Introduction to Control (00340040)
lecture no. 9

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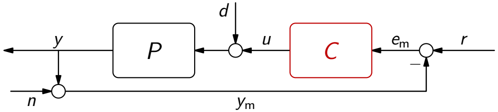
Loop shaping



Philosophy:

– analyze, and affect, properties of the closed-loop system via properties of the frequency response $L(\mathrm{j}\omega)$ of the loop transfer function $L = PC$,

i.e.

$$L(\mathrm{j}\omega) \quad \to \quad \begin{cases} S : r \mapsto e \\ T : r \mapsto y, -n \mapsto y \\ T_\mathrm{d} : d \mapsto y \\ T_\mathrm{c} : r \mapsto u, -n \mapsto u \end{cases}$$

Today:

– Internal stability via properties of the Nyquist (polar+) plot of $L(\mathrm{j}\omega)$.

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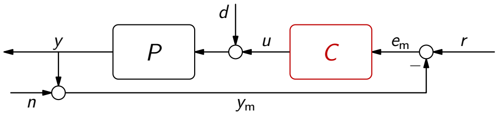

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- internal stability via properties of the Nyquist (polar +) plot of $L(\mathrm{j}\omega)$.

Outline

Nyquist stability criterion: no open-loop jω-poles

Nyquist stability criterion: including open-loop poles at the origin

Simplifications

Analysis of $kL(s)$ loops

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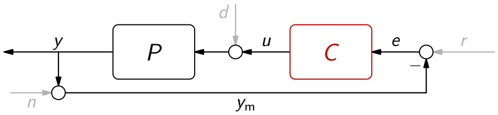
Preliminary: return difference



Let

$$L(s) := P(s)C(s) = \frac{N_L(s)}{D_L(s)}$$

(loop transfer function) and assume that there are no unstable pole / zero cancellations between $P(s)$ and $C(s)$. Consider the transfer function

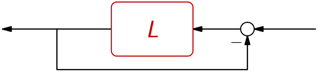
$$1 + L(s) = \frac{N_L(s) + D_L(s)}{D_L(s)} = \frac{\chi_{cl}(s)}{D_P(s)D_C(s)},$$

called return difference. It is readily seen that

- unstable poles of $1 + L(s)$ are open-loop unstable poles
- unstable zeros of $1 + L(s)$ are closed-loop unstable poles.

Assumptions

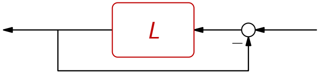
We consider the system



and assume that

- $L(s)$ is proper
- $L(s)$ has no jω-axis poles will be relaxed later on

Idea



Technical steps:

1. define a simple closed contour Γ_s containing all singularities of $1 + L(s)$ in the ORHP $\mathbb{C}_0 := \{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$;
2. determine the mapping Γ_{1+L} of Γ_s by the return difference;
3. count the number ν of clockwise encirclings the origin by Γ_{1+L}.

By the argument principle,

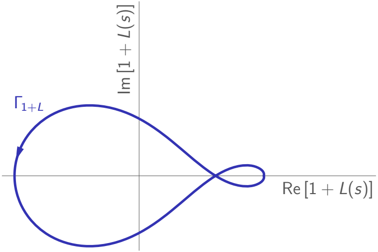
$$\nu = \pi_{\text{cl}}^+ - \pi_{\text{ol}}^+ \quad \iff \quad \pi_{\text{cl}}^+ = \nu + \pi_{\text{ol}}^+,$$

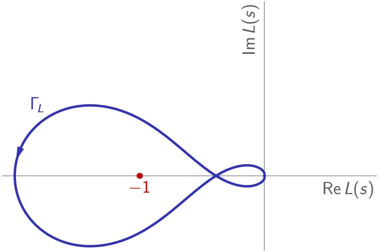
where

- π_{cl}^+ is the number of closed-loop poles in $\mathbb{C}_0$
- π_{ol}^+ is the number of open-loop poles (those of $L(s)$) in $\mathbb{C}_0$

Idea: slight modification

The difference between $[1 + L(s)]$-plane and $L(s)$-plane is just the horizontal shift by 1:





Hence,

$$\#\text{encirclings the origin by } \Gamma_{1+L} = \#\text{encirclings the point } -1 + j0 \text{ by } \Gamma_L$$

and we may replace steps 2. and 3. above with

2. determine the maping Γ_L of Γ_s by the loop transfer function;
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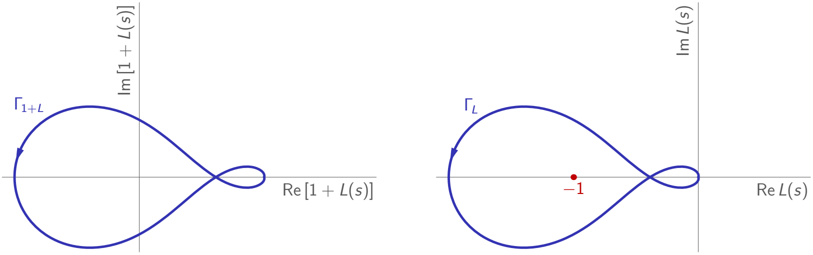

