Introduction to Control (00340040)
lecture no. 5

Leonid Mirkin

Faculty of Mechanical Engineering
Technion—IIT

Outline

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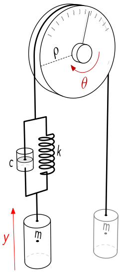
Control effort

Open-loop control: summary

(Naïve) introduction to feedback

Internal stability of feedback systems

Example 1



$\rho = 1.25\,\mathrm{m}$

$k = 40000\,\frac{\mathrm{N\,sec}}{\mathrm{m}}$

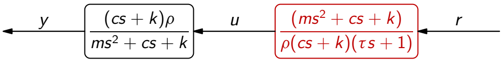
$c = 800\,\frac{\mathrm{N}}{\mathrm{m}}$

$m = 1410\,\mathrm{kg}$

With

$$T_{\mathrm{ref}}(s) = \frac{1}{\tau s + 1}$$

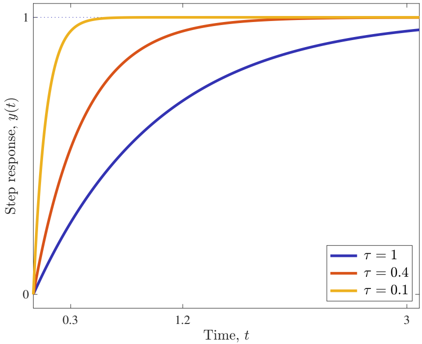
we have



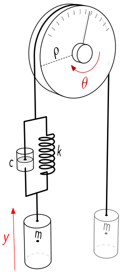
– implementable for every $\tau > 0$
– arbitrarily fast, if τ is small enough

⇓

– no limitations ?



Example 1



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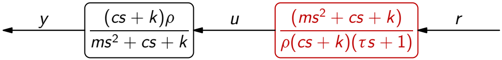

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$\Downarrow$

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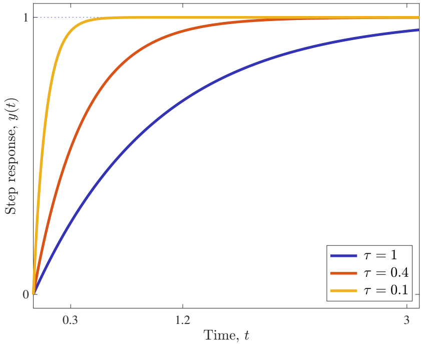
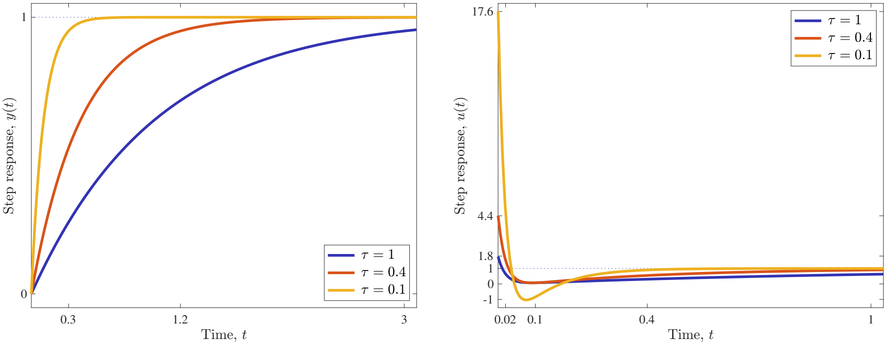


Example 1 (contd)

Take a look at the control signal:



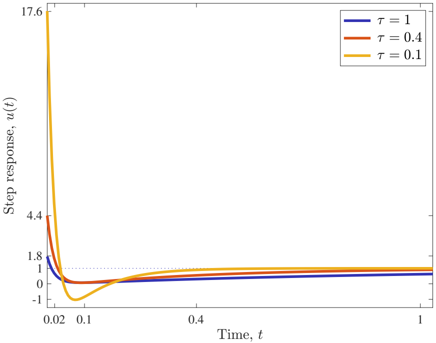
We can see that

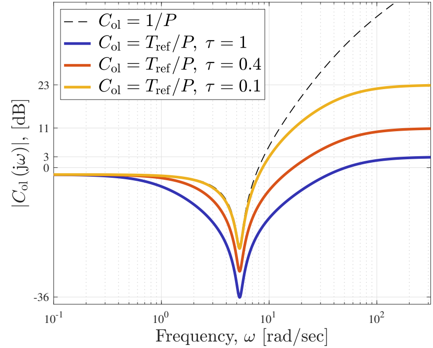
– accelerating y comes at a price of higher control efforts,

both amplitude- and velocity-wise.

Example 1 (contd)

Magnitude frequency response of $C_{\mathrm{ol}} : r \mapsto u$





reveals that

– as τ decreases, peak values of $|C_{\mathrm{ol}}(\mathrm{j}\omega)|$ increase,

causing higher peaks of u. But

– why do we get higher peaks of $|C_{\mathrm{ol}}(\mathrm{j}\omega)|$?

