Introduction to Control (00340040)
lecture no. 2

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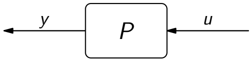
Abstract control problem to begin with

Setup:



where

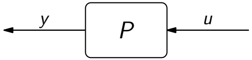
- P is a plant
  may comprise actual controlled process, actuators, sensors, et cetera
- u is a control signal (control input)
- y is a controlled (regulated) signal (output)

Problem: Given P, find u resulting in a desired y.

Outline

Outline

What is it about?



Model is a

- description of systems using an abstract (e.g. mathematical) language.

Modeling lets us handle problems of various nature, e.g.

- mechanical,
- electrical,
- biological,
- social,
- ...

in a unified manner. It must be realized though that

- models of real-world phenomena are never perfect,

they are just (more or less accurate) approximations of real processes.

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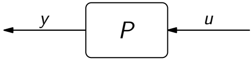

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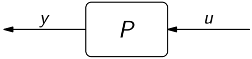
Why to model?



From control viewpoint, modeling is necessary just because

- if we do not know how the plant responds to our actions, then control tasks are hopeless.

In other words,

- model-free control is essentially a coin tossing.

Car example 1 (Myers, 1999)

Car example 2 (Butler, 2008)

How to model?

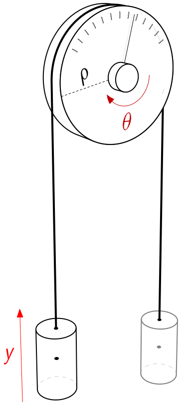
Essentially, three ways:

1. from first principles *ab initio*
2. phenomenological e.g. predator–prey, SIR
3. from observing experimental I/O relations *identification*

Outline

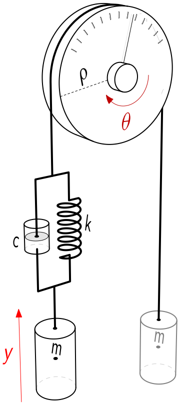
Case-studies

System Σ_1:



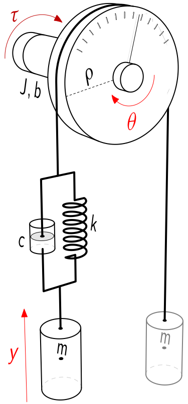
- control mass position y
- via pulley angle θ

System Σ_2:



- control mass position y
- via pulley angle θ

System Σ_3:



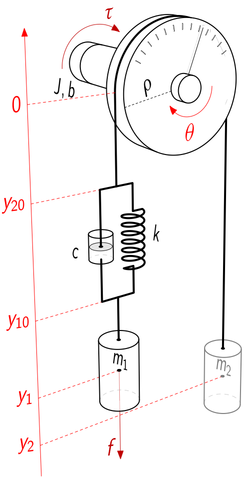
- control mass position y
- via torque τ

◦ elevator as motivation

◦ elevator with long hoistway as motivation

Data

Consider



where

ρ: pulley radius

m_1, m_2: masses

k: spring constant

c: damping coefficient

J: moment of inertia of pulley + shaft

b: friction coefficient of pulley + shaft

f: external force applied to m_1 (2nd input)

Denote also

f_{t1}, f_{t2}: tension forces at m_1 and m_2, respectively

$L_0 := y_{20} - y_2$ at $\theta = 0$

$L_1 := y_{10} - y_1$

$L_2 := \pi\rho - y_{20} - y_2$

$L := L_1 + L_2$ length of non-elastic part

Motion equations

Masses dynamics:

$$-m_1\ddot{y}_1(t) = m_1 g + f(t) - f_{\mathrm{t1}}(t) \quad \text{and} \quad -m_2\ddot{y}_2(t) = m_2 g - f_{\mathrm{t2}}(t)$$

Spring-damper dynamics:

$$c(\dot{y}_{20}(t) - \dot{y}_{10}(t)) + k(y_{20}(t) - y_{10}(t)) = f_{\mathrm{t1}}(t)$$