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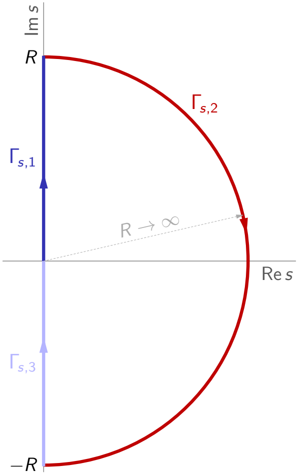
Nyquist contour



The contour

$$\Gamma_s = \Gamma_{s,1} \cup \Gamma_{s,2} \cup \Gamma_{s,3},$$

where

$$\begin{aligned} \Gamma_{s,1} &= \mathrm{j}\omega, & \omega &: 0 \to R, \\ \Gamma_{s,2} &= R\mathrm{e}^{\mathrm{j}\theta}, & \theta &: \tfrac{\pi}{2} \to -\tfrac{\pi}{2}, \\ \Gamma_{s,3} &= \mathrm{j}\omega, & \omega &: -R \to 0, \end{aligned}$$

with $R \to \infty$ is called the Nyquist contour. It contains all ORHP poles of transfer functions having a finite number of such poles.

Mapping the Nyquist contour by L

Consider each of the Nyquist contour segments separately:

$\Gamma_{s,1}$: In this case $s = \mathrm{j}\omega$ ($\omega \geq 0$) mapped to $L(\mathrm{j}\omega)$, so that

- $\Gamma_{L,1}$ is the polar plot of the frequency response of L.

$\Gamma_{s,2}$: As $L(s)$ is proper ($b_n = 0$ if $L(s)$ is strictly proper),

$$L(R\mathrm{e}^{\mathrm{j}\theta}) = \frac{b_n R^n \mathrm{e}^{\mathrm{j}\theta n} + b_{n-1} R^{n-1} \mathrm{e}^{\mathrm{j}\theta(n-1)} + \cdots + b_1 R \mathrm{e}^{\mathrm{j}\theta} + b_0}{R^n \mathrm{e}^{\mathrm{j}\theta n} + a_{n-1} R^{n-1} \mathrm{e}^{\mathrm{j}\theta(n-1)} + \cdots + a_1 R \mathrm{e}^{\mathrm{j}\theta} + a_0}$$

$$= \frac{b_n + b_{n-1} R^{-1} \mathrm{e}^{-\mathrm{j}\theta} + \cdots + b_1 R^{1-n} \mathrm{e}^{\mathrm{j}\theta(1-n)} + b_0 R^{-n} \mathrm{e}^{-\mathrm{j}\theta n}}{1 + a_{1-n} R^{-1} \mathrm{e}^{-\mathrm{j}\theta} + \cdots + a_1 R^{1-n} \mathrm{e}^{-\mathrm{j}\theta(n-1)} + a_0 R^{-n} \mathrm{e}^{-\mathrm{j}\theta n}}$$

Thus, $\lim_{R\to\infty} L(R\mathrm{e}^{\mathrm{j}\theta}) = b_n$ for every θ, i.e.

- $\Gamma_{L,2}$ collapses to a single point, b_n.

Since this point already belongs to $\Gamma_{L,1}$, we may effectively omit $\Gamma_{L,2}$.

$\Gamma_{s,3}$: As $L(s)$ has real coefficients, $L(-\mathrm{j}\omega) = \overline{L(\mathrm{j}\omega)}$, so that

- $\Gamma_{L,3}$ is the mirror of $\Gamma_{L,1}$ around the real axis.

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&= \frac{b_n + b_{n-1} R^{-1} \mathrm{e}^{-\mathrm{j}\theta} + \cdots + b_1 R^{1-n} \mathrm{e}^{\mathrm{j}\theta(1-n)} + b_0 R^{-n} \mathrm{e}^{-\mathrm{j}\theta n}}{1 + a_{1-n} R^{-1} \mathrm{e}^{-\mathrm{j}\theta} + \cdots + a_1 R^{1-n} \mathrm{e}^{-\mathrm{j}\theta(n-1)} + a_0 R^{-n} \mathrm{e}^{-\mathrm{j}\theta n}}
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Nyquist plot

The union of $\Gamma_{L,1}$ and $\Gamma_{L,3}$, which is the

- graph of $L(\mathrm{j}\omega)$ in polar coordinates as ω runs from $-\infty$ to $+\infty$,

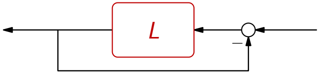
is called the Nyquist plot of $L(\mathrm{j}\omega)$.