Example 1 (contd)

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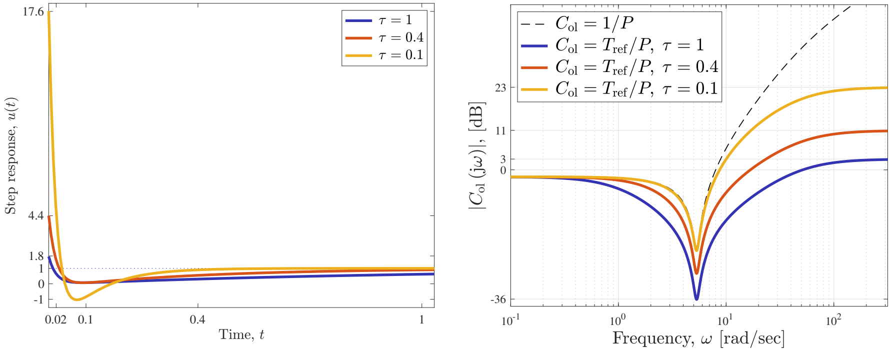

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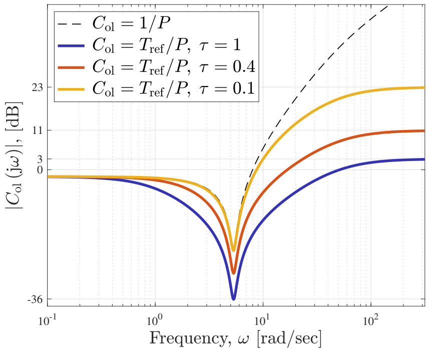
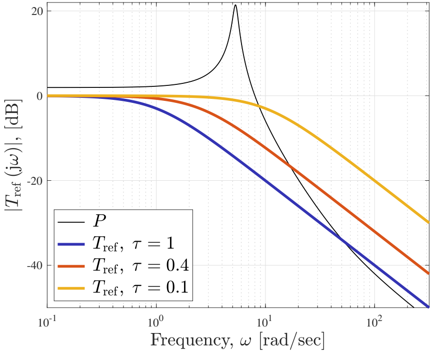
Example 1 (contd)

Compare





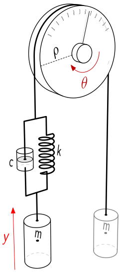
Clearly,

$$|C_{ol}(j\omega)| > 1 \iff |T_{ref}(j\omega)| > |P(j\omega)|$$

The question:

– is there a simple indicator of the latter?

Example 2



$\rho = 1.25\,\mathrm{m}$

$k = 40000\,\frac{\mathrm{N\,sec}}{\mathrm{m}}$

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This plant has

$$P(s) = \frac{(cs+k)\rho}{ms^2+cs+k} = \frac{1.25(0.1065\cdot 5.326s + 5.326^2)}{s^2+2\cdot 0.053\cdot 5.326s+5.326^2}$$

and the plot of $|P(\mathrm{j}\omega)|$ has a slope of about

- $-40\,\mathrm{dB/dec}$ for $5.311 < \omega < 50$ $\qquad \omega_\mathrm{n}/\alpha = 50$
- $-20\,\mathrm{dB/dec}$ for $\omega > 50$ $\qquad \sqrt{1-2\zeta^2}\omega_\mathrm{n} = 5.311$

Thus, the decay of $|P(\mathrm{j}\omega)|$ is way faster than that of $|T_\mathrm{ref}(\mathrm{j}\omega)|$ ($-20\,\mathrm{dB/dec}$ for all $\omega > 1/\tau$) over about a decade. Moreover, this imbalance exceeds the bandwidth $\omega_\mathrm{b} = 1/\tau$ of T_ref for all studied τ. This may suggest that

- if the required $\omega_\mathrm{b} < 50$, then we would better have $T_\mathrm{ref}(s)$ with a pole excess of at least 2,

to have it like P in, say, a decade beyond the bandwidth.

Example 2 (contd)

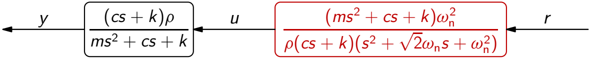
So choose

$$T_{\mathrm{ref}}(s) = \frac{\omega_{\mathrm{n}}^2}{s^2 + \sqrt{2}\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2},$$

which is a 2-order Butterworth, whose bandwidth $\omega_{\mathrm{b}} = \omega_{\mathrm{n}}$. In this case

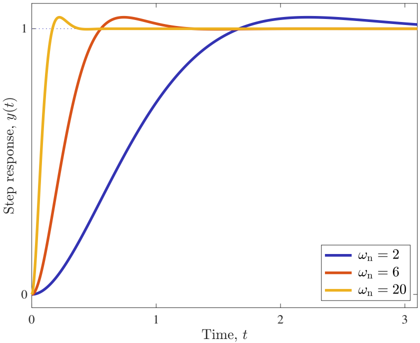
$$C_{\mathrm{ol}}(s) = \frac{T_{\mathrm{ref}}(s)}{P(s)} = \frac{(ms^2 + cs + k)\omega_{\mathrm{n}}^2}{\rho(cs + k)(s^2 + \sqrt{2}\omega_{\mathrm{n}} s + \omega_{\mathrm{n}}^2)}$$

