Introduction to Control (00340040)
lecture no. 4

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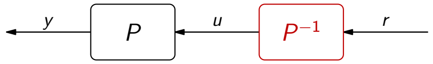
Plant inversion: so far



Aims at

- perfect control, $y = r$, for a given reference signal r.

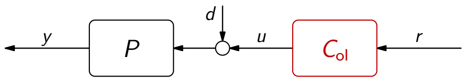
But

- is never attainable in practical situations
  because of uncertainty, like modeling errors and disturbances
- might be illegal
  because of internal instability caused by unstable cancellations
- might be too expensive
  in terms of control efforts, reasons not explained yet

Direction we explore today:

- how to formalize relaxing $y = r$ to $y \approx r$

Controlled dynamics



By linearity,

$$y = P(d + C_{\rm ol} r) = Pd + PC_{\rm ol} r$$

has two independent components,

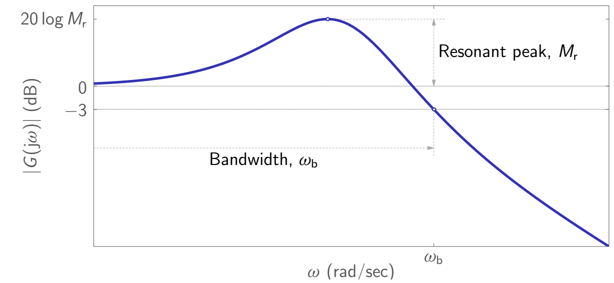
- disturbance response, $y_d = Pd$ — cannot be affected by $C_{\rm ol}$
- reference response, $y_r = PC_{\rm ol} r$ — can be affected by $C_{\rm ol}$

We concentrate on what we can affect, the controlled dynamics

$$T_{yr} = PC_{\rm ol},$$

and their steady-state and transient responses.

Magnitude frequency response of LPFs (from LS)



where

- bandwidth is the largest ω_b such that $|G(j\omega)| \geq 1/\sqrt{2}$ for all $\omega \leq \omega_b$
- resonance peak $M_r := \max_\omega |G(j\omega)| > 1$

and we assume that $|G(0)| = 1$.

Outline

Steady-state performance

Transient performance: modal perspective

Transient performance: frequency-response perspective (from LS)

Steady-state error in terms of T_{yr}

Let $y = T_{yr}r$ for a stable T_{yr}. We are interested in quantifying $e = r - y$ in steady state. In this case

$$r(t) = \sin(\omega t + \phi)\mathbb{1}(t) \quad\Longrightarrow\quad e_{\rm ss} = |1 - T_{yr}(\mathrm{j}\omega)|$$

(the step corresponds to $\omega = 0$). In other words,

- error equals the magnitude of the frequency response of $S_{yr} := 1 - T_{yr}$

"Small" error,

$$e_{\rm ss} \ll 1 \quad\Longrightarrow\quad |S_{yr}(\mathrm{j}\omega)| \ll 1,$$

In some situations it may be convenient to think of it as

$$T_{yr}(\mathrm{j}\omega) \approx 1$$

(i.e. both $|T_{yr}(\mathrm{j}\omega)| \approx 1$ and $\arg T_{yr}(\mathrm{j}\omega) \approx 0$).

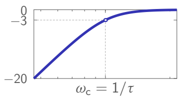
Steady-state error in terms of T_{yr}: example

Let

$$T_{yr}(s) = \frac{1}{\tau s + 1}$$

(low-pass filter with the bandwidth $\omega_{\rm b} = 1/\tau$). S_{yr} is a high-pass filter with the cut-off frequency $1/\tau$ then,

$$S_{yr}(s) = \frac{\tau s}{\tau s + 1} \quad\Longrightarrow\quad |S_{yr}(\mathrm{j}\omega)| = \frac{\tau\omega}{\sqrt{1 + \tau^2\omega^2}} =$$

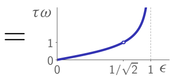


In terms of the asymptotic Bode plot,

- $e_{\rm ss} \le 0.1 \implies \omega \le 0.1/\tau \implies \tau \le 0.1/\omega \implies \omega_{\rm b} \ge 10\omega$
- $e_{\rm ss} \le 0.01 \implies \omega \le 0.01/\tau \implies \tau \le 0.01/\omega \implies \omega_{\rm b} \ge 100\omega$

If we need to be precise,

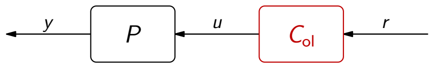
$$e_{\rm ss} = |S_{yr}(\mathrm{j}\omega)| \le \epsilon \in [0, 1] \quad\Longleftrightarrow\quad \tau\omega = \frac{\omega}{\omega_{\rm b}} \le \frac{\epsilon}{\sqrt{1 - \epsilon^2}} =$$



with the same qualitative conclusion:

- the larger ω we wanna follow, the larger bandwidth $\omega_{\rm b}$ of T_{yr} we need.

Controllers for zero steady-state errors: interpolation



If r is a sine wave with frequency $\omega \ge 0$ and $T_{yr} = PC_{\rm ol}$, then

$$e_{\rm ss} = |1 - T_{yr}(\mathrm{j}\omega)| = 0 \iff T_{yr}(\mathrm{j}\omega) = 1 \iff C_{\rm ol}(\mathrm{j}\omega) = \frac{1}{P(\mathrm{j}\omega)}.$$

This is

- plant inversion at one given frequency (or several frequencies)

which only requires $P(\mathrm{j}\omega) \ne 0$, milder than requirements for $C_{\rm ol} = P^{-1}$.