Pulley dynamics:

$$J\ddot{\theta}(t) + b\dot{\theta}(t) = \tau(t) - \rho f_{\mathrm{t1}}(t) + \rho f_{\mathrm{t2}}(t)$$

Algebraic constraints:

$$\begin{aligned} y_{10}(t) - y_1(t) &= L_1, \\ -y_{20}(t) - y_2(t) &= L_2 - \pi\rho, \\ y_{20}(t) - y_2(t) &= L_0 + 2\rho\theta(t) \end{aligned}$$

Elimination of variables

From algebraic constraints,

$$y_{10} = y_1 + L_1, \quad y_{20} = \rho\theta + \frac{L_0 - L_2 + \pi\rho}{2}, \quad y_2 = -\rho\theta - \frac{L_0 + L_2 - \pi\rho}{2}$$

Then, spring-damper verifies

$$c(\rho\dot{\theta} - \dot{y}_1) + k(\rho\theta - y_1) + k\left(\frac{L_0 - L_2 + \pi\rho}{2} - L_1\right) = f_{t1}$$

and the tension forces are

$$f_{t1} = m_1\ddot{y}_1 + m_1 g + f,$$
$$f_{t2} = m_2\ddot{y}_2 + m_2 g = -\rho m_2\ddot{\theta} + m_2 g$$

System equations

Thus, we have:

$$\begin{cases} m_1\ddot{y}_1 + c\dot{y}_1 + ky_1 + k\left(\dfrac{m_1 g}{k} + \dfrac{L + L_1 - L_0 - \pi\rho}{2}\right) = \rho(c\dot{\theta} + k\theta) - f \\ (J + \rho^2 m_2)\ddot{\theta} + b\dot{\theta} = \tau - \rho m_1\ddot{y}_1 - \rho(m_1 - m_2)g - \rho f \end{cases}$$

This is a nonlinear set of equations (superposition principle doesn't hold).

System equations

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This is a nonlinear set of equations (superposition principle doesn't hold[1]).

[1] Just think of the simpler system $y = u + 1$. Since $(u_1 + u_2) + 1 \neq (u_1 + 1) + (u_2 + 1)$, it's nonlinear. Such systems are called *affine* and can be linearized precisely via introducing deviation variables. For example, define $\tilde{u} = u + 1$, which yields linear system $y = \tilde{u}$.

Equilibrium

In equilibrium, the system satisfies algebraic equations

$$\begin{cases} y_1 = \rho\theta - \dfrac{L + L_1 - L_0 - \pi\rho}{2} - \dfrac{f + m_1 g}{k} \\ \tau = \rho(m_1 - m_2)g + \rho f \end{cases}$$

If we *assume* that $y_1 = y_2$ at $\theta = 0$ and $f = 0$, an additional constraint

$$-\frac{L + L_1 - L_0 - \pi\rho}{2} - \frac{m_1 g}{k} = -\frac{L_0 + L_2 - \pi\rho}{2}$$

yields $L_0 = L_1 + m_1 g/k$, so that

$$\begin{cases} y_1 = \rho\theta - \dfrac{L - \pi\rho + m_1 g/k}{2} - \dfrac{f}{k} \\ \tau = \rho(m_1 - m_2)g + \rho f \end{cases}$$

We then choose the equilibrium corresponding to $\theta = 0$ and $f = 0$.

Linearization

Defining deviation variables

$$\tilde{y}_1(t) := y_1(t) + \frac{L - \pi\rho + m_1 g/k}{2} \quad \text{and} \quad \tilde{\tau}(t) := \tau(t) - \rho(m_1 - m_2)g$$

we end up with the following linear model:

$$\begin{cases} m_1\ddot{\tilde{y}}_1(t) + c\dot{\tilde{y}}_1(t) + k\tilde{y}_1(t) = \rho(c\dot{\theta}(t) + k\theta(t)) - f(t) \\ (J + \rho^2 m_2)\ddot{\theta}(t) + b\dot{\theta}(t) = \tilde{\tau}(t) - \rho m_1\ddot{\tilde{y}}_1(t) - \rho f(t) \end{cases}$$