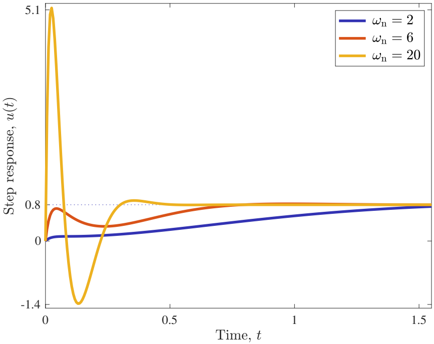
is stable for all ω_{n}, which is a tuning parameter. The control system is then



Example 2 (contd)

With step responses





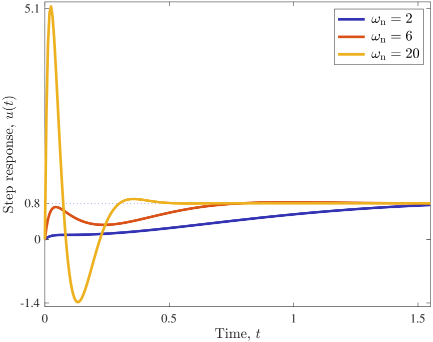
we can see that

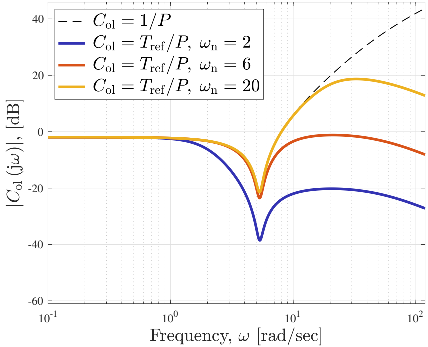
– accelerating y still comes at a price of higher control efforts,

both amplitude- and velocity-wise.

Example 2 (contd)

Magnitude frequency response of $C_{\mathrm{ol}} : r \mapsto u$





still sugegsts that

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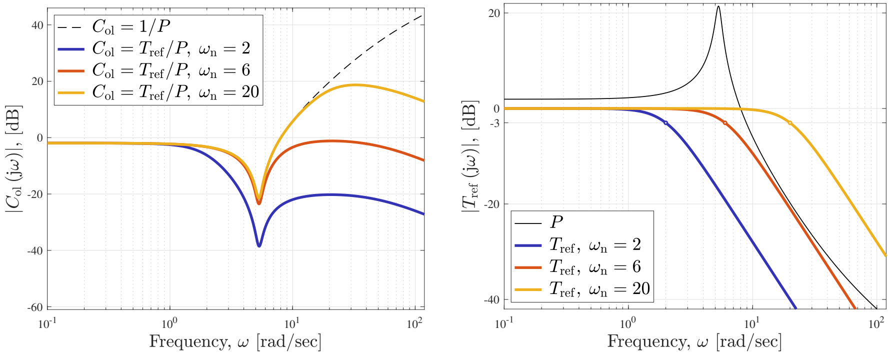
Example 2 (contd)

But comparing now frequency responses



we see that $|C_{\rm ol}(\mathrm{j}\omega)| = |T_{\rm ref}(\mathrm{j}\omega)|/|P(\mathrm{j}\omega)|$ starts to grow above $|C_{\rm ol}(0)|$ as

- the bandwidth $\omega_{\rm b}$ of $T_{\rm ref}(s)$ starts to exceeds that of the plant $P(s)$.

Required bandwidth vs. control efforts

Rule of thumb (quite practical):

- be careful in trying to achieve bandwidth wider than that of the plant.

It often requires extra control effort to render the controlled response faster than the natural response of the plant itself, provided they both exhibit low-pass structures.

Remark: All this makes sense only if we assume that $|C_{\mathrm{ol}}(\mathrm{j}\omega)| \leq 1$ determines "small" control effort. In principle, we may always assume that, which merely means that the control input is normalized. It is thus a healthy habit to normalize (regularize) the control signal u before starting to think about control effort.

Outline

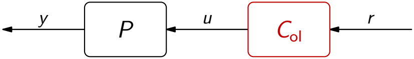
Control effort

Open-loop control: summary

(Naïve) introduction to feedback

Internal stability of feedback systems

Open-loop control: strategy



Plant inversion with reference model:

$$C_{\mathrm{ol}} = P^{-1} T_{\mathrm{ref}} \quad \text{i.e. } C_{\mathrm{ol}}(s) = \frac{T_{\mathrm{ref}}(s)}{P(s)},$$

where T_{ref} embodies requirements to reference response

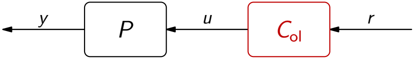
steady-state: – $T_{\mathrm{ref}}(\mathrm{j}\omega) \approx 1$ over the spectrum of r

transients: – either dominant poles of $T_{\mathrm{ref}}(s)$ are in "good" region
– or no high resonant peaks, sufficiently wide bandwidth

Technically, $T_{\mathrm{ref}}(s)$ is constrained to have

- all nonminimum-phase zeros of $P(s)$ as its own zeros
- pole excess $\geq$ poles excess of $P(s)$ (unless derivatives of r measurable)
- not too wide bandwidth (w.r.t. that of $P(s)$) if control effort is limited

Open-loop control: strategy



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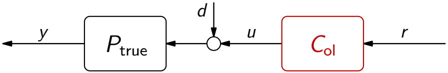
Open-loop control: limitations