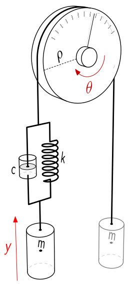
Remark: If we work with systems with real parameters, then $C_{\rm ol}(\mathrm{j}\omega) = 1/P(\mathrm{j}\omega)$ must be complemented by $C_{\rm ol}(-\mathrm{j}\omega) = 1/P(-\mathrm{j}\omega) = \overline{C_{\rm ol}(\mathrm{j}\omega)}$ whenever $\omega > 0$.

If $\omega = 0$, i.e. $r = \mathbb{1}$, then

$$C_{\rm ol}(0) = \frac{1}{P(0)},$$

meaning all we need is to set a right static gain to the controller.

Example 1



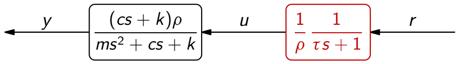
$\rho = 1.25\,\text{m}$

$k = 40000\,\frac{\text{N sec}}{\text{m}}$

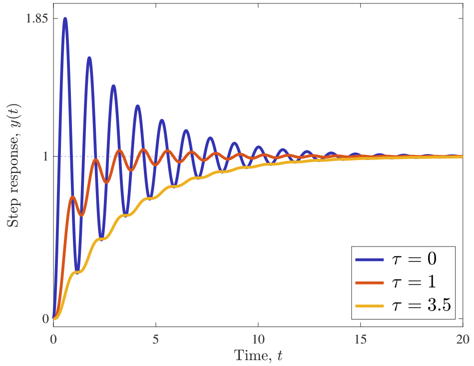
$c = 800\,\frac{\text{N}}{\text{m}}$

$m = 1410\,\text{kg}$

For every $\tau > 0$ the controller in

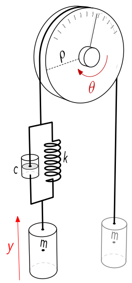


inverts $P(0) = \rho$ and renders $e_{ss} = 0$:



although with different transients.

Example 2



If now $r(t) = \sin(\omega t + \phi)\mathbb{1}(t)$, then $e_{ss} = 0$ iff

$$C_{\mathrm{ol}}(\mathrm{j}\omega) = \frac{1}{P(\mathrm{j}\omega)} \quad \text{and} \quad C_{\mathrm{ol}}(-\mathrm{j}\omega) = \frac{1}{P(-\mathrm{j}\omega)} = \overline{C_{\mathrm{ol}}(\mathrm{j}\omega)}.$$

A way to solve that is to fix, for whatever $\tau > 0$,

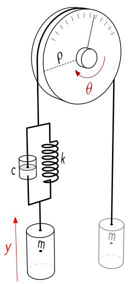
$$C_{\mathrm{ol}}(s) = \frac{b_1 s + b_0}{\tau s + 1}$$

and find b_0 and b_1 via $b_0 \pm b_1 \mathrm{j}\omega = (1 \pm \mathrm{j}\tau\omega)/P(\pm\mathrm{j}\omega)$, i.e.

$$\underbrace{\begin{bmatrix} 1 & \mathrm{j}\omega \\ 1 & -\mathrm{j}\omega \end{bmatrix}}_{\text{Vandermonde matrix}} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} (1 + \mathrm{j}\tau\omega)/P(\mathrm{j}\omega) \\ (1 - \mathrm{j}\tau\omega)/P(-\mathrm{j}\omega) \end{bmatrix}$$

This equation is solvable for all $\omega \neq 0$.

Example 2 (contd)



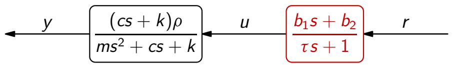
$\rho = 1.25\,\text{m}$

$k = 40000\,\frac{\text{N sec}}{\text{m}}$

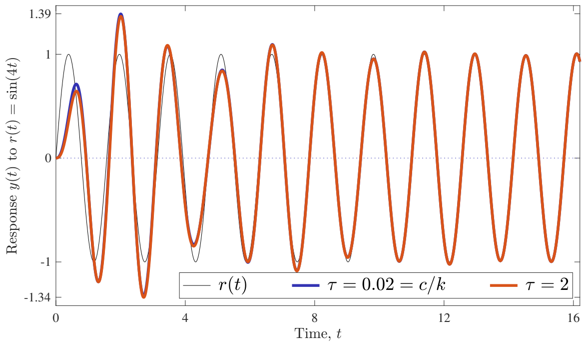
$c = 800\,\frac{\text{N}}{\text{m}}$

$m = 1410\,\text{kg}$

Let $\omega = 4$. For every $\tau > 0$ the controller in



inverts $P(\pm\mathrm{j}4)$ and renders $e_{ss} = 0$:



$(\begin{bmatrix} b_0 & b_1 \end{bmatrix} = \begin{bmatrix} 0.436 & 0.02 \end{bmatrix}$ and $\begin{bmatrix} b_0 & b_1 \end{bmatrix} = \begin{bmatrix} 0.081 & -0.89 \end{bmatrix})$.

Outline

1st order systems revised

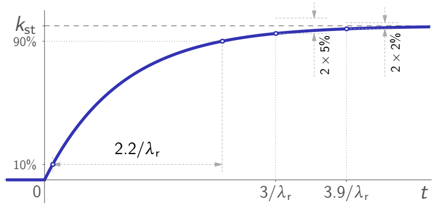
The transfer function

$$G(s) = \frac{k_{st}}{\tau s + 1}$$

has one (real) pole at

$$s = -\frac{1}{\tau} =: -\lambda_r,$$

where $\lambda_r > 0$ is the absolute value (of the real part) of the pole. Therefore,



and

– the larger λ_r is, the faster the transients are.