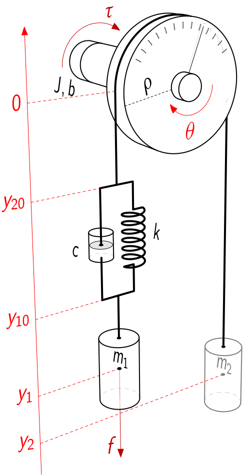
or

$$\begin{cases} \tilde{Y}_1(s) = \dfrac{1}{m_1 s^2 + cs + k}\left((cs + k)\rho\Theta(s) - F(s)\right) \\ \Theta(s) = \dfrac{1}{(J + \rho^2 m_2)s^2 + bs}\left(\tilde{T}(s) - \rho m_1 s^2 \tilde{Y}_1(s) - \rho F(s)\right) \end{cases}$$

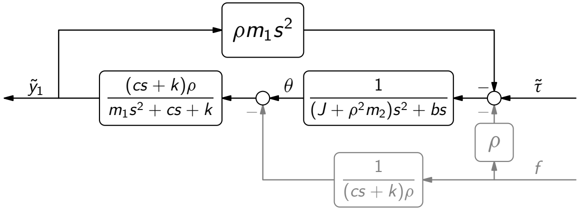
in the Laplace domain.

Block-diagram

Thus,



can be modeled as



in terms of deviations

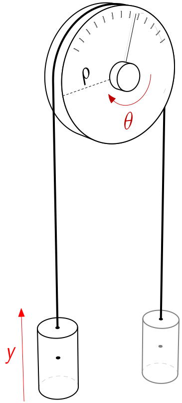
$$\tilde{y}_1(t) = y_1(t) + \frac{L - \pi\rho + m_1 g/k}{2},$$

$$\tilde{\tau}(t) = \tau(t) - \rho(m_1 - m_2)g$$

from the equilibrium corresponding to $y_1 = y_2$ at $\theta = 0$ and $f = 0$.

System Σ_1 (Atwood machine)

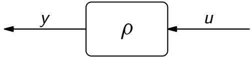
Let



Assuming inelastic string, no slippage, and $y = 0$ at $\theta = 0$,

$$y(t) = \rho\theta(t) \qquad \text{(with } \theta \text{ in rad)},$$

so the system can be presented by the following block diagram:

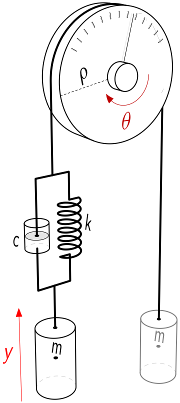


and the plant transfer function

$$P(s) = \rho$$

System Σ_2

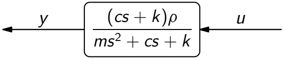
Let



Now, the system motion satisfies

$$m\ddot{y}(t) + c(\dot{y}(t) - \rho\dot{\theta}(t)) + k(y(t) - \rho\theta(t)) = 0$$

(assuming no slippage and $y = 0$ at $\theta = 0$), which leads to



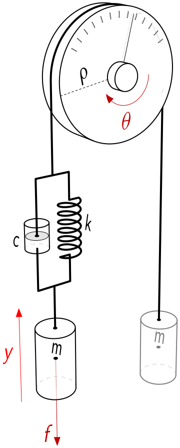
and the plant transfer function

$$P(s) = \frac{(cs + k)\rho}{ms^2 + cs + k}$$

(note that $\lim_{k\to\infty} P(s) = \rho$).