Inefficient in handling uncertainties, like

- ☹ modeling inaccuracies in P
- ☹ disturbances

Cannot be applied

- ☹ if plant P is unstable

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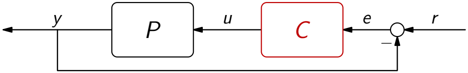
Unity feedback configuration

Let y be measurable. Consider the following control configuration:



In this scheme control signal u is formed on basis of the

– mismatch between the regulated signal y and the reference signal r

(denoted as e). This setup

– called unity feedback configuration

and is the simplest feedback control strategy. Basic relations:

$$e = r - y = r - PCe \iff (1 + PC)e = r \iff e = (1 + PC)^{-1}r,$$

so that

$$u = Ce = C(1 + PC)^{-1}r \quad \text{and} \quad y = Pu = PC(1 + PC)^{-1}r.$$

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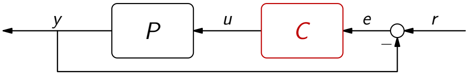

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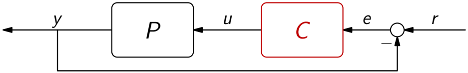
Closed-loop transfer functions



Thus,

$$r \mapsto u:\ \frac{C(s)}{1+P(s)C(s)} =: T_c(s) \qquad \text{control sensitivity}$$

$$r \mapsto y:\ \frac{P(s)C(s)}{1+P(s)C(s)} =: T(s) = P(s)T_c(s) \qquad \text{complementary sensitivity}$$

$$r \mapsto e:\ \frac{1}{1+P(s)C(s)} =: S(s) = 1 - T(s) \qquad \text{sensitivity}$$

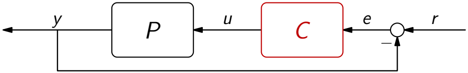
We assume hereafter that $P(s)$ and $C(s)$ are proper. The loop is said to be

- well posed if $1 + P(\infty)C(\infty) \neq 0 \implies S(s)$ is proper.

We may think of

- T_c as the closed-loop counterpart of C_{ol}
- T as the closed-loop counterpart of $T_{yr} = PC_{ol}$

Closed-loop transfer functions



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$r \mapsto y$: $\dfrac{P(s)C(s)}{1+P(s)C(s)} =: T(s) = P(s)T_c(s)$ complementary sensitivity

$r \mapsto e$: $\dfrac{1}{1+P(s)C(s)} =: S(s) = 1 - T(s)$ sensitivity

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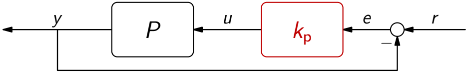
Static high-gain controller

Choose $C(s) = k_{\mathrm{p}}$ for some gain k_{p} (simplest choice):



Then,

$$T_{\mathrm{c}}(s) = \frac{k_{\mathrm{p}}}{1 + k_{\mathrm{p}} P(s)} = \frac{1}{1/k_{\mathrm{p}} + P(s)}.$$

Important is that

– if $k_{\mathrm{p}} \to \infty$, then $T_{\mathrm{c}} \to P^{-1}$,

i.e. $u \to P^{-1} r$ and $y \to r$. Thus, *loosely speaking*

although in the feedback scheme controller does not depend on $P(s)$!!!

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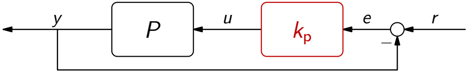

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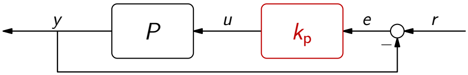

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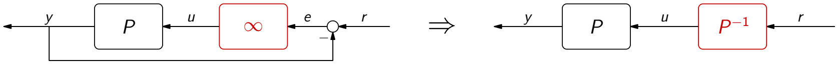

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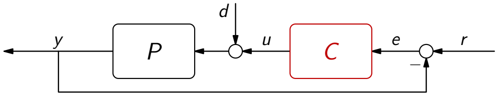
Effect of disturbances

Now assume that there is an input disturbance signal:



Closed-loop transfer functions

$d \mapsto u$: $-\dfrac{P(s)C(s)}{1+P(s)C(s)} = -T(s)$

$d \mapsto y$: $\dfrac{P(s)}{1+P(s)C(s)} =: T_{\mathrm{d}}(s)$ disturbance sensitivity

The four systems

— S, T, T_{d}, and T_c,

colloquially known as the Gang of Four, completely determine properties of the controlled closed-loop system.

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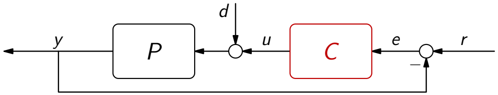

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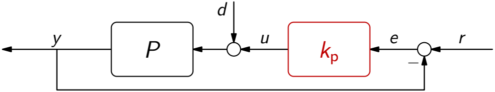
Static high-gain controller and disturbances

Choose again $C(s) = k_\text{p}$:



leading to

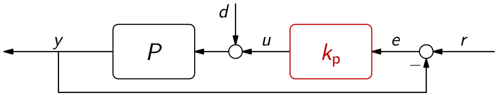
$$T(s) = \frac{k_\text{p} P(s)}{1 + k_\text{p} P(s)} = \frac{1}{1/(k_\text{p} P(s)) + 1}.$$

If $k_\text{p} \to \infty$, then $T(s) \to 1$ ($\forall s \in \mathbb{C}$ such that $P(s) \neq 0$), which effectively implies that $u \to P^{-1} r - d$ and then, again, $y \to r$. Thus, *loosely speaking*

although in the feedback scheme we do not (directly) measure d !!!