2nd order underdamped systems revised

The transfer function

$$G(s) = \frac{k_{st}\,\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k_{st}\,\omega_n^2}{(s + \zeta\omega_n)^2 + (1 - \zeta^2)\omega_n^2}$$

has two poles at

$$s = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n =: -\lambda_r \pm j\lambda_i,$$

i.e. λ_r and λ_i are the absolute values of the real and imaginary parts of the poles. It is readily seen that

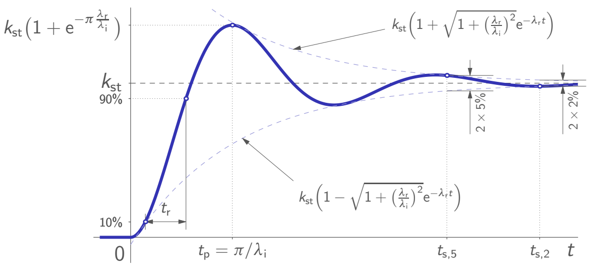
$$\frac{\lambda_r}{\lambda_i} = \frac{\zeta}{\sqrt{1-\zeta^2}} \qquad \text{and} \qquad \lambda_r^2 + \lambda_i^2 = \omega_n^2$$

i.e.

– the ratio between pole real and imaginary parts depends only on ζ

– the absolute value of the pole depends only on ω_n

2nd order underdamped systems revised (contd)



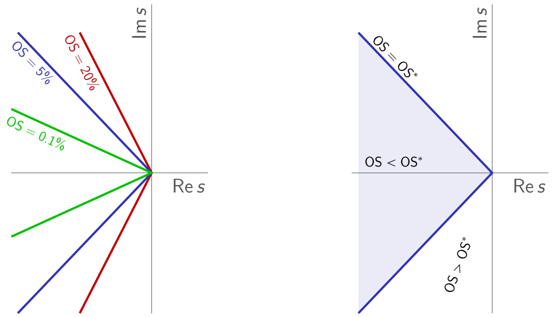
Thus,

– OS depends only on the ratio $\frac{\lambda_r}{\lambda_i}$ (in fact, $\text{OS} = e^{-\pi(\lambda_r/\lambda_i)} \cdot 100\%$)

– speed of transients proportional to the absolute value of the poles

Overshoot level curves

Constant OS $\iff$ constant ratio $\frac{\lambda_r}{\lambda_i}$. Hence, OS level curves are radial lines:



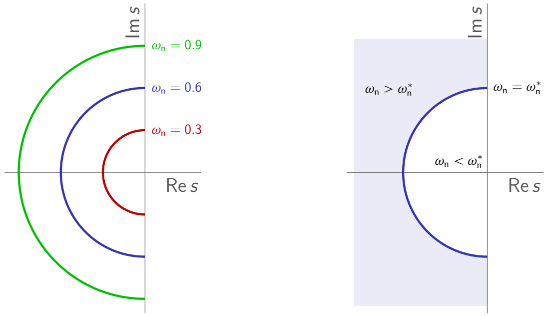
Given some $\text{OS}^* \in (0\%, 100\%)$, the shaded area contains

– all poles producing $\text{OS} < \text{OS}^*$.

Note that damping factor level ($\zeta = \text{const}$) curves are the same radial lines.

Natural frequency level curves

Constant $\omega_n \iff$ constant $\lambda_r^2 + \lambda_i^2$. Hence, ω_n level curves are concentric semi-circles:



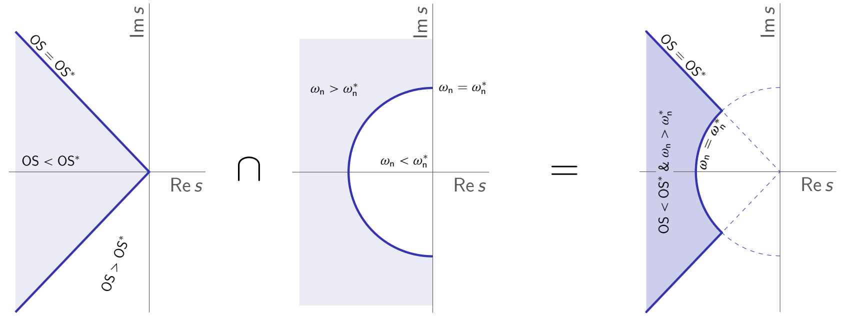
Given some $\omega_n^* > 0$, the shaded area contains

- all poles producing $\omega_n > \omega_n^*$.

Area of (relatively) fast and smooth transients

Assume we need both fast ($\omega_n > \omega_n^*$) and not too oscillatory ($\mathrm{OS} < \mathrm{OS}^*$) transients. In terms of pole location, we need to

- use the intersection of these regions:



In other words, the

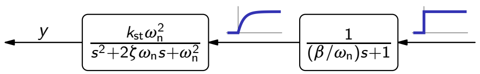
- shaded area is where poles shall be placed to have "fast enough" and "smooth enough" step responses.