System Σ_2 with disturbance

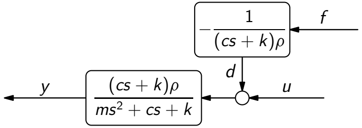
Let



Now, the system motion satisfies

$$m\ddot{y}(t) + c(\dot{y}(t) - \rho\dot{\theta}(t)) + k(y(t) - \rho\theta(t)) = -f(t)$$

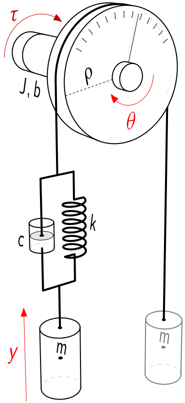
(assuming no slippage and $y = 0$ at $\theta = 0$ and $f = 0$), which leads to



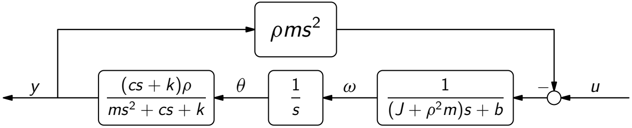
where signal d called input disturbance.

System Σ_3

Let



In this case, assuming no slippage and $y = 0$ at $\theta = 0$ we have the following block diagram:



and the plant transfer function

$$P(s) = \frac{(cs+k)\rho}{s((J+\rho^2 m)s+b)(ms^2+cs+k)+\rho^2 ms^2(cs+k)}$$

(it is unstable because of the pole at the origin).

Outline

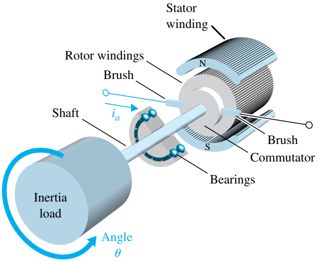
Modeling

Case-study systems and their first-principles modeling

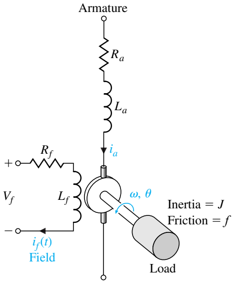
Actuation: DC motors and their first-principles modeling

Parameter identification of DC motor

DC motors



sketch



wiring diagram

Electric motors are devices converting

- electrical energy into mechanical energy.

DC motors run on DC electric power. There are many types of DC motors, we study armature-controlled brushed DC motors.

DC motors (contd)

Advantages:

- high torque
- position / speed / torque controllability over a wide range
- portability
- well-behaved speed-torque characteristics
- ...

Applications (actuators):

- robotic manipulators
- tape transport mechanisms
- disk drivers
- ...

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Modeling voltage-controlled DC motors

Important things:

1. Torque τ_{m} generated by the motor proportional to armature current i_{a}:

$$\tau_{\mathrm{m}}(t) = K_{\mathrm{m}} i_{\mathrm{a}}(t) \qquad \text{or} \qquad T_{\mathrm{m}}(s) = K_{\mathrm{m}} I_{\mathrm{a}}(s),$$

where $K_{\mathrm{m}}\left[\frac{\mathrm{Nm}}{\mathrm{A}}\right]$ is the motor constant (torque constant).

2. Armature current satisfies

$$L_{\mathrm{a}} \dot{i}_{\mathrm{a}}(t) + R_{\mathrm{a}} i_{\mathrm{a}}(t) = v_{\mathrm{a}}(t) - v_{\mathrm{b}}(t) \quad \text{or} \quad I_{\mathrm{a}}(s) = \frac{V_{\mathrm{a}}(s) - V_{\mathrm{b}}(s)}{L_{\mathrm{a}} s + R_{\mathrm{a}}},$$

where v_{a} is the applied input voltage and v_{b} is the back electromotive force (back emf) voltage proportional to motor angular velocity ω_m:

$$v_{\mathrm{b}}(t) = K_{\mathrm{b}} \omega_{\mathrm{m}}(t) \qquad \text{or} \qquad V_{\mathrm{b}}(s) = K_{\mathrm{b}} \Omega_{\mathrm{m}}(s),$$

where $K_{\mathrm{b}}\left[\frac{\mathrm{V\,sec}}{\mathrm{rad}}\right]$ is the motor back emf constant $\left(9.55 \frac{\mathrm{V\,sec}}{\mathrm{rad}} \approx 1 \frac{\mathrm{V}}{\mathrm{rpm}}\right)$.