Static high-gain controller and disturbances

Choose again $C(s) = k_p$:



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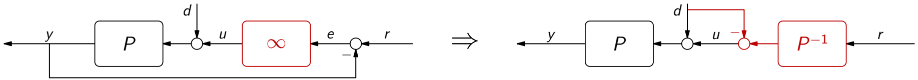

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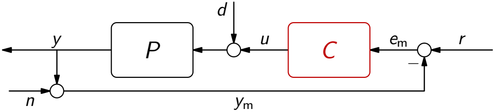
Effect of measurement noise

The availability of perfect measurements of y is an unrealistic assumption. Sensor imperfections may be modeled as measurement noise:



Closed-loop transfer functions

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$$n \mapsto y: \; -\frac{P(s)C(s)}{1+P(s)C(s)} = -T(s)$$

As a matter of fact, summarizing the effects of all inputs we have:

$$\boxed{u = T_c r - T d - T_c n} \quad \text{and} \quad \boxed{y = T r + T_d d - T n}.$$

Do *memorize* these relations well.

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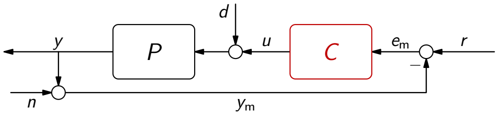


Closed-loop transfer functions

$$n \mapsto u: \; -\frac{C(s)}{1+P(s)C(s)} = -T_c(s)$$

$$n \mapsto y: \; -\frac{P(s)C(s)}{1+P(s)C(s)} = -T(s)$$

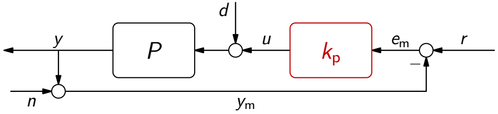
As a matter of fact, summarizing the effects of all inputs we have:

$$\boxed{u = T_c r - T d - T_c n} \quad \text{and} \quad \boxed{y = T r + T_d d - T n}.$$

Do *memorize* these relations well.

Static high-gain controller and measurement noise

Choose $C(s) = k_p$ yet again:



If now $k_p \to \infty$, we end up with $u \to P^{-1}r - P^{-1}n - d$, so that $y \to r - n$.

Thus, *loosely speaking*

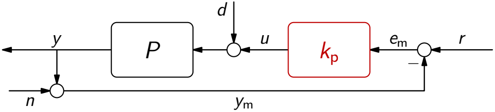
and we actually follow corrupted reference $r - n$ rather than r. This is not what we need. Moreover, the term

$-P^{-1}n$ might cause high-magnitude fast oscillations of control signal u

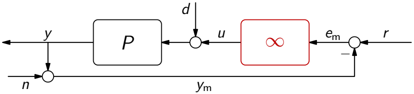
because P^{-1} is typically a high-pass filter and n is fast. This is undesirable.

Static high-gain controller and measurement noise

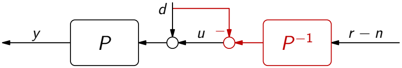
Choose $C(s) = k_\mathrm{p}$ yet again:



If now $k_\mathrm{p} \to \infty$, we end up with $u \to P^{-1}r - P^{-1}n - d$, so that $y \to r - n$. Thus, *loosely speaking*



$\Rightarrow$



and we actually follow corrupted reference $r - n$ rather than r. This is not what we need. Moreover, the term

$- P^{-1}n$ might cause high-magnitude fast oscillations of control signal u because P^{-1} is typically a high-pass filter and n is fast. This is undesirable.

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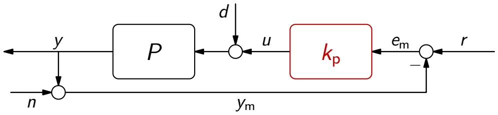

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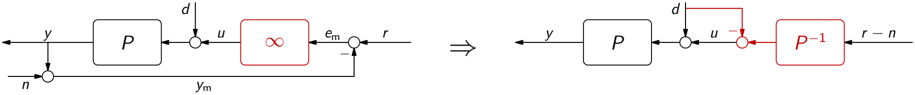

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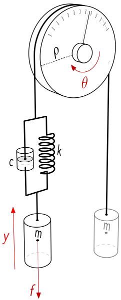
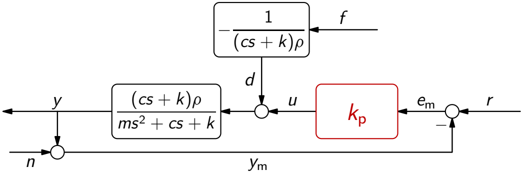
Example 3



Unity feedback configuration with a static C is then



where n is inaccuracy in measuring y. We choose

$$k_p = 1000,$$

which should be high enough.

$\rho = 1.25\,\mathrm{m}$

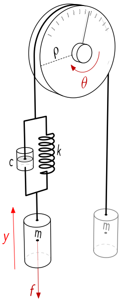
$k = 40000\,\frac{\mathrm{N\,sec}}{\mathrm{m}}$

$c = 800\,\frac{\mathrm{N}}{\mathrm{m}}$

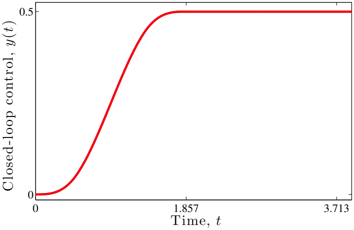
$m = 1410\,\mathrm{kg}$

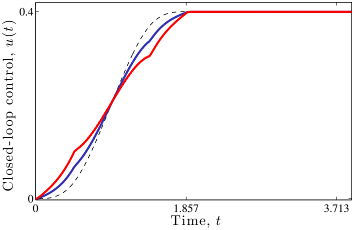
$m = 2820\,\mathrm{kg}$

Example 3: $f \equiv 0$, $n \equiv 0$

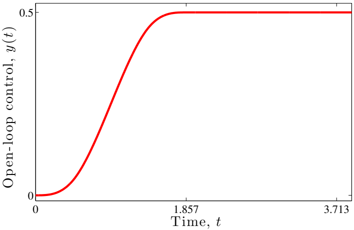


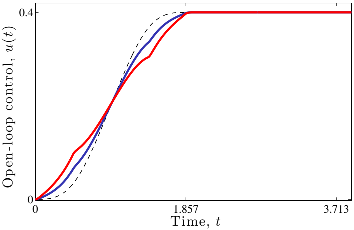
Closed-loop control (controller is independent of m):





Open-loop control (controller depends on m):





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$k = 40000\,\frac{\mathrm{N\,sec}}{\mathrm{m}}$

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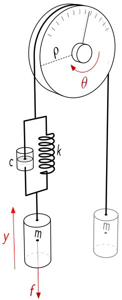
Example 3: $f(t) = -250g(\mathbb{1}(t-1) - \mathbb{1}(t-2.2))$, $n \equiv 0$



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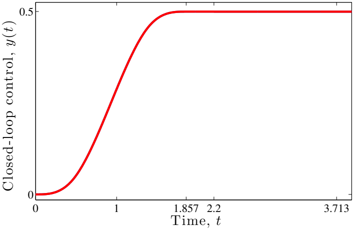
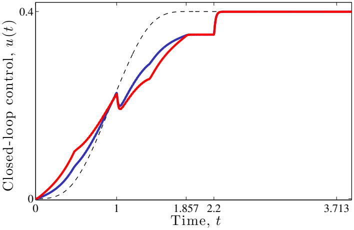
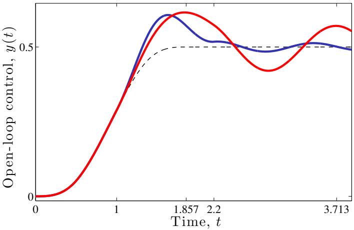
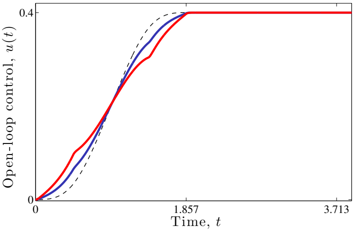
Closed-loop control (senses f via y):



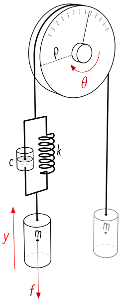


Open-loop control (unaware of f):





Example 3: $f(t) = 0,\ n(t) = 0.01\sin(20\pi t)$



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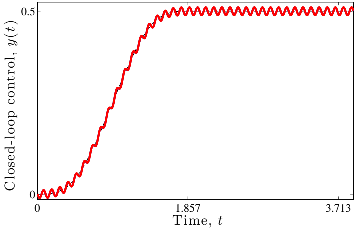
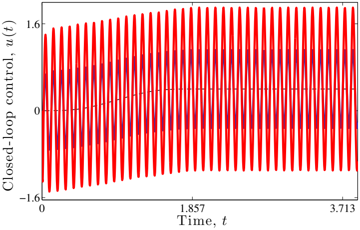
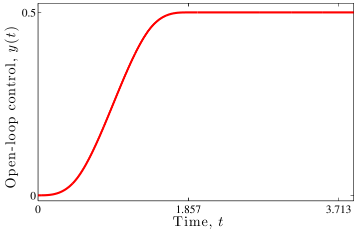
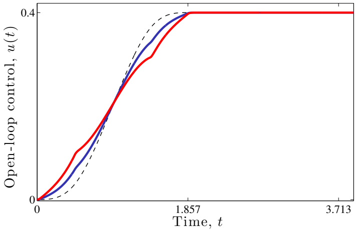




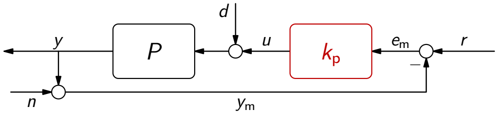
Open-loop control (unaffected by n):





Static low-gain controller

Return to the unity feedback scheme



and assume that $k_p \to 0$ (low-gain feedback). In this case

$$T \to 0, \quad S \to 1, \quad T_c \to 0, \quad \text{and} \quad T_d \to P.$$

In other words,

$$u \to 0 \quad \text{and} \quad y \to Pd.$$

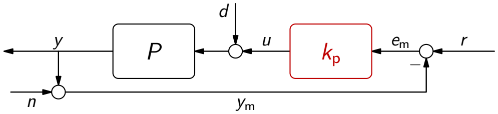
Thus,

$\Longrightarrow$ low-gain feedback does nothing, effectively opening the loop.

Yet this "does nothing" also includes "does not let the measurement noise pass through," which is what we might need (more about this later on).

Static low-gain controller

Return to the unity feedback scheme



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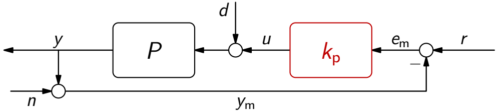
Feedback control: is it a panacea?



Feedback does what open-loop control can never do:

- performs in spite of modeling inaccuracies in P
- performs in spite of disturbances

Sounds like a miracle . . . or something too good to be true.

Up to this point we did not mention the stability of the closed-loop system, which is

– the first thing to take care of.

Without addressing (and guaranteeing) stability any analysis is meaningless.

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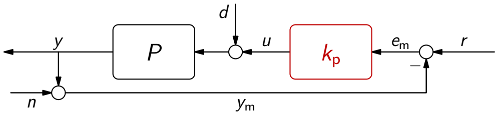

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Outline

Control effort

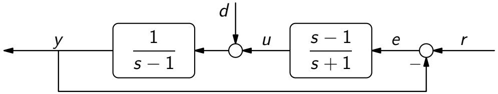
Open-loop control: summary

(Naïve) introduction to feedback

Internal stability of feedback systems

Stability of closed-loop systems: Example 1

Consider



Then

$$y = Tr + T_{\mathrm{d}} d =: y_r + y_d.$$

Since

$$y_r = \frac{1}{s+2}\, r,$$

it is bounded whenever so is r. However,

$$y_d = \frac{s+1}{(s-1)(s+2)}\, d,$$

i.e. there is (arbitrarily small) d that causes unbounded y.

Stability of closed-loop systems: Example 1

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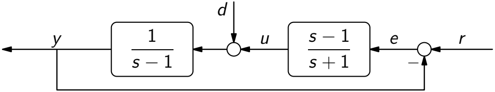

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Stability of closed-loop systems: Example 2

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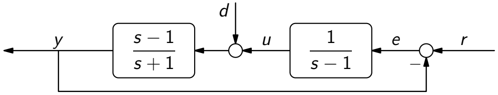

Then

$$y = Tr + T_{\mathrm{d}} d = \frac{1}{s+2}\, r + \frac{s-1}{s+2}\, d,$$

i.e. it is bounded whenever so are r and d. However,

$$u = T_{\mathrm{c}} r - T d = \frac{s+1}{(s-1)(s+2)}\, r - \frac{1}{s+2}\, d,$$

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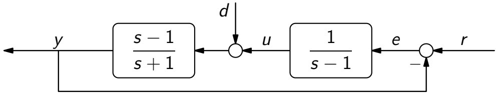

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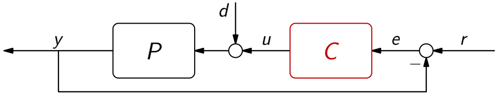
Moral



Even in this relatively simple closed-loop system

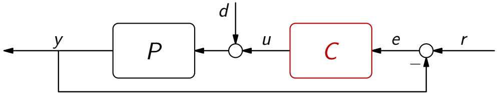
– the stability of any single closed-loop system is not enough

to conclude about the stability of the whole system.

Like in open-loop control, this is sorted out by the use of the notion of

– internal stability.

Moral



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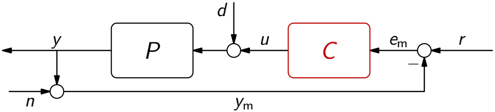
Internal stability of closed-loop systems: definition

The closed-loop system



said to be internally stable if

– transfer functions from all possible inputs to all possible outputs stable.

There are four possible closed-loop system for this configuration, S, T, T_d, and T_c, with the relationship

$$\begin{bmatrix} e \\ u \end{bmatrix} = \begin{bmatrix} S & T_d \\ T_c & T \end{bmatrix} \begin{bmatrix} r \\ -d \end{bmatrix}.$$

So we need to check only these four.

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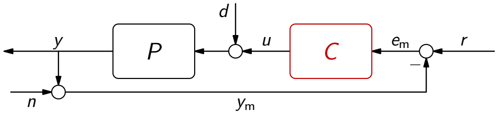

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So we need to check only these four[1].

[1] In fact, only three, because $S = 1 - T$ is stable iff T is stable.

Poles of closed-loop transfer functions

Let

$$P(s) = \frac{N_P(s)}{D_P(s)} \qquad \text{and} \qquad C(s) = \frac{N_C(s)}{D_C(s)}$$

be proper, with coprime[2] $N_P(s)$ and $D_P(s)$ and coprime $N_C(s)$ and $D_C(s)$. In this case

$$T(s) = \frac{N_P(s)N_C(s)}{\chi_{\text{cl}}(s)}, \quad T_{\text{d}}(s) = \frac{N_P(s)D_C(s)}{\chi_{\text{cl}}(s)}, \quad T_{\text{c}}(s) = \frac{D_P(s)N_C(s)}{\chi_{\text{cl}}(s)},$$

where

$$\chi_{\text{cl}}(s) := N_P(s)N_C(s) + D_P(s)D_C(s).$$

These transfer functions have the same denominator, $\chi_{\text{cl}}(s)$, unless there are

– pole/zero cancellation between their numerators and $\chi_{\text{cl}}(s)$.

If there are no cancellations, then all these transfer functions have the same poles and the situation is simple. Otherwise ...

[2] I.e. they have no common roots.

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Pole/zero cancellations in $T(s)$

In

$$T(s) = \frac{N_P(s)N_C(s)}{\chi_{\text{cl}}(s)}$$

polynomials

$$N_P(s)N_C(s) \quad \text{and} \quad \chi_{\text{cl}}(s) = N_P(s)N_C(s) + D_P(s)D_C(s)$$

have common roots iff

- $N_P(s)$ and $D_C(s)$ have common roots

or

- $N_C(s)$ and $D_P(s)$ have common roots.

Thus,

- common roots of $N_P(s)$ and $D_C(s)$ and those of $N_C(s)$ and $D_P(s)$ are not poles of $T(s)$.

Pole/zero cancellations in $T_{\mathrm{d}}(s)$

In

$$T_{\mathrm{d}}(s) = \frac{N_P(s)D_C(s)}{\chi_{\mathrm{cl}}(s)}$$

polynomials

$$N_P(s)D_C(s) \quad \text{and} \quad \chi_{\mathrm{cl}}(s) = N_P(s)N_C(s) + D_P(s)D_C(s)$$

have common roots

- iff $N_P(s)$ and $D_C(s)$ have common roots,
- but not if $N_C(s)$ and $D_P(s)$ have common roots.

Thus,

- common roots of $N_C(s)$ and $D_P(s)$ are still poles of $T_{\mathrm{d}}(s)$.

Pole/zero cancellations in $T_c(s)$

In

$$T_c(s) = \frac{D_P(s)N_C(s)}{\chi_{cl}(s)}$$

polynomials

$$D_P(s)N_C(s) \quad \text{and} \quad \chi_{cl}(s) = N_P(s)N_C(s) + D_P(s)D_C(s)$$

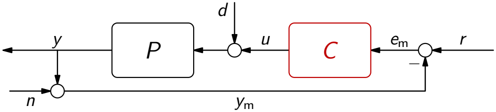
have common roots

- iff $N_C(s)$ and $D_P(s)$ have common roots,
- but not if $N_P(s)$ and $D_C(s)$ have common roots.

Thus,

- common roots of $N_P(s)$ and $D_C(s)$ are still poles of $T_c(s)$.

Internal stability criteria



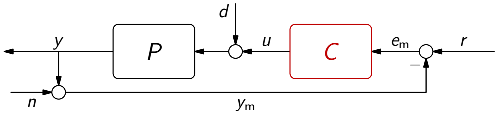
Theorem (pole/zero cancellations)

If $P(s)$ and $C(s)$ are proper and $1 + P(\infty)C(\infty) \neq 0$, then the system is internally stable iff

1. *there are no unstable pole/zero cancellations between $P(s)$ and $C(s)$*
2. *either of the closed-loop systems is stable.*

In terms of the $\chi_{\mathrm{cl}}(s) = N_P(s)N_C(s) + D_P(s)D_C(s)$:

Internal stability criteria



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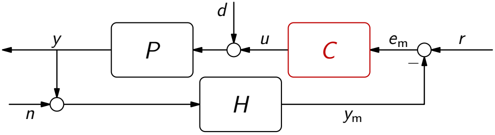
In terms of the $\chi_{\mathrm{cl}}(s) = N_P(s)N_C(s) + D_P(s)D_C(s)$:

Theorem (characteristic polynomial)

If $P(s)$ and $C(s)$ are proper and $\deg \chi_{\mathrm{cl}}(s) = \deg D_P(s) + \deg D_C(s)$, then the system is internally stable iff its characteristic polynomial $\chi_{\mathrm{cl}}(s)$ has no roots in the closed RHP $\bar{\mathbb{C}}_0 = \{s \in \mathbb{C} \mid \operatorname{Re} s \geq 0\}$.

More general setup

Consider



where H can be interpreted as a sensor. This

— changes almost nothing

in definitions / criteria for internal stability. The only modification we need is to redefine the characteristic polynomial as

$$\chi_{cl}(s) := N_P(s)N_C(s)N_H(s) + D_P(s)D_C(s)D_H(s),$$

where $N_H(s)$ and $D_H(s)$ are coprime numerator and denominator of

$$H(s) = \frac{N_H(s)}{D_H(s)}.$$

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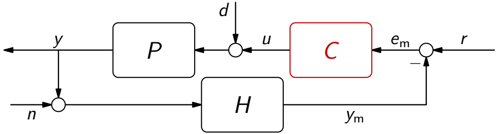

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$$\chi_{\text{cl}}(s) := N_P(s)N_C(s)N_H(s) + D_P(s)D_C(s)D_H(s),$$

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