Introduction to Control (00340040)
lecture no. 6

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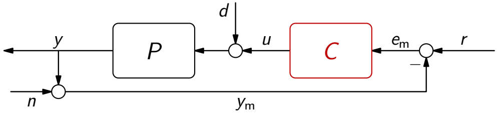
Internal stability of feedback systems



Let

$$P(s) = \frac{N_P(s)}{D_P(s)} \qquad \text{and} \qquad C(s) = \frac{N_C(s)}{D_C(s)}$$

with

$$\chi_{cl}(s) = N_P(s)N_C(s) + D_P(s)D_C(s)$$

(the characteristic polynomial of the closed loop system).

Theorem

If $P(s)$ and $C(s)$ are proper and $\deg \chi_{cl}(s) = \deg D_P(s) + \deg D_C(s)$, then the closed-loop system is (internally) stable iff $\chi_{cl}(s)$ is Hurwitz, i.e. has no roots in the closed RHP $\bar{\mathbb{C}}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\}$.

Outline

Stability and feedback

Root locus: motivation

Some root locus rules

Outline

Qualitative observations

Example 1: With $C(s) = k_{\rm p}$,

$$P(s) = \frac{1}{s-1} \quad \Longrightarrow \quad \chi_{\rm cl}(s) = s + (k_{\rm p} - 1).$$

It is Hurwitz iff $(\chi_0 > 0)$ $k_{\rm p} > 1$. Hence,

- $\ddot\smile$ feedback can stabilize unstable systems.

Example 2: With $C(s) = k_{\rm p}$,

$$P(s) = \frac{1}{(s+0.1)^3} \quad \Longrightarrow \quad \chi_{\rm cl}(s) = s^3 + \frac{3}{10}s^2 + \frac{3}{100}s + \Big(k_{\rm p} + \frac{1}{1000}\Big).$$

It is Hurwitz iff $(\chi_i > 0$ and $\chi_2\chi_1 > \chi_3\chi_0)$ $-0.001 < k_{\rm p} < 0.008$. Hence

- $\ddot\frown$ feedback can destabilize stable systems.

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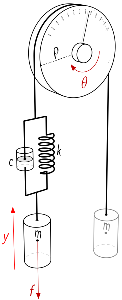

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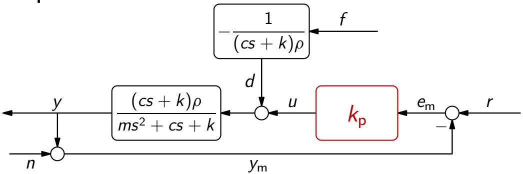

Characteristic polynomial

$$\begin{aligned}\chi_\mathrm{cl}(s) &= ms^2 + cs + k + (cs+k)\rho k_\mathrm{p} \\ &= ms^2 + c(1+\rho k_\mathrm{p})s + k(1+\rho k_\mathrm{p})\end{aligned}$$

stable iff $k_\mathrm{p} > -1/\rho$, so we may use high-gain feedback here.

Note that

– d is bounded whenever so is f (as $1/(cs+k)\rho$ is stable)

so we do not need to account for the form of disturbance d.

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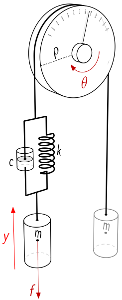

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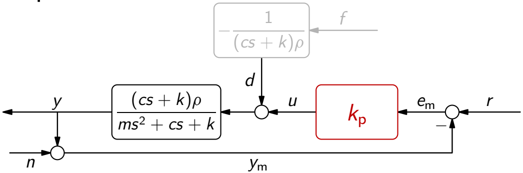

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Example 4: stabilizing DC motor

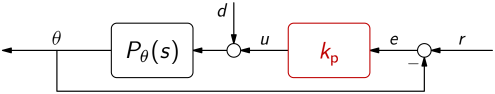
We remember, Lecture 2, that the t.f. of a DC motor (voltage $\mapsto$ angle) is

$$P_\theta(s) = \frac{K_m}{s\big((L_a s + R_a)(Js + f) + K_b K_m\big)}$$

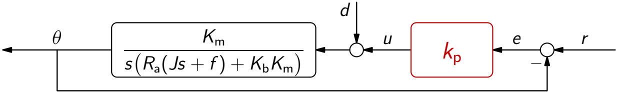
or, if we neglect L_a,

$$P_\theta(s) \approx \frac{K_m}{s\big(R_a(Js + f) + K_b K_m\big)}.$$

Let's try to stabilize it by feedback with the controller $C(s) = k_p$:



Example 4: stabilizing approximate model



Characteristic polynomial

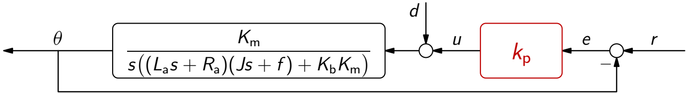
$$\begin{aligned}\chi_{cl}(s) &= s(R_a(Js+f)+K_bK_m)+K_m k_p \\ &= R_a J s^2 + (R_a f + K_b K_m)s + K_m k_p\end{aligned}$$

is Hurwitz iff $k_p > 0$. This suggests that we

- can use high-gain feedback

for this system and effectively implement the plant inversion strategy.

Example 4: stabilizing "full" model



Characteristic polynomial

$$\chi_{cl}(s) = s\big((L_a s + R_a)(Js + f) + K_b K_m\big) + K_m k_p$$
$$= L_a J s^3 + (L_a f + R_a J)s^2 + (R_a f + K_b K_m)s + K_m k_p$$

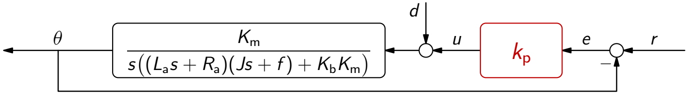
is Hurwitz iff

$$k_p > 0 \quad \text{and} \quad (L_a f + R_a J)(R_a f + K_b K_m) > L_a J K_m k_p$$

i.e.

$$0 < k_p < k_{p,sup} := \frac{(L_a f + R_a J)(R_a f + K_b K_m)}{L_a J K_m}.$$

Example 4: stabilizing "full" model (contd)



Thus

– the conclusion derived on the basis of approximate model is erroneous

and we can increase the gain only up to (not including)

$$k_{p,sup} = \frac{(L_a f + R_a J)(R_a f + K_b K_m)}{L_a J K_m} \quad (\approx 1412.8 \text{ for motor in Lecture 2})$$

Thus, stability requirement might impose limitations on feedback gains. We might be interested to understand,

– if this is a general property.

The answer is affirmative, but to apprehend this we need to *learn more*.

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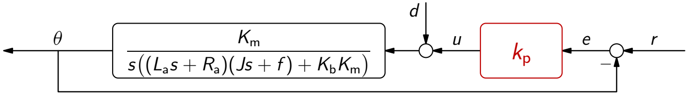

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Example

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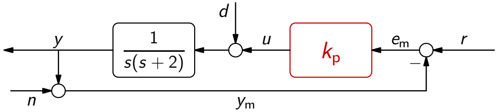

The closed-loop characteristic polynomial $\chi_{cl}(s) = s^2 + 2s + k_p$ has roots at $p_{1,2} = -1 \pm \sqrt{1 - k_p}$. Some examples (for different values of k_p):

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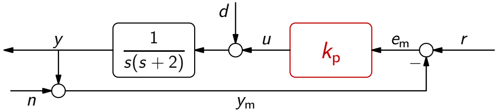

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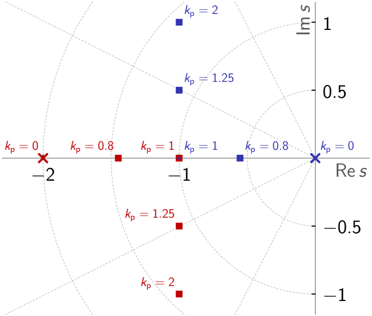
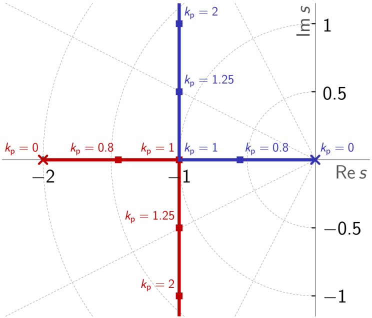