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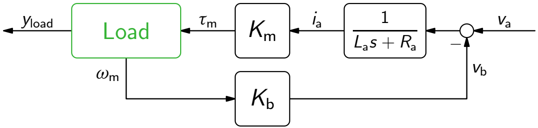
[2] Normally, $K_b = K_m$ if measured in compatible units.

Modeling voltage-controlled DC motors (contd)

Resulting system can be presented as the following block diagram:



Here y_{load} is a (controlled) load output, not necessarily coinciding with the motor shaft angular velocity ω_{m}.

The dependence of ω_{m} on the load and internal feedback loop (back emf) renders voltage-controlled motors

strongly dependent on load dynamics.

We shall explicitly have ω_{m} as an output of the load model to incorporate the load into the motor model.

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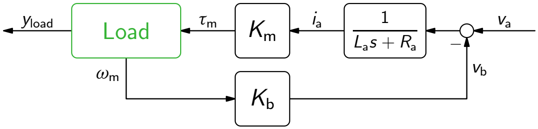

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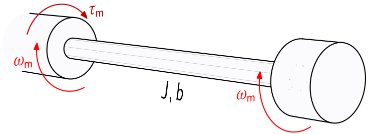
– strongly dependent on load dynamics.

We shall explicitly have ω_{m} as an output of the load model to incorporate the load into the motor model.

[3] Models substantially simplified in the *current-controlled* case (no back emf loop).

Example 1: rigid mechanical load



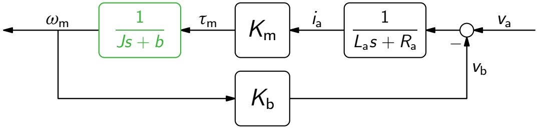
Consider a rigid load (e.g. the rotor itself) with $y_{\text{load}} = \omega_{\text{m}}$ and satisfying

$$J\dot{\omega}_{\text{m}}(t) + b\omega_{\text{m}}(t) = \tau_{\text{m}}(t) \quad \text{or} \quad \Omega_{\text{m}}(s) = \frac{1}{Js + b} T_{\text{m}}(s),$$

where J is its moment of inertia and b is the friction coefficient.

Example 1: rigid mechanical load (contd)

In this case we have:



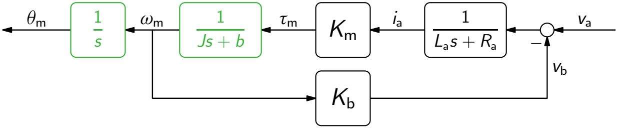
which results in the following transfer function from v_a to ω_m:

$$P_\omega(s) = \frac{K_m}{(L_a s + R_a)(Js + b) + K_b K_m},$$

which is always stable (2nd order denominator with positive coefficients).

Example 1: rigid mechanical load (contd)

If $y_{\text{load}} = \theta_{\text{m}}$ (motor shaft angle), then system becomes



with the transfer function

$$P_\theta(s) = \frac{K_{\text{m}}}{s\big((L_{\text{a}}s + R_{\text{a}})(Js + b) + K_{\text{b}}K_{\text{m}}\big)} = \frac{1}{s}P_\omega(s),$$

which is unstable (pole at the origin). If armature (electrical) time constant is significantly smaller than mechanical time constant, i.e. if $\frac{L_{\text{a}}}{R_{\text{a}}} \ll \frac{J}{b}$, then

$$P_\omega(s) \approx \frac{K_{\text{m}}}{R_{\text{a}}(Js + b) + K_{\text{b}}K_{\text{m}}} \quad \text{and} \quad P_\theta(s) \approx \frac{K_{\text{m}}}{s(R_{\text{a}}(Js + b) + K_{\text{b}}K_{\text{m}})}$$

(where L_{a} neglected) are sufficiently accurate.