Effect of additional pole

Let

$$G_\tau(s) = \frac{k_{st}\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)((\beta/\omega_n)s + 1)}$$

for $\beta > 0$, which may be viewed as the series of 1- and 2-order systems[1]. If we consider the resulting response as

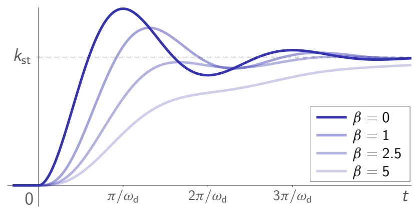


we may view the step response of G_τ as the response of a standard 2-order system to a "smoothed step" input. The response may then be expected to be

- slower
- smoother (less oscillatory)

[1] We have $\tau = \beta/\omega_n$ to have ω_n scaling the time in all components of the response.

Effect of additional pole (contd)



As β (and therefore $\tau = \beta/\omega_n$) grows,

- the overshoot OS decreases
- the raise time t_r increases

Effect of additional zero

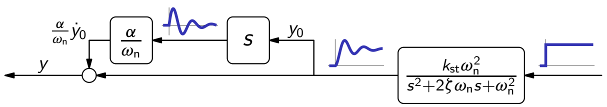
Let

$$G_\alpha(s) = \frac{k_{st}\omega_n^2((\alpha/\omega_n)s+1)}{s^2+2\zeta\omega_n s+\omega_n^2},$$

for $\alpha \in \mathbb{R}$. In this case

$$Y_\alpha(s) = G_\alpha(s)\frac{1}{s} = Y_0(s) + \frac{\alpha}{\omega_n}sY_0(s) \iff y_\alpha(t) = y_0(t) + \frac{\alpha}{\omega_n}\dot{y}_0(t)$$

where y_0 is the response with $\alpha = 0$ (no zeros). In other words,

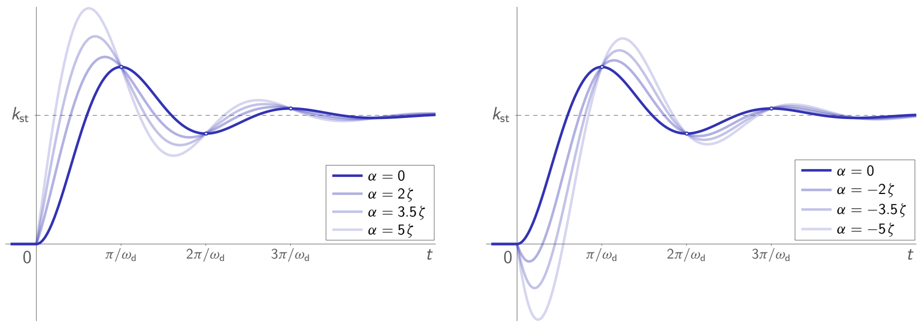


As a matter of fact,

$$\frac{\alpha}{\omega_n}\dot{y}_0(t) = \frac{\alpha}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\omega_n t)$$

(and sin $\to$ sinh if $\zeta > 1$).

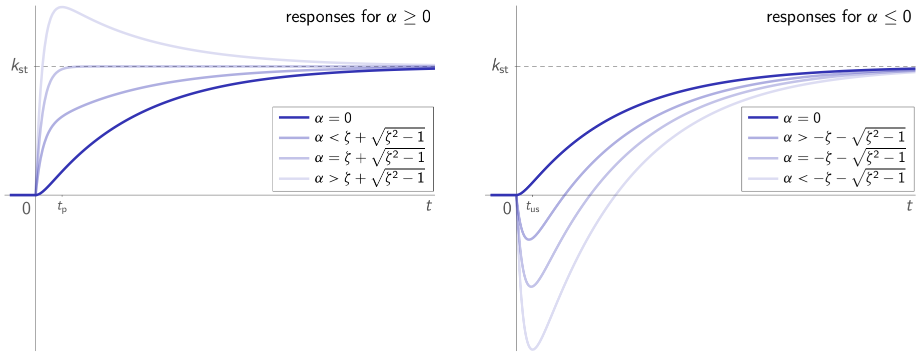
Effect of additional zero on underdamped systems



As $|\alpha|$ grows,

- the overshoot OS increases
- the undershoot US increases, if $\alpha < 0$
- the raise time t_r decreases
- the settling time t_s increases

Effect of additional zero on overdamped systems



As $|\alpha|$ grows,

- the overshoot OS increases, provided $\alpha > \zeta + \sqrt{\zeta^2-1}$
- the undershoot US increases, provided $\alpha < 0$
- the raise time t_r decreases

Modal analysis: beyond 1st and 2nd order dynamics

As we just saw,

- adding more poles and / or zeros may render modal analysis void.

For example,

- we may have (large) overshoot for systems with only real poles / zeros
- we may have no overshoot for systems with lightly damped poles

In some cases, however, we may extend modal insight of low-order systems to higher-order systems. This is possible if

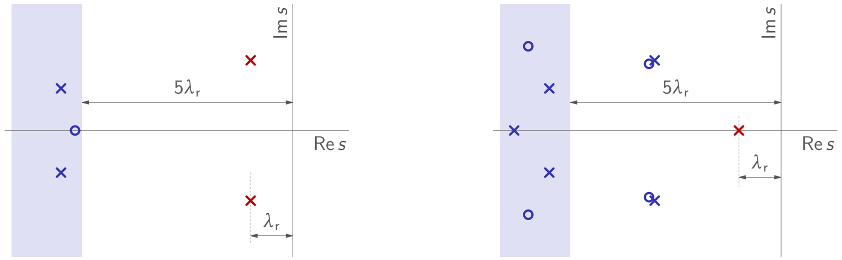
- dominant dynamics of a system is low order.

Dominant poles

A group of poles / zeros is said to be dominant if either of below holds:

- all other poles / zeros are at least 5 times further away from the $j\omega$-axis
- the closer poles / zeros "almost cancel" each other

e.g.[2]



Non-dominant poles and zeros may be safe to neglect in the modal analysis (still, required caution).

[2] Hereafter we denote a pole by "×" and a zero by "○" on pole-zero maps.

Dominant poles: Example 1

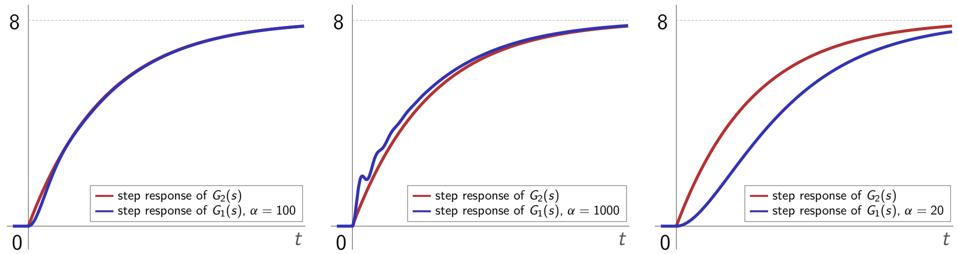
For any $\alpha > 0$, define

$$G_1(s) = \frac{\alpha(s+8)}{(s+1)(s^2+12s+\alpha)} = \frac{8}{s+1} \times \frac{s/8+1}{s^2/\alpha + 12s/\alpha + 1}$$

(poles at $s \in \{-1, -6 \pm \sqrt{36-\alpha}\}$ and zero at $s = -8$) and

$$G_2(s) = \frac{8}{s+1}.$$

Then:



Dominant poles: Example 2

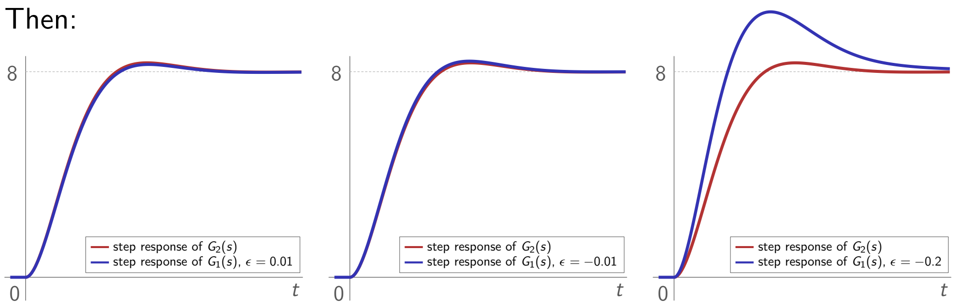
For any $|\epsilon| < 1$, define

$$G_1(s) = \frac{64(s/(1+\epsilon)+1)^2}{(s^2+4s+8)(s+1)^2} = \frac{8\times 8}{s^2+4s+8} \times \left(\frac{\frac{1}{1+\epsilon}s+1}{s+1}\right)^2$$

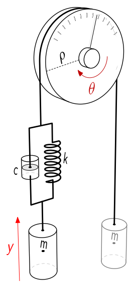
(poles at $s \in \{-2 \pm j2, -1, -1\}$ and zeros at $s \in \{-1-\epsilon, -1-\epsilon\}$) and

$$G_2(s) = \frac{8\times 8}{s^2+4s+8}.$$

Then:



Example 1 (contd)



$\rho = 1.25$ m

$k = 40000\,\frac{\text{N sec}}{\text{m}}$

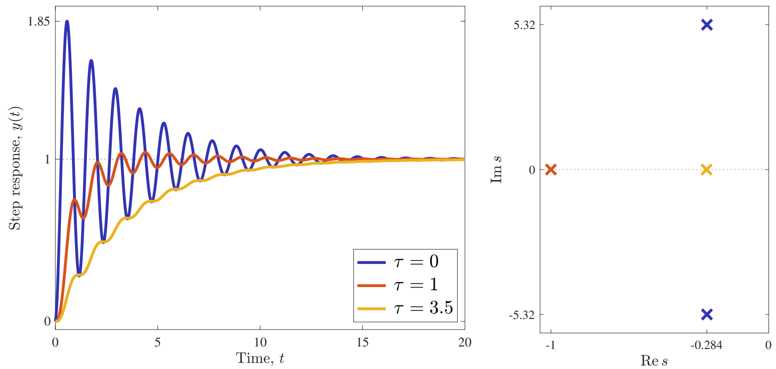
$c = 800\,\frac{\text{N}}{\text{m}}$

$m = 1410$ kg

For every $\tau > 0$ the controller in

$$y \leftarrow \boxed{\frac{(cs+k)\rho}{ms^2+cs+k}} \xleftarrow{u} \boxed{\frac{1}{\rho}\,\frac{1}{\tau s+1}} \leftarrow r$$

inverts $P(0) = \rho$ and renders $e_{ss} = 0$:



(the zero at $s = -50$ has virtually no effect on transients).

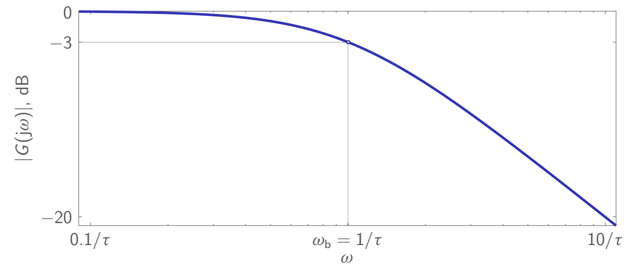
Outline

Steady-state performance

Transient performance: modal perspective

Transient performance: frequency-response perspective (from LS)

Magnitude frequency response of 1-order systems



Bandwidth ω_b

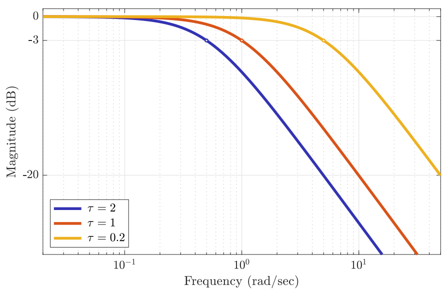
- increases as τ decreases (and the step response becomes faster)

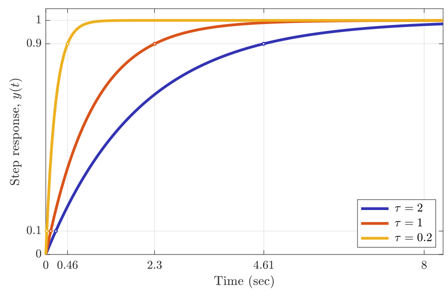
1-order systems: bandwidth vs. raise time

If

$$G(s) = \frac{1}{\tau s + 1}$$

then with $\tau \in \{0.2, 1, 2\}$,

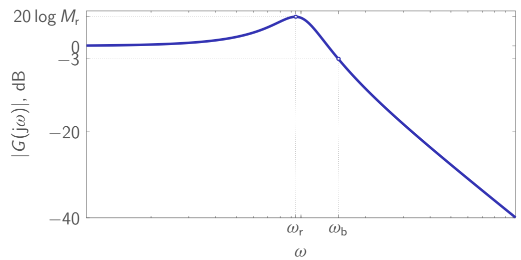




showing that

- wider $\omega_b \implies$ shorter t_r (faster transients)

Magnitude frequency response of 2-order systems



$$\omega_b = \sqrt{1 - 2\zeta^2 + \sqrt{1 + (1 - 2\zeta^2)^2}}\,\omega_n$$

$\omega_b > \omega_n$ iff $\zeta < 1/\sqrt{2}$

$$\omega_r = \sqrt{1 - 2\zeta^2}\,\omega_n \quad \text{(defined iff } \zeta < 1/\sqrt{2})$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad (M_r > 1 \text{ iff } \zeta < 1/\sqrt{2})$$

Bandwidth ω_b

- increases as ω_n increases (and the step response becomes faster)
- increases, a bit, as ζ decreases (and the step response becomes faster)

Resonant peak M_r

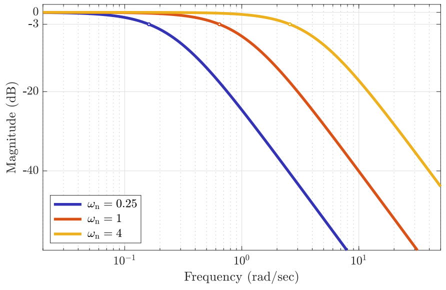
- increases as ζ decreases (and the step response becomes more shaky)

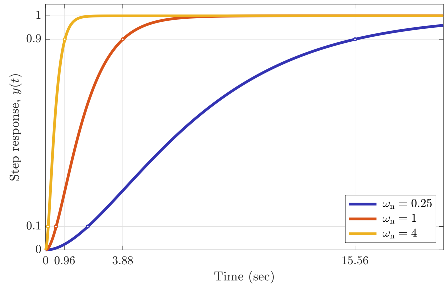
2-order systems: bandwidth vs. raise time

If

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

then with $\zeta = 1$ and $\omega_n \in \{0.25, 1, 4\}$,





showing that

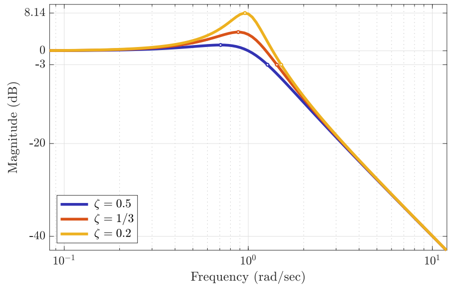
- wider $\omega_b \implies$ shorter t_r (faster transients)

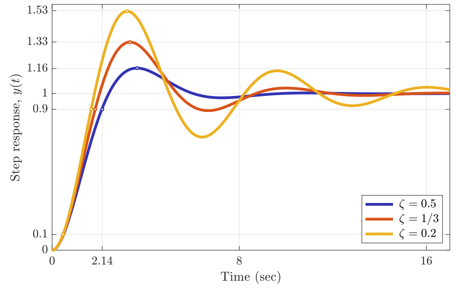
2-order systems: resonance vs. overshoot

If

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

then with $\zeta \in \{0.5, 1/3, 0.2\}$ and $\omega_n = 1$,





showing that

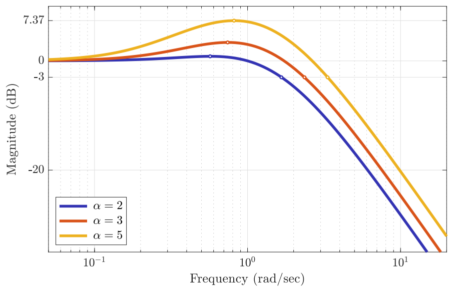
- larger $M_r \implies$ larger OS
- wider $\omega_b \implies$ shorter t_r (faster transients)

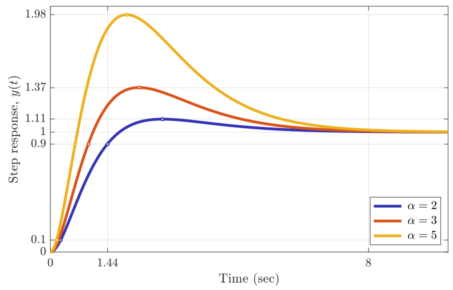
3-order systems with zeros

If

$$G(s) = \frac{\alpha\omega_n s + \omega_n^2}{(s/2+1)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

then with $\zeta = 1$, $\omega_n = 1$, and $\alpha \in \{2, 3, 5\}$,





showing that

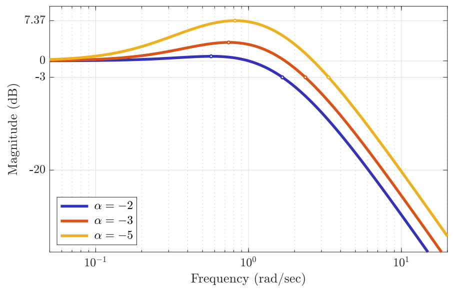
- larger $M_r \implies$ larger OS
- wider $\omega_b \implies$ shorter t_r (faster transients)

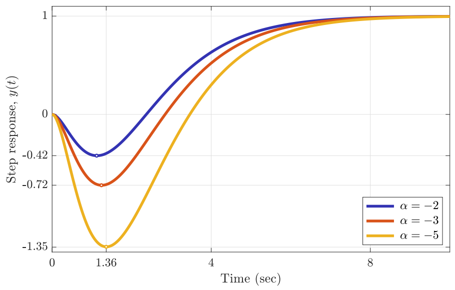
3-order systems with zeros (contd)

If

$$G(s) = \frac{\alpha\omega_n s + \omega_n^2}{(s/2+1)(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

then with $\zeta = 1$, $\omega_n = 1$, and $\alpha \in \{-2, -3, -5\}$,





showing that

- larger $M_r \implies$ larger US
- wider $\omega_b \implies$ faster leap (transients)

Rules of thumb

In general, we may *expect* that

- the higher M_r is, the larger the OS / US might be
  typically,
  - narrow peaks indicate oscillatory responses, with oscillation frequencies close to frequencies of peak on the Bode magnitude plot
  - wide peaks indicate overshoot / undershoot without oscillations
- the larger ω_b is, the faster time response is
  think of the Fourier transform frequency scaling property[3], $\mathfrak{F}\{\mathbb{P}_\varsigma y\} = \frac{1}{\varsigma}\,\mathbb{P}_{1/\varsigma}(\mathfrak{F}\{y\})$

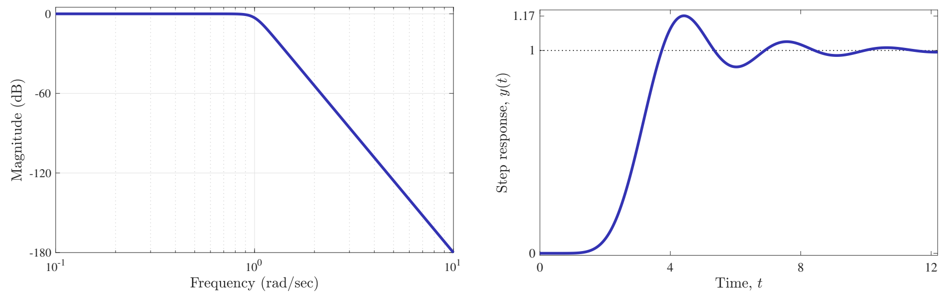
[3] The time scale operator $\mathbb{P}_\varsigma$ acts as $(\mathbb{P}_\varsigma x)(t) = x(\varsigma t)$ for every $\varsigma \in \mathbb{R}$.

Rules of thumb (contd)

Relation should be taken with a grain of salt. For example, consider the 9-order low-pass Butterworth filter with the transfer function

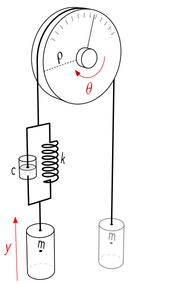
$$\frac{1}{(s+1)(s^2+0.347s+1)(s^2+s+1)(s^2+1.532s+1)(s^2+1.879s+1)},$$

whose frequency response has no resonant peaks. . .



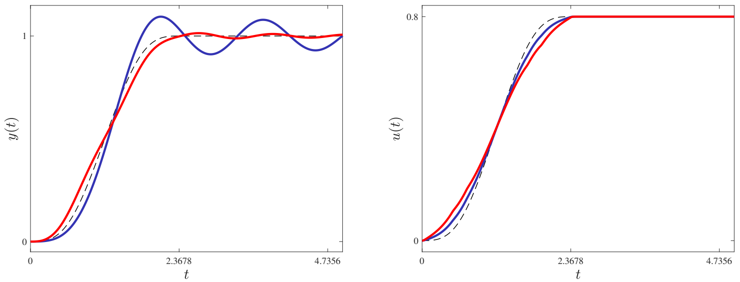
. . . yet whose step response exhibits an overshoot of 17%

Modeling uncertainty & plant inversion: example



$\rho = 1.25\,\mathrm{m}$
$k = 40000\,\frac{\mathrm{N\,sec}}{\mathrm{m}}$
$c = 800\,\frac{\mathrm{N}}{\mathrm{m}}$
$m = 1410\,\mathrm{kg}$
$m = 2820\,\mathrm{kg}$

If the actual mass mismatches that assumed in the design of u:



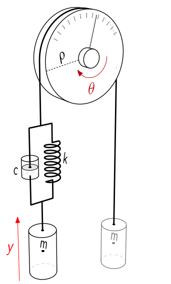
- u calculated for $m = 1410$ applied to $m = 2820$
- u calculated for $m = 2820$ applied to $m = 1410$

Curiously,

- "blue" oscillations are substantially larger than "red" ones.

Why?

Modeling uncertainty & plant inversion: example (contd)

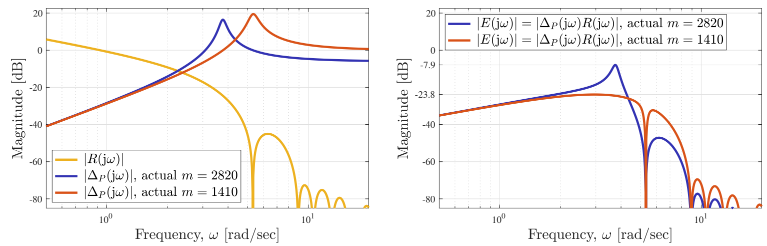


$\rho = 1.25\,\mathrm{m}$
$k = 40000\,\frac{\mathrm{N\,sec}}{\mathrm{m}}$
$c = 800\,\frac{\mathrm{N}}{\mathrm{m}}$
$m = 1410\,\mathrm{kg}$
$m = 2820\,\mathrm{kg}$

The error due to modeling uncertainty is

$$e = (1 - P_{\mathrm{true}}P^{-1})r := \Delta_P r$$

Inspecting frequency responses of Δ_P and the spectrum of r,

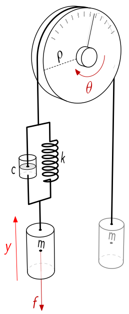


reveals that

- $R(j\omega)$ vanishes at the resonance of Δ_P at $\omega = 5.31$

Therefore, this resonance isn't excited by r (incidentally). Yet the resonance of Δ_P at $\omega = 3.76$ isn't canceled.

Disturbances & plant inversion: example



$\rho = 1.25\,\mathrm{m}$

$k = 40000\,\frac{\mathrm{N\,sec}}{\mathrm{m}}$

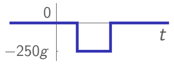
$c = 800\,\frac{\mathrm{N}}{\mathrm{m}}$

$m = 1410\,\mathrm{kg}$

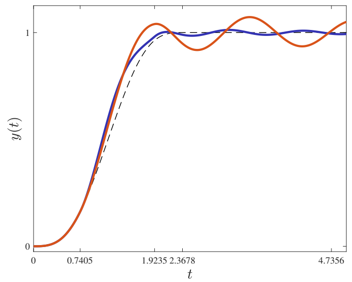
$m = 2820\,\mathrm{kg}$

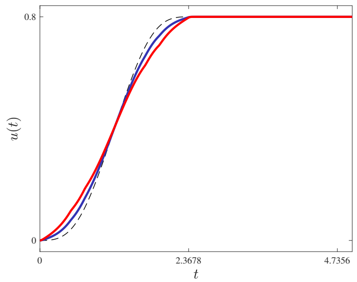
Let

$$f = -250g(\mathbb{S}_{-0.74}\mathbb{1} - \mathbb{S}_{-1.92}\mathbb{1}) =$$



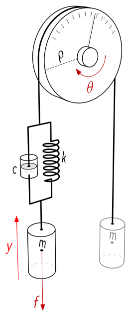
(think of a jump of somebody heavy, like , in an elevator). Responses:





Now "red" oscillations are substantially larger than "blue" ones. Why?

Disturbances & plant inversion: example (contd)



$\rho = 1.25\,\mathrm{m}$

$k = 40000\,\frac{\mathrm{N\,sec}}{\mathrm{m}}$

$c = 800\,\frac{\mathrm{N}}{\mathrm{m}}$

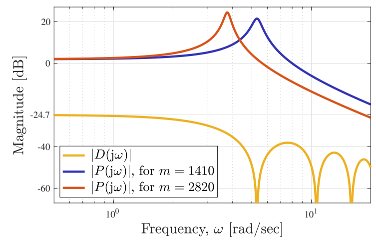
$m = 1410\,\mathrm{kg}$

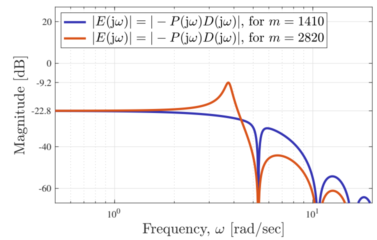
$m = 2820\,\mathrm{kg}$

The error due to disturbances is

$$e = -Pd.$$

Our $D(s) = -\frac{1}{(cs+k)\rho}F(s)$ (independent of the mass). From





we can see that $D(\mathrm{j}\omega)$ vanishes at $\omega = 5.31$ (not incidentally), which is exactly the resonance of P. Hence, this resonance is not excited in P. The resonance of P at $\omega = 3.76$ is exited.