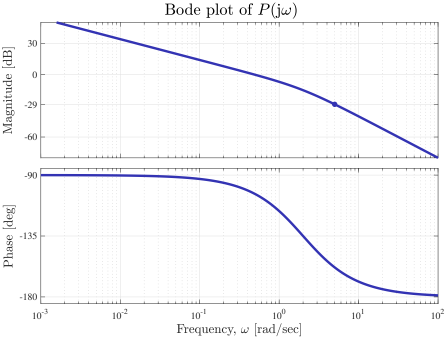


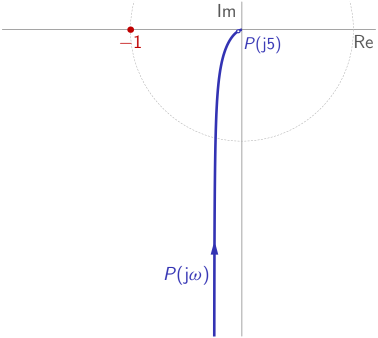
To note:

– closed-loop bandwidth ≈ open-loop bandwidth $\implies$ moderate control effort

Example 3: the plant

Let $\omega_c = 5$ now:

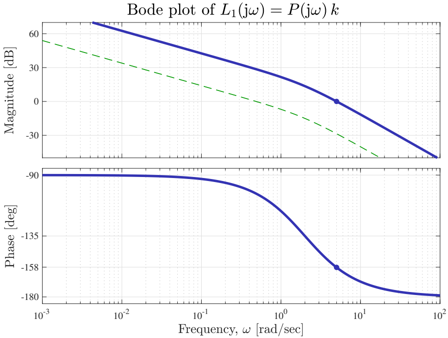


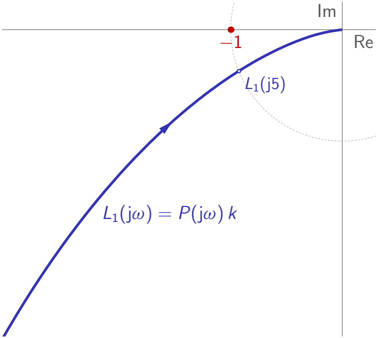


This is below the actual crossover, so can be attained by the gain $k \approx 26.9$.

Example 3: adjusting crossover

We get:





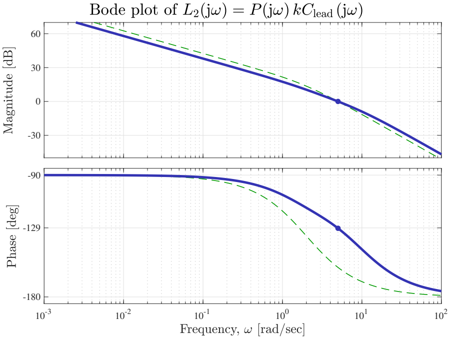
Here $\mu_{ph} \approx 22^\circ$ and

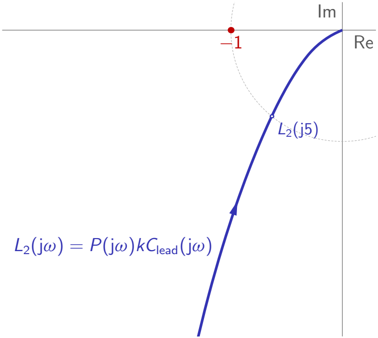
– we do need a phase lead of $45^\circ - 22^\circ + 5.7^\circ = 28.7^\circ$,

for which one lead is enough.

Example 3: adjusting phase around crossover

We get:

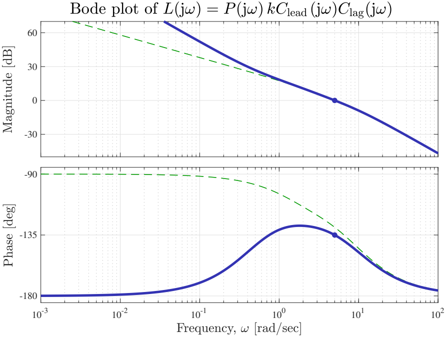


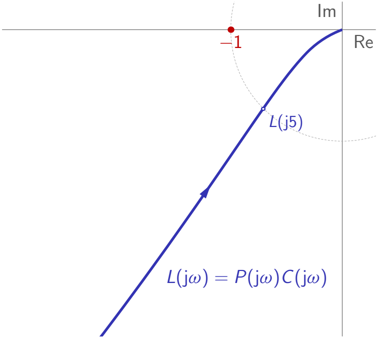


Here $\mu_{\text{ph}} \approx 50.7^\circ$, which is exactly what we need before the lag controller adds its 5.7°.

Example 3: adjusting low-frequency gain

Use the lag controller with $\omega_m = 5$ and $\beta = \infty$:

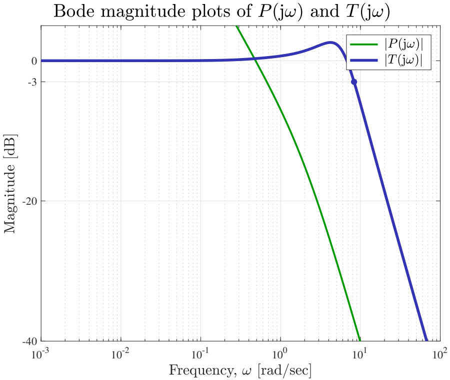


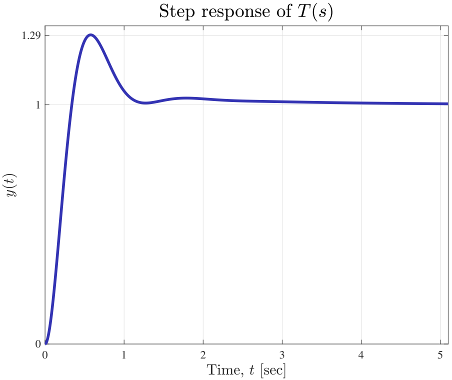


It yields $\mu_{\mathrm{ph}} \approx 45^\circ$, which is exactly what we need. Resulting controller:

$$C(s) = kC_{\mathrm{lead}}(s)C_{\mathrm{lag}}(s) = \frac{45.619(s+2.951)(s+0.5)}{s(s+8.471)}.$$

Example 3: closed-loop command response

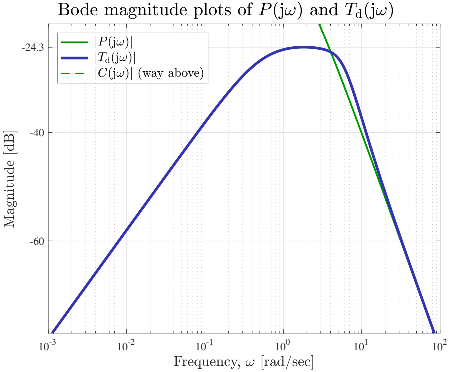


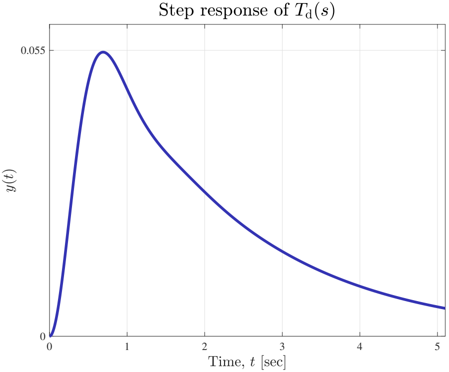


To note:

- resonance peak becomes larger $\implies$ larger OS
- closed-loop bandwidth $\omega_b \approx 8.3176$, which is a bit above the designed $\omega_c = 5$ and higher than the open-loop bandwidth

Example 3: closed-loop disturbance response

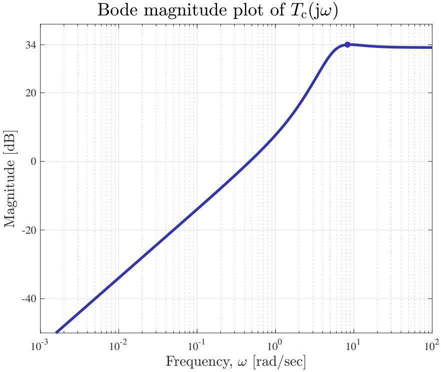


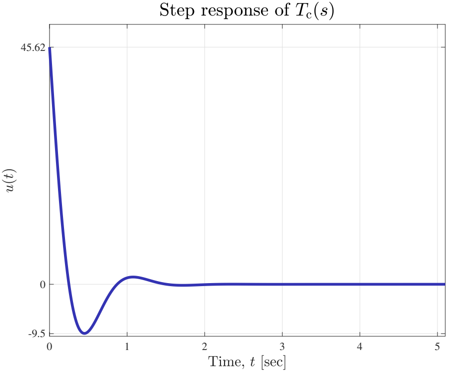


To note:

- high controller gain $\Longrightarrow$ low disturbance sensitivity

Example 3: closed-loop control signal





To note:

– closed-loop bandwidth $\gg$ open-loop bandwidth $\implies$ high control effort