Example 1: rigid mechanical load (contd)

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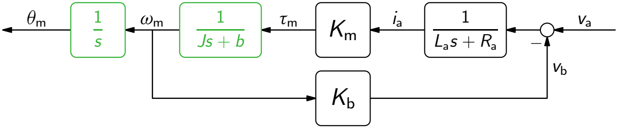

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(where L_{a} neglected) are sufficiently accurate.

Example 1: electrical time constant

Consider a motor (in fact, MINIMOTOR 2342) with

$K_{\mathrm{m}}\,[\frac{\mathrm{N\,m}}{\mathrm{A}}]$	$K_{\mathrm{b}}\,[\frac{\mathrm{V\,sec}}{\mathrm{rad}}]$	$J\,[\mathrm{kg\,m^2}]$	$b\,[\frac{\mathrm{kg\,m^2}}{\mathrm{sec}}]$	$R_{\mathrm{a}}\,[\Omega]$	$L_{\mathrm{a}}\,[\mathrm{H}]$
0.0261	0.0261	$5.8 \cdot 10^{-7}$	$9.67 \cdot 10^{-5}$	7.1	$2.65 \cdot 10^{-4}$

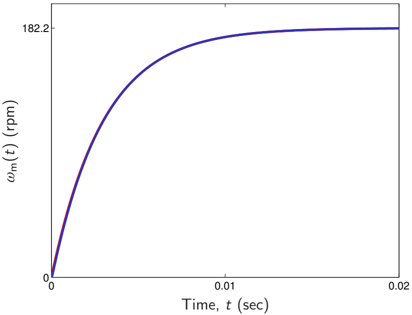
with mechanical and electrical time constants of $6 \cdot 10^{-3}$ and $3.73 \cdot 10^{-5}$, respectively. This results in

$$P_\omega(s) = \frac{19.082}{(0.003s + 1)(3.756 \cdot 10^{-5}s + 1)}$$

or, if we neglect L_{a},

$$P_\omega(s) \approx \frac{19.082}{0.003s + 1}.$$

Example 1: electrical time constant (contd)

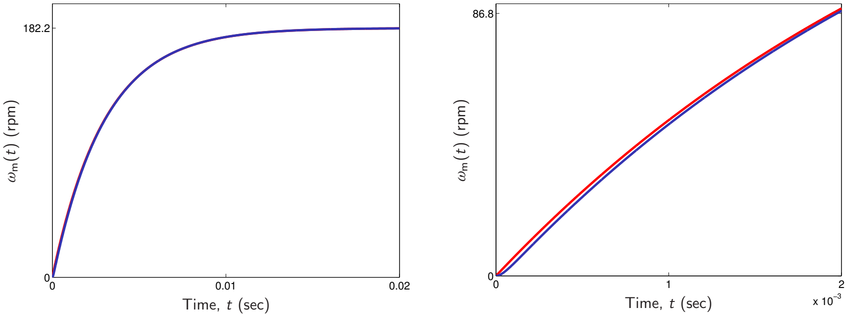


Step responses of the second- and first-order systems

— almost indistinguishable,

which justifies neglecting the dynamics of the armature circuit in this case.

Example 1: electrical time constant (contd)



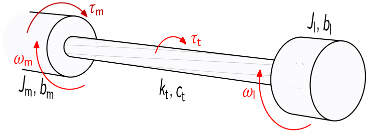
Step responses of the second- and first-order systems

– almost indistinguishable[4],

which justifies neglecting the dynamics of the armature circuit in this case.

[4] Except for a small difference at the start, see the close-up on the right.