Example (contd)

The complete picture is obtained if we plot

– locations of the roots of $\chi_{\mathrm{cl}}(s)$ as a function of all k_{p}'s

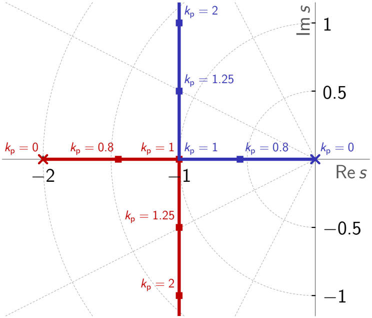


This plot is called the root-locus plot, with

– root loci are paths of the roots of $\chi_{\mathrm{cl}}(s) = 0$ in the s-plane

as some parameter changes.

Example: what can we learn from root locus



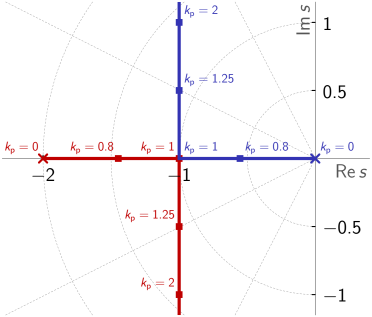
For small k_p ($k_p \leq 1$, overdamped 2-order system): as k_p increases,

- the closed-loop system becomes faster (slowest mode moves leftward)

For large k_p ($k_p > 1$, underdamped 2-order system): as k_p increases,

- the system becomes faster (ω_n increases)
- the system becomes less damped (ζ decreases)

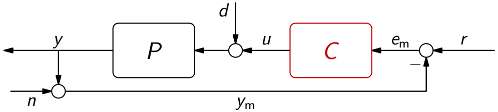
Example: what can we learn from root locus (contd)



Thus, qualitatively,

– limitations on OS $\implies$ limitations on feedback gain k_p
– "faster" might conflict with "non-oscillatory"

Why root locus?



Through root locations we can

- analyze stability
  (closed-loop system stable $\iff$ all roots of $\chi_{\text{cl}}(s)$ are in the open LHP)
- analyze transient performance
  (damping ratio/natural frequency of poles connected with OS/speed of transients)
- design controller to meet performance specifications
- understand limitations of high-gain feedback

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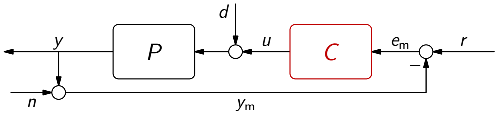

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- *design* controller to meet performance specifications
- understand limitations of high-gain feedback

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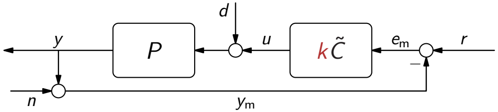
Stability and feedback

Root locus: motivation

Some root locus rules

Root-locus form of characteristic equation

Consider



for some given P and $\tilde{C}$ and $k > 0$ to be played with. The characteristic equation, $\chi_{cl}(s) = 0$, writes then

$$kN_P(s)N_{\tilde{C}}(s) + D_P(s)D_{\tilde{C}}(s) = 0$$

or, equivalently, as

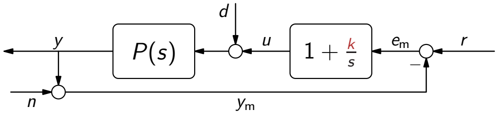
$$-\frac{1}{k} = G_k(s) := P(s)\tilde{C}(s).$$

This form called the root-locus form of the characteristic equation and we assume hereafter w.l.o.g.[1] that $G_k(s)$ is proper.

[1] Otherwise, we can replace k with $1/k$ and end up with a proper $G_k(s)$.

Root-locus form of characteristic equation: remark

Root-locus form can be obtained from other parameters too. For example,



yields

$$\chi_{\text{cl}}(s) = (s+k)N_P(s) + sD_P(s) = 0,$$

which leads to the following root-locus form:

$$-\frac{1}{k} = G_k(s) = \frac{N_P(s)}{s(N_P(s)+D_P(s))}.$$

Root-locus procedure

Given

$$-\frac{1}{k} = G_k(s) \quad \text{or, equivalently,} \quad D_k(s) + kN_k(s) = 0$$

with

$$G_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} = \frac{b_m \prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)},$$

where $n \geq m$, we are interested to

- sketch root locus as k changes from 0 to ∞.

To this end, we'll develop a set of rules, called the root-locus procedure.

Remark 1: there are exactly n root loci, each represents a closed-loop pole

Remark 2: root locus is symmetric with respect to the real axis

Remark 3: if negative k's are required, just replace $G_k \to -G_k$

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Gain and phase rules

Rewrite root-locus form as

$$\frac{1}{k}\,e^{j\pi} = |G_k(s)|e^{j\arg G_k(s)}.$$

Thus, the following equalities must hold for every $s \in \mathbb{C}$ belonging to root locus:

- $|G_k(s)| = 1/k$ (gain rule)
- $\arg G_k(s) \equiv \pi \pmod{2\pi}$ (phase rule[2])

Gain $k > 0$ satisfying the gain rule always exists. Hence, to check whether a point $s_0 \in \mathbb{C}$ belongs to the root locus of G_k,

- we only need to check whether $\arg G_k(s_0) \equiv \pi \pmod{2\pi}$.

If no, this s_0 is not a part of the locus. If yes, it is for

- $k = 1/|G_k(s_0)|$.

[2]Notation $a \equiv b \pmod{c}$ reads: $\exists i \in \mathbb{Z}$ so that $a = b + ci$ (e.g. $-3\pi \equiv \pi \pmod{2\pi}$).

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Analytic determination of gain and phase of $G_k(s_0)$

Consider

$$G_k(s) = \frac{b_m \prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}$$

for some $b_m \neq 0$. Then for any $s_0 \in \mathbb{C}$ we have:

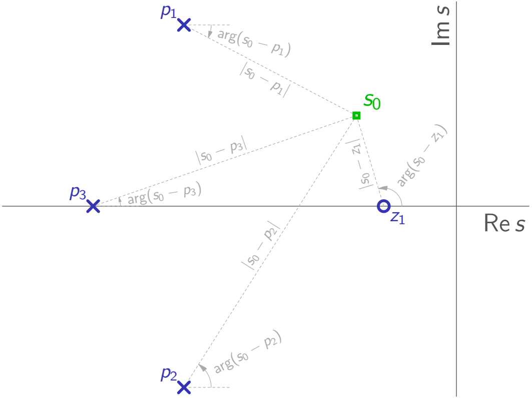
- $\arg G_k(s_0) = \arg b_m + \sum_{i=1}^{m} \arg(s_0 - z_i) - \sum_{i=1}^{n} \arg(s_0 - p_i)$

and

- $|G_k(s_0)| = \dfrac{|b_m| \prod_{i=1}^{m} |s_0 - z_i|}{\prod_{i=1}^{n} |s_0 - p_i|}$.

Remark: because $b_m \in \mathbb{R} \setminus \{0\}$, we have that $\arg b_m = \begin{cases} 0 & \text{if } b_m > 0 \\ \pi & \text{if } b_m < 0 \end{cases}$.

Graphic determination of gain and phase of $G_k(s_0)$



- angles measured in the counterclockwise direction
  (so that $\arg(s_0 - p_1) < 0$, $\arg(s_0 - p_2) > 0$, $\arg(s_0 - p_3) > 0$, and $\arg(s_0 - z_1) > 0$)
- does not account for b_m

Start points ($k = 0$)

From

$$D_k(s) + kN_k(s) = 0$$

it follows that

- all n loci begin at roots of $D_k(s) = 0$.

These points marked on the s-plane by "$\times$".

Remark: Because $-1/k = G_k(s)$, roots of $D_k(s)$ can belong to a loci only at $k = 0$. For all nonzero k we have $1/k < \infty$, so no equality.

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End points ($k = \infty$)

From

$$\frac{1}{k} D_k(s) + N_k(s) = 0$$

it follows that

- m loci end at roots of $N_k(s) = 0$.

These points marked on the s-plane by "∘".

From

$$-\frac{1}{k} = G_k(s)$$

we can guess that

- as $k \to \infty$, the other $n - m$ loci go to infinity in the s-plane

(because $G_k(s)$ has $n - m$ "zeros at infinity"). The question is how?

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Asymptotes ($k \to \infty$)

It can be shown that

– $n - m$ loci approach infinity along asymptotes centered at

$$\sigma_c = \frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n - m}$$

(called the center of gravity) and directed with angles

$$\phi_i = \frac{\arg b_m - \pi + 2\pi i}{n - m}, \qquad i = 0, 1, \ldots, n - m - 1.$$

For example:

$b_m > 0$:

$n - m$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
1	π				
2	$\frac{\pi}{2}$	$\frac{3\pi}{2}$			
3	$\frac{\pi}{3}$	π	$\frac{5\pi}{3}$		
4	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	
5	$\frac{\pi}{5}$	$\frac{3\pi}{5}$	π	$\frac{7\pi}{5}$	$\frac{9\pi}{5}$

$b_m < 0$:

$n - m$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
1	0				
2	0	π			
3	0	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$		
4	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	
5	0	$\frac{2\pi}{5}$	$\frac{4\pi}{5}$	$\frac{6\pi}{5}$	$\frac{8\pi}{5}$

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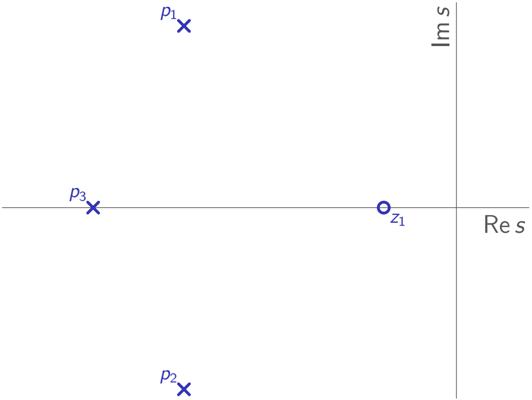
3	$\frac{\pi}{3}$	π	$\frac{5\pi}{3}$		
4	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	
5	$\frac{\pi}{5}$	$\frac{3\pi}{5}$	π	$\frac{7\pi}{5}$	$\frac{9\pi}{5}$

$b_m < 0$:

$n - m$	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5
1	0				
2	0	π			
3	0	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$		
4	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	
5	0	$\frac{2\pi}{5}$	$\frac{4\pi}{5}$	$\frac{6\pi}{5}$	$\frac{8\pi}{5}$

Asymptotes ($k \to \infty$): example

Assume the following pole / zero map of $G_k(s)$ (and that $b_m > 0$):



In this case, $n - m = 2$, $\sigma_c = \frac{p_1+p_2+p_3-z_1}{2}$, and $\phi_1 = \frac{\pi}{2}$ and $\phi_2 = \frac{3\pi}{2}$.

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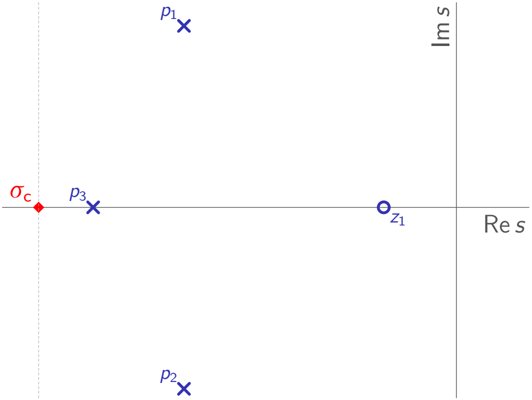
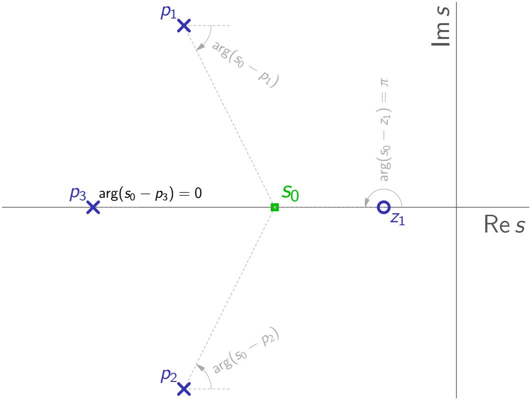


In this case, $n - m = 2$, $\sigma_c = \frac{p_1+p_2+p_3-z_1}{2}$, and $\phi_1 = \frac{\pi}{2}$ and $\phi_2 = \frac{3\pi}{2}$.

Root locus on real axis



- net sum of arg's from any pair of complex-conjugate singularities is 0 (e.g. $\arg(s_0 - p_1) = -\arg(s_0 - p_2)$)
- arg of any real singularity to the right of s_0 is π
- arg of any real singularity to the left of s_0 is 0

Root locus on real axis (contd)

Thus, for every $s_0 \in \mathbb{R}$

- only singularities (poles and zeros) to the right of s_0 matter.

Therefore,

if $b_m > 0$, we have that

$$\arg G(s_0) = \pi \cdot (\text{no. of real singularities to the right of } s_0)$$

and, by the phase rule, root locus lies in

- all sections of $\mathbb{R}$ to the left of an odd number of singularities

if $b_m < 0$, we have that

$$\arg G(s_0) = \pi + \pi \cdot (\text{no. of real singularities to the right of } s_0)$$

and, by the phase rule, root locus lies in

- all sections of $\mathbb{R}$ to the left of an even number of singularities

Root locus on real axis (contd)

Thus, for every $s_0 \in \mathbb{R}$

- only singularities (poles and zeros) to the right of s_0 matter.

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if $b_m > 0$, we have that

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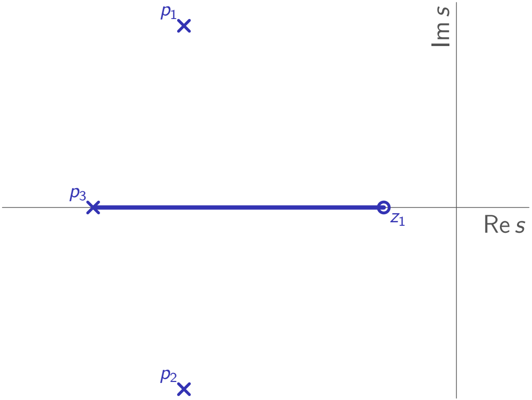
Root locus on real axis: example

Assuming $b_m > 0$, in this case there is one real axis segment:



Breakaway / break-in points on real axis

Breakaway / break-in points can be characterized as

- points on $\mathbb{R}$ where $\chi_{\mathrm{cl}}(s)$ has multiple roots.

Hence, $\sigma \in \mathbb{R}$ is breakaway / break-in points for some $k \neq 0$ if

$$\chi_{\mathrm{cl}}(\sigma) = 0 \qquad \text{and} \qquad \frac{\mathrm{d}}{\mathrm{d}\sigma}\chi_{\mathrm{cl}}(\sigma) = 0.$$

It can be shown that at each σ such that $\chi_{\mathrm{cl}}(\sigma) = D_k(\sigma) + kN_k(\sigma) = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\chi_{\mathrm{cl}}(\sigma) = kD_k(\sigma) \cdot \frac{\mathrm{d}}{\mathrm{d}\sigma}G_k(\sigma).$$

Because $D_k(\sigma) \neq 0$ for all $k > 0$, the breakaway/in condition reads

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}G_k(\sigma) = 0 \quad \text{for } \sigma \in \mathbb{R} \text{ belonging to the root locus.}$$

Remark: alternatively, we may look for extremal points of the real function $G_k(\sigma)$ located within real axis segments of the root locus.

Crossing jω-axis

Points where root locus crosses jω-axis may be especially important as they

– may indicate boundaries of stabilizing k.