The Nyquist plot can be constructed in two steps:

1. construct the polar plot of $L(\mathrm{j}\omega)$
   (which is the graph of $L(\mathrm{j}\omega)$ in polar coordinates as ω runs from 0 to $+\infty$)
2. add the reflection of the polar plot about the real axis

The critical point, $-1 + \mathrm{j}0$, on the $L(s)$-plane is also presented, owing to its importance in the stability analysis.

Nyquist stability criterion



Theorem (Nyquist)

The closed-loop system is stable iff the Nyquist plot of $L(j\omega)$

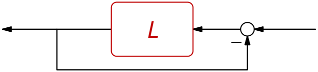
- *does not intersect the critical point* $-1 + j0$
- *encircles the critical point* π_{ol}^+ *times in the counterclockwise direction*

as ω *grows from* $-\infty$ *to* ∞, *where* π_{ol}^+ *is the number of poles of* $L(s)$ *in* $\mathbb{C}_0$.

Proof:

- If $\exists\omega_0$ s.t. $L(j\omega_0) = -1$, $\chi_{cl}(j\omega_0) = 0$ and closed-loop system unstable.
- We know that the number of clockwise encirclings of the critical point by the mapping of the Nyquist contour by $L(s)$ (i.e. the Nyquist plot) is $\nu = \pi_{cl}^+ - \pi_{ol}^+$. The stability of the closed-loop system is equivalent to $\pi_{cl}^+ = 0$, hence the system stable iff $\nu = -\pi_{ol}^+$. □

Nyquist stability criterion (contd)



Corollary 1: If the Nyquist plot of $L(\mathrm{j}\omega)$ does not cross the critical point, then the number of closed-loop unstable poles

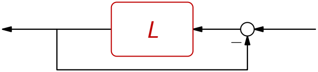
$$\pi^+_{\mathrm{cl}} = \pi^+_{\mathrm{ol}} + \nu,$$

where ν is the number of clockwise encirclings of the critical point $-1 + \mathrm{j}0$ by the Nyquist plot.

Corollary 2: If the Nyquist plot of $L(\mathrm{j}\omega)$ intersects the critical point at some frequency, say $\omega = \omega_0$, then $\chi_{\mathrm{cl}}(s)$ has at least one root at $s = \mathrm{j}\omega_0$.

Corollary 3: If L is stable itself, then the closed-loop system is stable iff the Nyquist plot of $L(\mathrm{j}\omega)$ neither intersects nor encircles the critical point as ω increases from $-\infty$ to ∞.

Nyquist stability criterion (contd)



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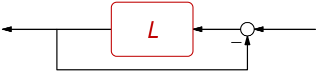

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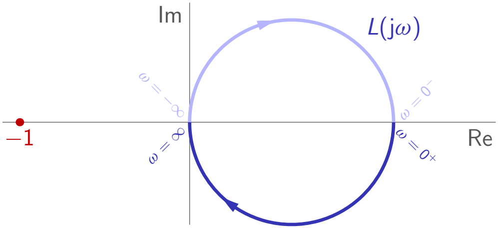
Example 1

Let $L(s) = \dfrac{1.2}{s+1}$. Its Nyquist plot is



We have:

- L is stable (i.e. $\pi_{\text{ol}}^{+} = 0$)
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Hence,

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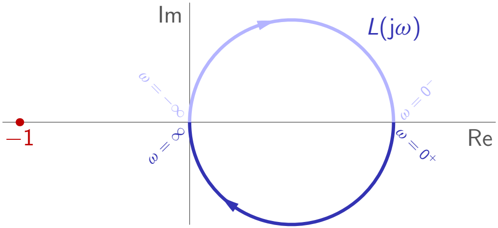

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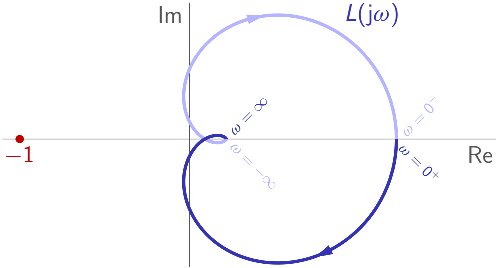
Example 2

Let $L(s) = \dfrac{(3s+8)(9s^2+3s+19)}{125(s+1)^3}$. Its Nyquist plot is



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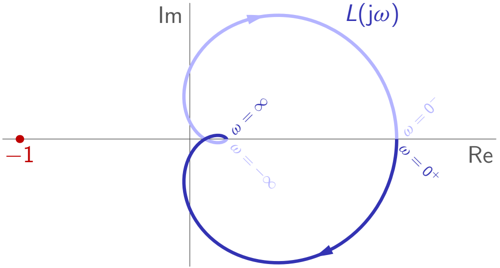

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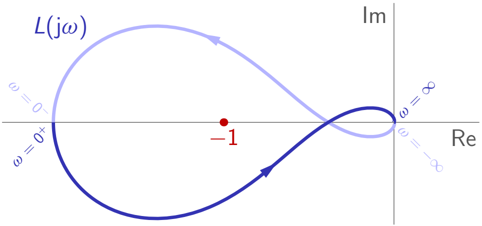
Example 3

Let $L(s) = \dfrac{2}{(s-1)(0.125s+1)^3}$. Its Nyquist plot is



We have:

- L is unstable with $L(s)$ havingd one unstable pole (i.e. $\pi_{ol}^{+} = 1$)
- Nyquist plot encircles the critical point once in the counterclockwise direction

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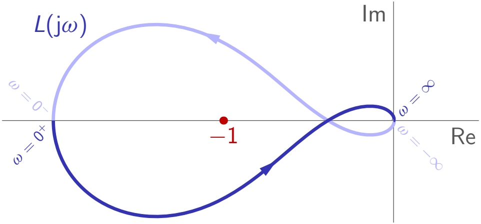

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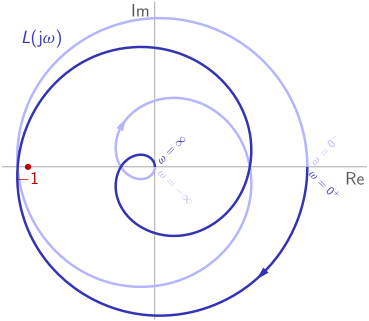
Example 4

Let $L(s) = \dfrac{(-s+1)^3}{(s+1)^4}$. Its Nyquist plot is



We have:

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Hence,

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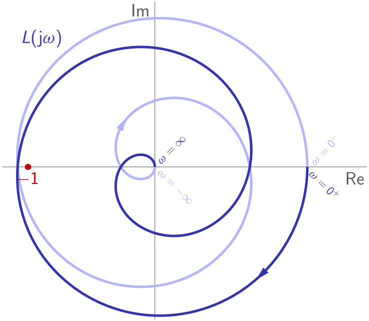

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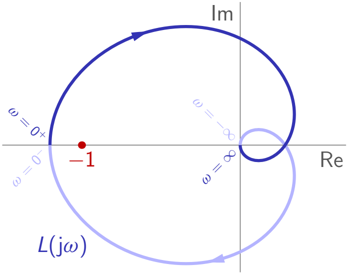
Example 5

Let $L(s) = \dfrac{1.2(-0.5s+1)^2}{(0.5s+1)^2(s-1)}$. Its Nyquist plot is



We have:

- L is unstable with $L(s)$ having one unstable pole (i.e. $\pi_{ol}^{+} = 1$)
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Hence,

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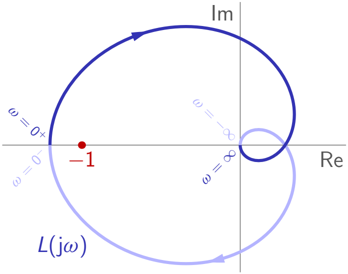

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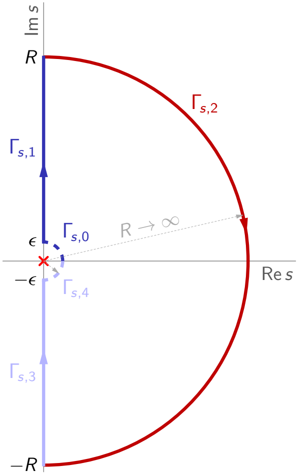
Nyquist stability criterion: no open-loop jω-poles

Nyquist stability criterion: including open-loop poles at the origin

Simplifications

Analysis of $kL(s)$ loops

Nyquist contour if $L(s)$ has pole(s) at the origin



The modified contour is

$$\Gamma_s = \Gamma_{s,0} \cup \Gamma_{s,1} \cup \Gamma_{s,2} \cup \Gamma_{s,3} \cup \Gamma_{s,4},$$

where

$$\begin{aligned}
\Gamma_{s,0} &= \epsilon e^{j\theta}, & \theta &: 0 \to \tfrac{\pi}{2}, \\
\Gamma_{s,1} &= j\omega, & \omega &: \epsilon \to R, \\
\Gamma_{s,2} &= R e^{j\theta}, & \theta &: \tfrac{\pi}{2} \to -\tfrac{\pi}{2}, \\
\Gamma_{s,3} &= j\omega, & \omega &: -R \to -\epsilon, \\
\Gamma_{s,4} &= \epsilon e^{j\theta}, & \theta &: -\tfrac{\pi}{2} \to 0,
\end{aligned}$$

with $R \to \infty$ and $\epsilon \to 0$. It still contains all ORHP poles of transfer functions having a finite number of such poles.

Mapping $\Gamma_{s,0}$ and $\Gamma_{s,4}$ by $L(s)$

The mapping of $\Gamma_{s,1} \cup \Gamma_{s,2} \cup \Gamma_{s,3}$ are practically unchanged. So to complete the picture we only need

$\Gamma_{s,0}$: Let $L(s) = \frac{1}{s^n}\tilde{L}(s)$, where $n \in \mathbb{N}$ and $\tilde{L}(0) \neq 0$ is finite. In this case

$$L(\epsilon e^{j\theta}) = \frac{1}{\epsilon^n e^{j\theta n}}\tilde{L}(\epsilon e^{j\theta}) \xrightarrow{\epsilon \to 0} \frac{\tilde{L}(0)}{\epsilon^n} e^{-j\theta n},$$

so that

- $\Gamma_{L,0}$ is an arc of ∞ radius, starting at either positive (if $\tilde{L}(0) > 0$) or negative (if $\tilde{L}(0) < 0$) real axis and going $n\pi/2$ [rad] clockwise

until it intersect the beginning of $\Gamma_{s,1}$.

$\Gamma_{s,4}$: As $L(s)$ has real coefficients, $L(-j\omega) = \overline{L(j\omega)}$, hence

- $\Gamma_{L,4}$ is the mirror of $\Gamma_{L,0}$ around the real axis.

Remark: $\Gamma_{L,4} \cup \Gamma_{L,0}$ is an arc connecting the end of $\Gamma_{L,3}$ ($\omega \to 0^-$) and the beginning of $\Gamma_{L,1}$ ($\omega \to 0^+$) through the angle πn in the clockwise direction.

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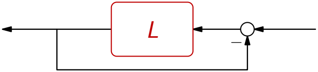
Nyquist stability criterion



Since the modified Nyquist contour excludes the poles of $L(s)$ at the origin,

- these poles are not counted in π_{ol}^{+}.

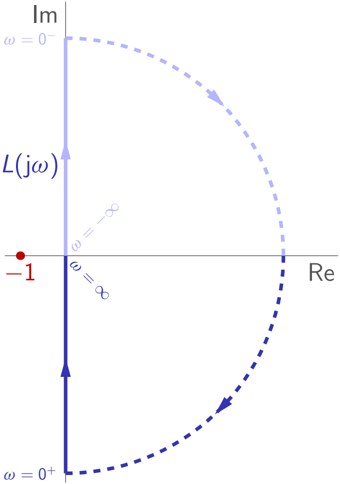
That may be done since $\chi_{\text{cl}}(s)$ cannot[1] have roots at $s = 0$, so all unstable closed-loop poles are separated from the origin and, therefore, lie inside the modified Nyquist contour if ϵ is sufficiently small.

The rest of the criterion remains unchanged . . .

[1] Common roots of $D_L(s)$ and $\chi_{\text{cl}}(s) = N_L(s) + D_L(s)$ are common roots of $D_L(s)$ and $N_L(s)$, so cannot be unstable under our assumption.

Example 6

Let $L(s) = \dfrac{1}{s}$. Its Nyquist plot is



We have:

- L is "stable" (i.e. $\pi_{ol}^{+} = 0$)
- Nyquist plot does not encircle the critical point

Hence,

- the closed-loop system is stable

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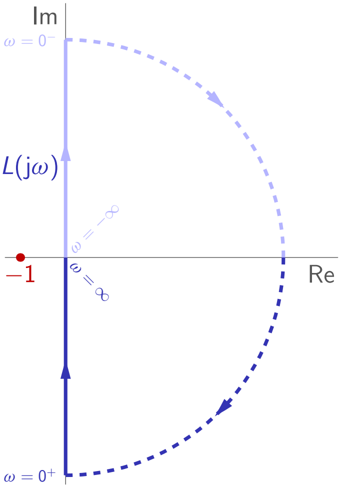

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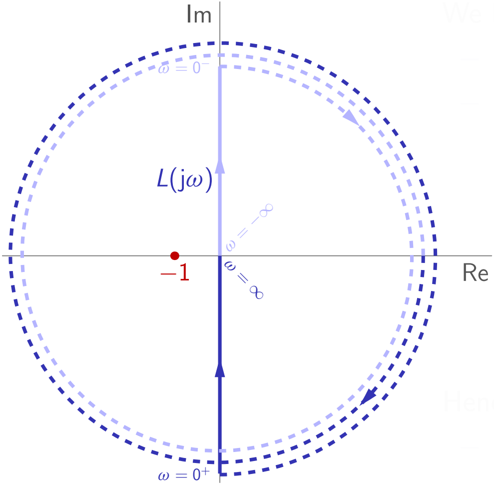
Example 7

Let $L(s) = \dfrac{1}{s^5}$. Its Nyquist plot is



We have:

- L is "stable" (i.e. $\pi_{ol}^+ = 0$)
- Nyquist plot encircles the critical point twice in the clockwise dir.

Hence,

- the closed-loop system is unstable ($\pi_{cl}^+ = 2$)

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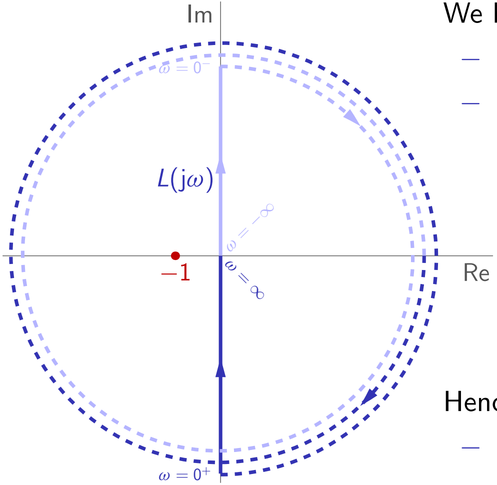
Let $L(s) = \dfrac{1}{s^5}$. Its Nyquist plot is



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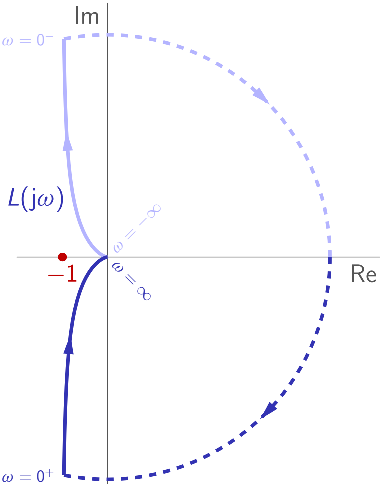
Example 8

Let $L(s) = \dfrac{1}{s(s+1)}$. Its Nyquist plot is



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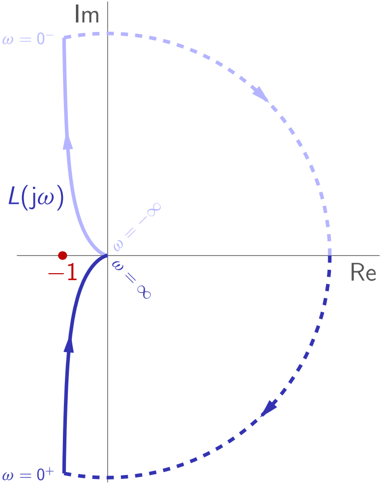


We have:

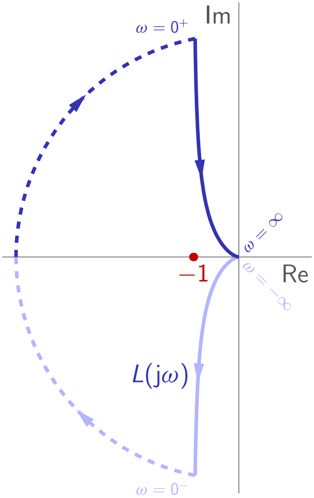
- L is "stable" (i.e. $\pi_{\mathrm{ol}}^{+} = 0$)
- Nyquist plot does not encircle the critical point

Hence,

- the closed-loop system is stable

Example 9

Let $L(s) = \dfrac{1}{s(s-1)}$. Its Nyquist plot is



We have:

- $L(s)$ has one unstable pole (i.e. $\pi^+_{\mathrm{ol}} = 1$)
- Nyquist plot encircles the critical point once in the clockwise dir.

Hence,

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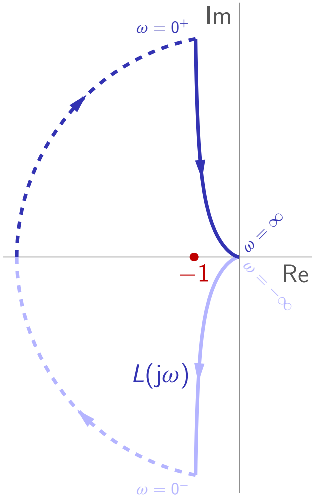
Let $L(s) = \dfrac{1}{s(s-1)}$. Its Nyquist plot is



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- Nyquist plot encircles the critical point once in the clockwise dir.

Hence,

- the closed-loop system is unstable ($\pi^+_{\text{cl}} = 2$)

Outline

Nyquist stability criterion: no open-loop jω-poles

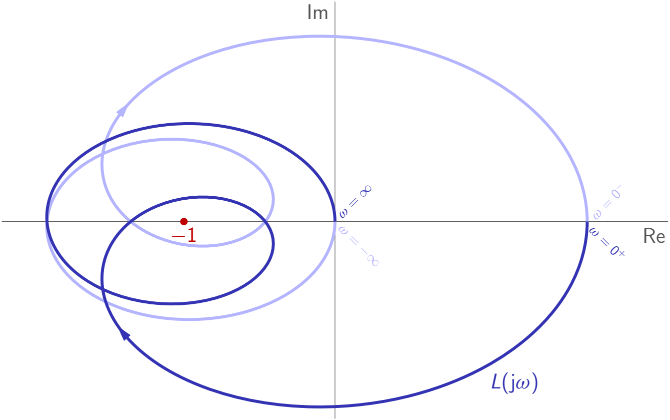
Nyquist stability criterion: including open-loop poles at the origin

Simplifications

Analysis of $kL(s)$ loops

Counting encirclements

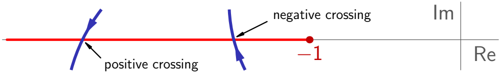
Counting counterclockwise encirclements may be tedious (-4 in this case):



But there is a way to simplify this procedure . . .

Positive and negative crossings

Consider the ray $(-\infty, -1]$ in the $L(j\omega)$-plane:



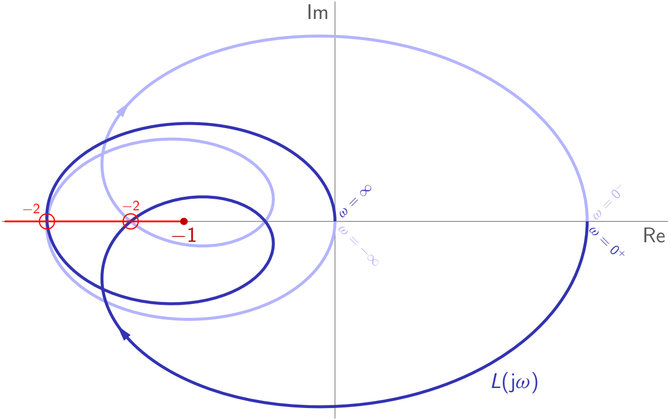
A crossing occurs when the plot of $L(j\omega)$ intersects the ray. It is said to be

- *positive* if the direction of $L(j\omega)$ is downward
- *negative* if the direction of $L(j\omega)$ is upward

Encirclements via the Nyquist plot

Lemma

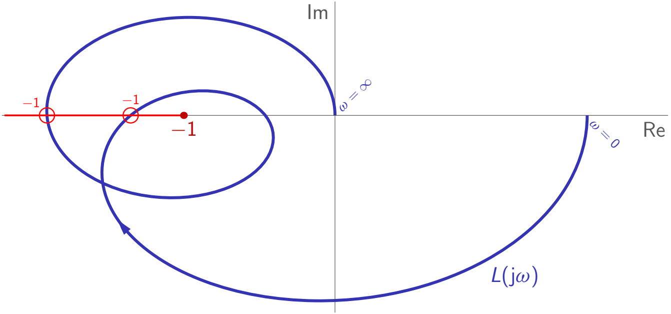
The number of counterclockwise encirclements around the critical point by the Nyquist plot of $L(\mathrm{j}\omega)$ equals the net sum of crossings the ray $(-\infty, -1]$ by the Nyquist plot of $L(\mathrm{j}\omega)$.



Encirclements via the polar plot

Lemma

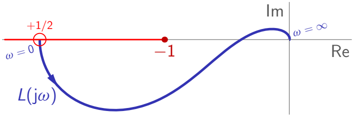
The number of counterclockwise encirclements around the critical point by the Nyquist plot of $L(\mathrm{j}\omega)$ *equals twice*[2] *the net sum of crossings the ray* $(-\infty, -1]$ *by the polar plot of* $L(\mathrm{j}\omega)$*.*



[2] One should be careful with the cases when the polar plot starts or / and ends at the ray. This situation, if happens, may be counted as "half crossings."

Example 3 (contd)

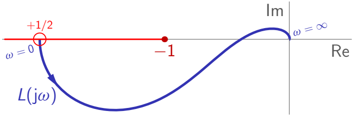
If $L(s) = \dfrac{2}{(s-1)(0.125s+1)^3}$:



If $L(s) = \dfrac{6}{(s-1)(0.125s+1)^3}$:

Example 3 (contd)

If $L(s) = \dfrac{2}{(s-1)(0.125s+1)^3}$ (one counterclockwise encirclement):



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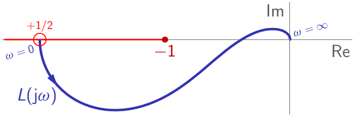

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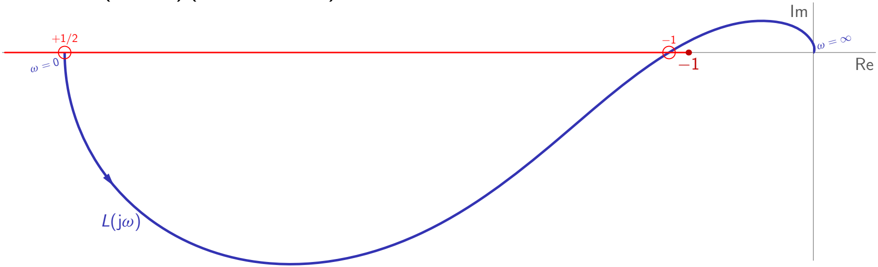

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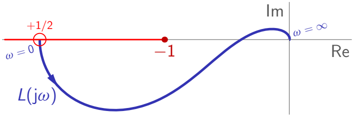

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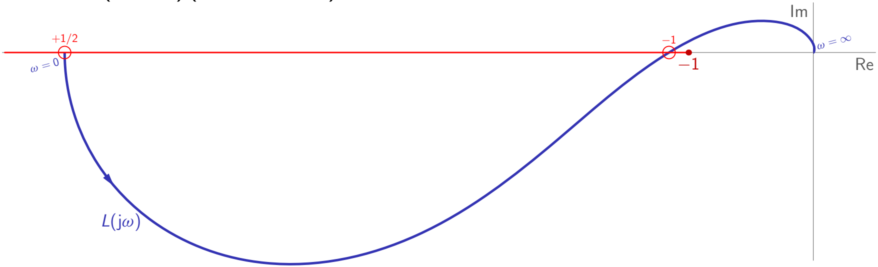


Outline

Nyquist stability criterion: no open-loop jω-poles

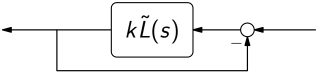
Nyquist stability criterion: including open-loop poles at the origin

Simplifications

Analysis of $kL(s)$ loops

Problem

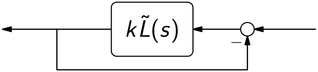
Consider (in fact, $L(s) = k\tilde{L}(s)$)



where $\tilde{L}(s)$ is a proper transfer function and k is the parameter we want to choose. The problem is to

– find all k stabilizing the system.

Nyquist criterion revisited



The return difference transfer function rewrites as

$$1 + L(s) = 1 + k\tilde{L}(s) = k\left(\frac{1}{k} + \tilde{L}(s)\right)$$

Since the multiplication by k affects neither poles nor zeros of this transfer function, we may analyze stability by mapping the Nyquist contour by $\frac{1}{k} + \tilde{L}(s)$. This, in turn, implies that we should

- analyze the Nyquist plot of $\tilde{L}(j\omega)$ with the critical point $-\frac{1}{k} + j0$.

In other words, different critical point is the only modification we need

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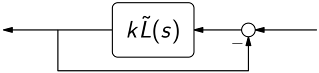

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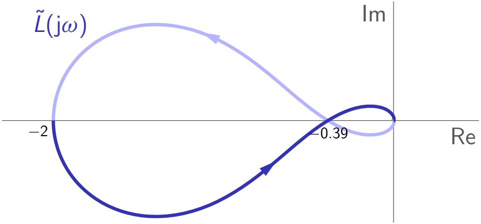
In other words, different critical point is the only modification we need.

Example 3 (contd)

Let $\tilde{L}(s) = \dfrac{2}{(s-1)(0.125s+1)^3}$. Its Nyquist plot[3] is



In this case $\pi_{ol}^+ = 1$ and there are four intervals of k:

$$-\tfrac{1}{k} < -2:\ \text{no encirclings} \implies \pi_{cl}^+ = 1;$$

$$-2 < -\tfrac{1}{k} < -\tfrac{529}{1372} \approx -0.39:\ \text{1 counterclockwise encircling} \implies \pi_{cl}^+ = 0;$$

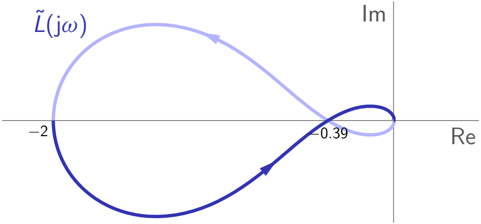
$$-0.39 \approx -\tfrac{529}{1372} < -\tfrac{1}{k} < 0:\ \text{1 clockwise encircling} \implies \pi_{cl}^+ = 2;$$

$$-\tfrac{1}{k} > 0:\ \text{no encirclings} \implies \pi_{cl}^+ = 1.$$

[3] Points of intersection with the real axis found by solving $\tilde{L}(j\omega) = \tilde{L}(-j\omega)$ in ω.

Example 3 (contd)

Let $\tilde{L}(s) = \dfrac{2}{(s-1)(0.125s+1)^3}$. Its Nyquist plot is



In this case $\pi_{ol}^{+} = 1$ and there are three intervals of k:

$$k < \tfrac{1}{2}:\ \text{no encirclings} \implies \pi_{cl}^{+} = 1;$$

$$\tfrac{1}{2} < k < \tfrac{1372}{529} \approx 2.59:\ \text{1 counterclockwise encircling} \implies \pi_{cl}^{+} = 0;$$

$$k > \tfrac{1372}{529} \approx 2.59:\ \text{1 clockwise encircling} \implies \pi_{cl}^{+} = 2.$$