Example 2: load with flexible transmission



Consider now a load (J_l and b_l) connected to the motor shaft (J_m and b_m) by a flexible inertialess transmission with the dynamics

$$\tau_t(t) = k_t\theta_\delta(t) + c_t\,\omega_\delta(t) \quad \text{or} \quad T_t(s) = \frac{c_t\,s + k_t}{s}\,\Omega_\delta(s),$$

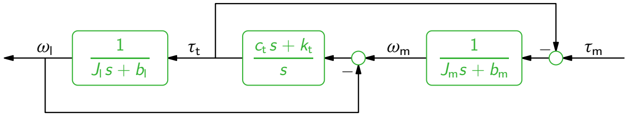
where $\theta_\delta := \theta_m - \theta_l$, $\omega_\delta := \dot{\theta}_\delta = \omega_m - \omega_l$ and k_t and c_t are the stiffness and damping coefficient, respectively, of the transmission. The other equations:

$$J_m\dot{\omega}_m(t) + b_m\omega_m(t) = \tau_m(t) - \tau_t(t) \quad \text{or} \quad \Omega_m(s) = \frac{T_m(s) - T_t(s)}{J_m s + b_m},$$

$$J_l\dot{\omega}_l(t) + b_l\omega_l(t) = \tau_t(t) \quad \text{or} \quad \Omega_l(s) = \frac{1}{J_l\,s + b_l}\,T_t(s).$$

Example 2: load with flexible transmission (contd)

This system corresponds to the following block-diagram:

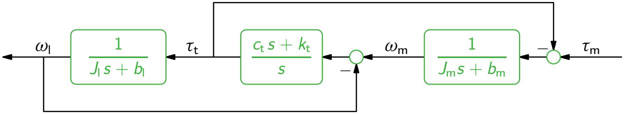


Combining load and motor block diagrams, we end up with

The transfer function of the system can be derived by routine block-diagram manipulations, as shown in Lecture 1.

Example 2: load with flexible transmission (contd)

This system corresponds to the following block-diagram:



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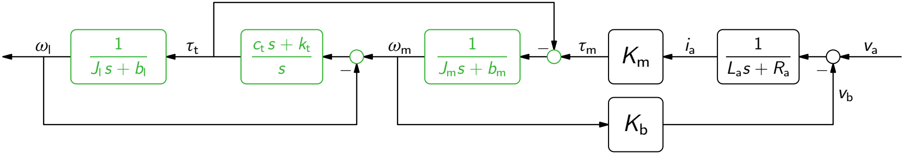
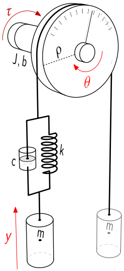
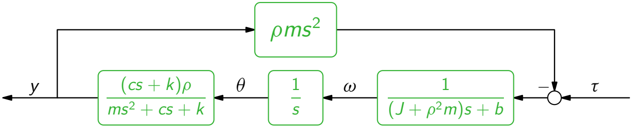


The transfer function of the system can be derived by routine block-diagram manipulations, as shown in Lecture 1.

System Σ_3 as load



Remember, this system described as

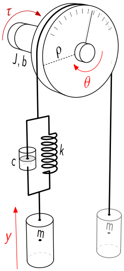


If T is generated by a DC motor, we end up with the plant

having the following t.f. (assuming $L_a = 0$ and denoting $\varkappa := K_m K_b / R_a$):

$$P(s) = \frac{(cs+k)\rho K_m/R_a}{m(J+m\rho^2)s^4+(\varkappa m+cJ+bm+2cm\rho^2)s^3+(\varkappa c+cb+Jk+2km\rho^2)s^2+k(\varkappa+b)s}$$

System Σ_3 as load



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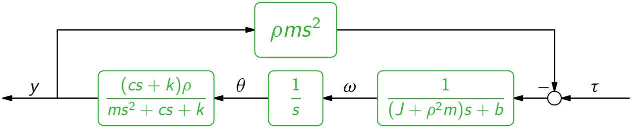

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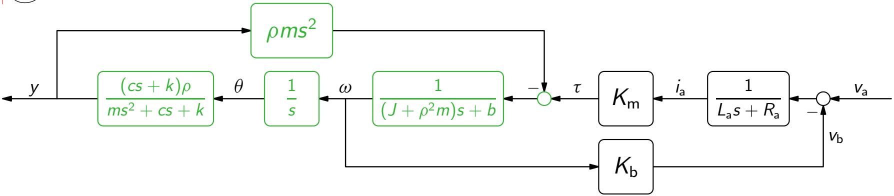