These points solve

$$-\frac{1}{k} = G_k(\mathrm{j}\omega)$$

and effectively depend on the phase of the frequency response of G_k only[2]. There are two alternative approaches to determine these points:

1. via the phase rule $\arg G_k(\mathrm{j}\omega) \equiv \pi \pmod{2\pi}$
   (results in transcendental equations, thus efficiently can be solved only for low-order systems + whatever number of integrators)
2. via the Routh-Hurwitz test
   (results in polynomial equations, also not so simple task for high-order polynomials)

[2]If the phase rule $\arg G_k(\mathrm{j}\omega) \equiv \pi \pmod{2\pi}$ holds for some finite $\omega > 0$, there always is k for which the gain rule $k|G_k(\mathrm{j}\omega)| = 1$ holds.

Crossing jω-axis: frequency plots insight

Note that the condition

$$\arg G_k(j\omega) \equiv \pi \pmod{2\pi}$$

can be verified using frequency response plots. Required ω are frequencies at which

- polar plot of $G_k(j\omega)$ crosses the negative real semi-axis
- phase Bode plot of $G_k(j\omega)$ crosses any of the levels $-180 \pmod{360}$

Departure and arrival angles (simple poles)

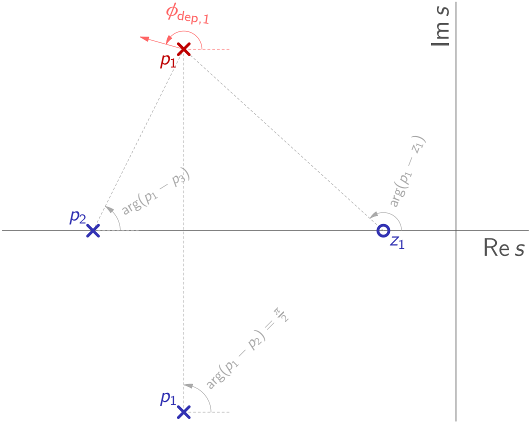
Let p_i $(i = 1, \ldots, n)$ be a simple (i.e. of multiplicity 1) pole of $G_k(s)$. The corresponding locus departs then this pole at the angle

$$\phi_{\text{dep},i} = \pi - \arg b_m - \sum_{\substack{j=1 \\ j \neq i}}^{n} \arg(p_i - p_j) + \sum_{j=1}^{m} \arg(p_i - z_j).$$

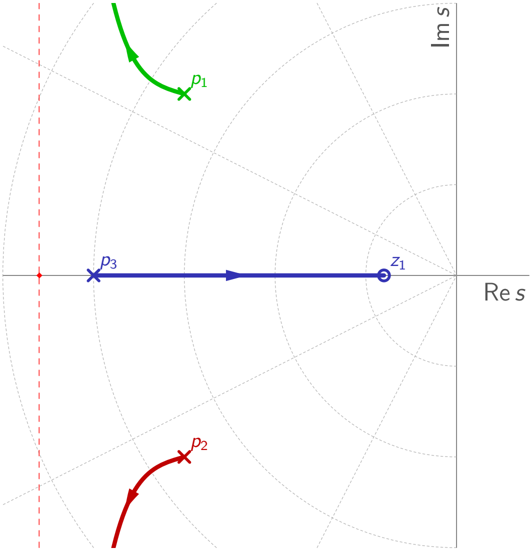
Let z_i $(i = 1, \ldots, m)$ be a simple (i.e. of multiplicity 1) zero of $G_k(s)$. The corresponding locus arrives then at this zero at the angle

$$\phi_{\text{arr},i} = \pi + \arg b_m + \sum_{j=1}^{n} \arg(z_i - p_j) - \sum_{\substack{j=1 \\ j \neq i}}^{m} \arg(z_i - z_j).$$

Departure and arrival angles (simple poles): example



Example: final sketch



Useful MATLAB commands

- `rlocus` (calculates and plots root locus of its argument)
- `rlocfind` (finds root-locus gain for a given set of roots, interactively)
- `sisotool` (could be fun)

Root locus for the example above with MATLAB:

```
Gk = zpk([-.4],[-1.5+j,-1.5-j,-2],1);
rlocus(Gk)
grid on        % plots grid (const "zeta" and "wn" curves)
rlocfind(Gk)   % prompts to select a point in the plot and
               %  then returns the gain corresponding to the
               %  selected point and marks the other points
               %  having the same gain on each locus
```