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Outline

Modeling

Case-study systems and their first-principles modeling

Actuation: DC motors and their first-principles modeling

Parameter identification of DC motor

The problem

We know that

$$P_\omega(s) \approx \frac{K_\mathrm{m}}{R_\mathrm{a}(Js+b)+K_\mathrm{b}K_\mathrm{m}}.$$

but we might not know the parameters,

- some of them (K_m, K_b, R_a) can be taken from the catalog
- the others (load's J and b) are harder to calculate

Alternative to the first-principles approach:

- determining parameters from experiments (system identification)

To that end, rewrite

$$\frac{K_\mathrm{m}}{R_\mathrm{a}(Js+b)+K_\mathrm{b}K_\mathrm{m}} = \frac{k_\mathrm{st}}{\tau s+1},$$

where

$$k_\mathrm{st} := \frac{K_\mathrm{m}}{K_\mathrm{b}K_\mathrm{m}+R_\mathrm{a}b} \quad \text{and} \quad \tau := \frac{R_\mathrm{a}J}{K_\mathrm{b}K_\mathrm{m}+R_\mathrm{a}b}.$$

The problem

We know that

$$P_\omega(s) \approx \frac{K_m}{R_a(Js + b) + K_b K_m}.$$

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Alternative to the first-principles approach:

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To that end, rewrite

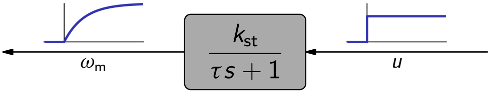
$$\frac{K_m}{R_a(Js + b) + K_b K_m} = \frac{k_{st}}{\tau s + 1},$$

where

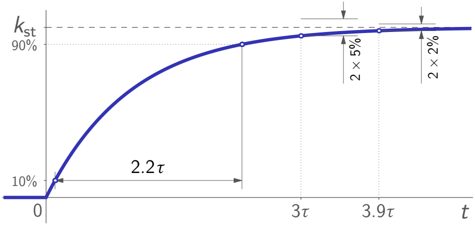
$$k_{st} := \frac{K_m}{K_b K_m + R_a b} \quad \text{and} \quad \tau := \frac{R_a J}{K_b K_m + R_a b}.$$

Experimental setup

We try to identify parameters k_{st} and τ from the step response

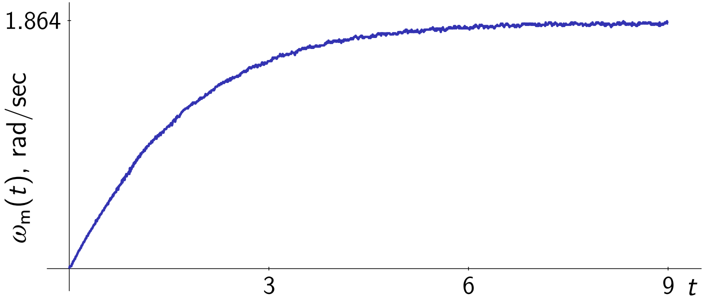


taking into account that it is relatively simple, viz. $y(t) = k_{st}(1 - e^{-t/\tau})$:



Experimental data

Reality (response to the step voltage of a magnitude of 1.2V):



is not exactly according to the theory. Reasons:

- measurement noise (sensor is an encoder, hence quantization)
- nonlinearities (e.g. mechanical friction)
- additional dynamics (inductance, eccentricity, et cetera)

Still, it closely resembles the step response of a 1-order system.

Fitting 1-order response to experimental data

Brute-force parametric search over possible values of k_{st} and τ to fit

$$\omega_m(t) = 1.2k_{st}(1 - e^{-t/\tau})$$

into experimental data yields

$$k_{st} = 1.5533 \quad \text{and} \quad \tau = 1.7$$

with a reasonably good fit: