Control Theory (00840738)

tutorial 0, review of some "Dynamic Systems" material

Question 1. Draw the asymptotic Bode magnitude plots of the transfer function

$$G(s) = \frac{k}{(\tau_1 s + 1)(\tau_2 s + 1)},$$

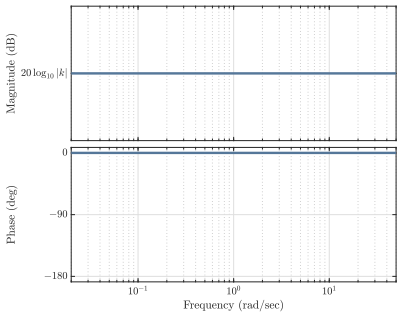
where $\tau_1 > 0$ and $\tau_2 > 0$.

Question 2. Draw the Bode and polar plots for the following transfer functions:

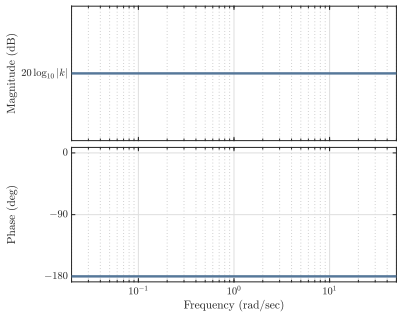
1. $G_1(s) = \dfrac{1}{(\tau s + 1)^2}$ for $\tau > 0$;
2. $G_2(s) = \dfrac{k}{s(\tau s + 1)}$ for $\tau > 0$ and $k > 0$;
3. $G_3(s) = \dfrac{\tau_2 s + 1}{\tau_1 s + 1}$ for $\tau_1 = \frac{1}{3}$ and $\tau_2 = \frac{5}{3}$ and then for $\tau_1 = \frac{5}{3}$ and $\tau_2 = \frac{1}{3}$.

A Bode diagrams of elementary systems

- static gain, $G(s) = k$:

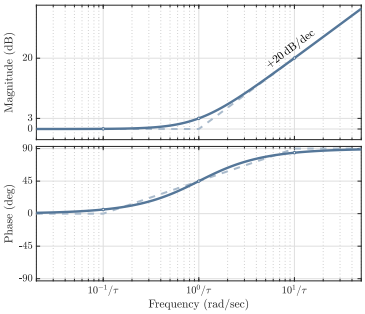


if $k > 0$

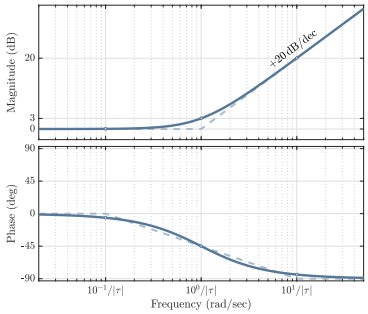


if $k < 0$

- 1-order factor, $G(s) = \tau s + 1$ for $\tau \neq 0$:

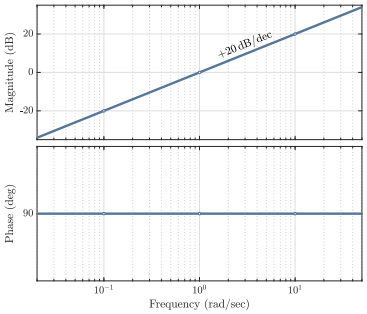


if $\tau > 0$

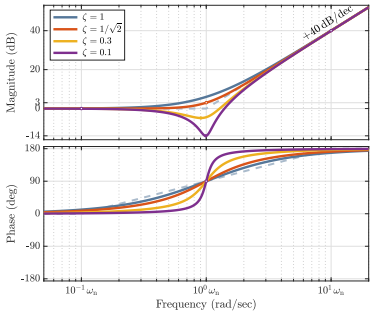


if $\tau < 0$

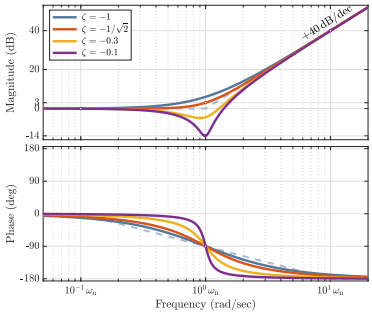
- 1-order differentiator, $G(s) = s$:



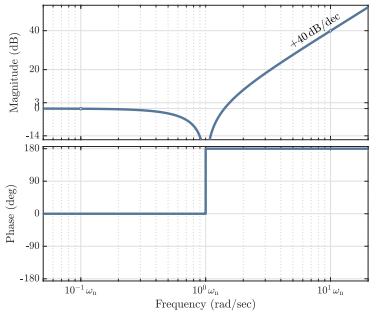
- 2-order factor, $G(s) = (s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1$ for $\omega_n > 0$:



if $\zeta > 0$

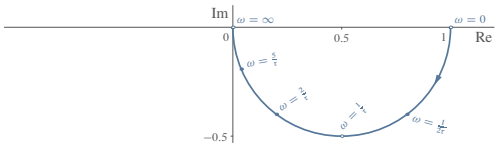


if $\zeta < 0$

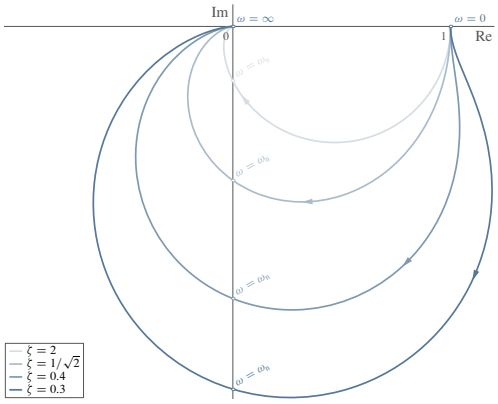


if $\zeta = 0$

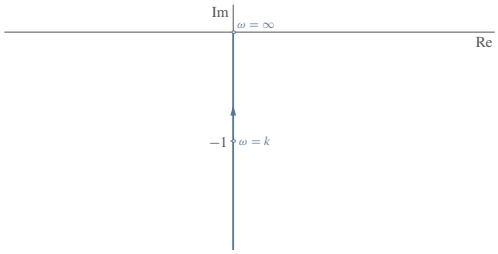
B Polar plots of elementary systems



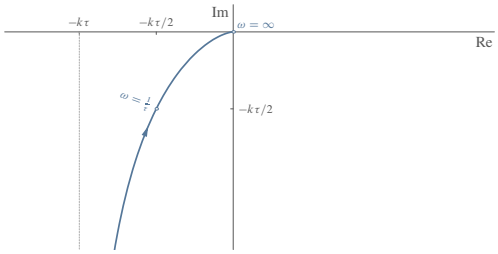
$$G(s) = \frac{1}{\tau s + 1}$$



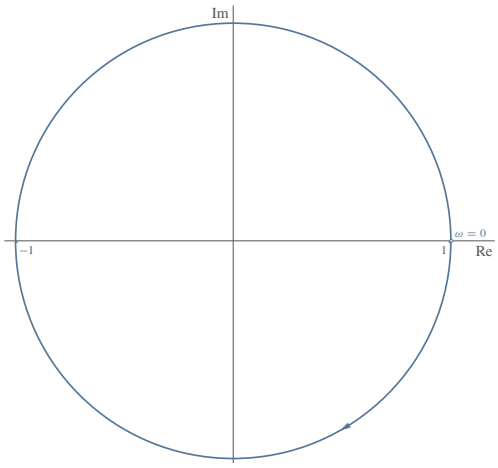
$$G(s) = \frac{1}{s^2/\omega_\mathrm{n}^2 + 2\zeta s/\omega_\mathrm{n} + 1}$$



$$G(s) = \frac{k}{s}, k > 0$$



$$G(s) = \frac{k}{s(\tau s + 1)}, k > 0$$



$$G(s) = \mathrm{e}^{-\tau s}$$ infinite number of encirclements

or

$$G(s) = \left(\frac{-\tau s + 1}{\tau s + 1}\right)^2$$ one encirclement

C Other useful facts learned at Dynamic Systems

Conditions on low-order polynomials to be Hurwitz:

- $a_1 s + a_0$ with $a_1 \neq 0$ is Hurwitz iff all $a_i \neq 0$ and have the same sign
- $a_2 s^2 + a_1 s + a_0$ with $a_2 \neq 0$ is Hurwitz iff all $a_i \neq 0$ and have the same sign
- $a_3 s^3 + a_2 s^2 + a_1 s + a_0$ with $a_3 \neq 0$ is Hurwitz iff all $a_i \neq 0$, have the same sign, and $a_1 a_2 > a_0 a_3$

Step responses of elementary systems $\mathbb{1} \mapsto y$:

- first-order: $\dfrac{k}{\tau s + 1}$ with $\tau > 0$ has

$$y(t) = k(1 - \mathrm{e}^{-t/\tau})\mathbb{1}(t)$$

- second-order overdamped: $\dfrac{k\omega_\mathrm{n}^2}{s^2 + 2\zeta\omega_\mathrm{n} s + \omega_\mathrm{n}^2}$ with $\omega_\mathrm{n} > 0$ and $\zeta > 1$ has

$$y(t) = k\left(1 - \frac{1}{2}\left(\frac{\zeta}{\sqrt{\zeta^2 - 1}} + 1\right)\mathrm{e}^{\lambda_1 t} + \frac{1}{2}\left(\frac{\zeta}{\sqrt{\zeta^2 - 1}} - 1\right)\mathrm{e}^{\lambda_2 t}\right)\mathbb{1}(t),$$

  where $\lambda_1 = -(\zeta - \sqrt{\zeta^2 - 1})\omega_\mathrm{n} < 0$ and $\lambda_2 = -(\zeta + \sqrt{\zeta^2 - 1})\omega_\mathrm{n} < 0$

- second-order critically damped: $\dfrac{k\omega_\mathrm{n}^2}{s^2 + 2\zeta\omega_\mathrm{n} s + \omega_\mathrm{n}^2}$ with $\omega_\mathrm{n} > 0$ and $\zeta = 1$ has

$$y(t) = k\big(1 - (1 + \omega_\mathrm{n} t)\mathrm{e}^{-\omega_\mathrm{n} t}\big)\mathbb{1}(t)$$

- second-order underdamped: $\dfrac{k\omega_\mathrm{n}^2}{s^2 + 2\zeta\omega_\mathrm{n} s + \omega_\mathrm{n}^2}$ with $\omega_\mathrm{n} > 0$ and $0 \leq \zeta < 1$ has

$$y(t) = k\left(1 - \frac{1}{\sqrt{1 - \zeta^2}}\mathrm{e}^{-\zeta\omega_\mathrm{n} t}\sin\big(\sqrt{1 - \zeta^2}\,\omega_\mathrm{n} t + \arccos\zeta\big)\right)\mathbb{1}(t)$$

Bilateral Laplace transforms of some signals:

$f(t)$	$\delta(t)$	$\mathbb{1}(t)$	$t\,\mathbb{1}(t)$	$\mathrm{e}^{\lambda t}\mathbb{1}(t)$	$t\mathrm{e}^{\lambda t}\mathbb{1}(t)$	$\sin(\omega t + \phi)\mathbb{1}(t)$	$\mathrm{e}^{\lambda t}\sin(\omega t + \phi)\mathbb{1}(t)$
$F(s)$	1	$\dfrac{1}{s}$	$\dfrac{1}{s^2}$	$\dfrac{1}{s - \lambda}$	$\dfrac{1}{(s - \lambda)^2}$	$\dfrac{s\sin\phi + \omega\cos\phi}{s^2 + \omega^2}$	$\dfrac{(s - \lambda)\sin\phi + \omega\cos\phi}{(s - \lambda)^2 + \omega^2}$

The frequency-response theorem:

- if an LTI system G has a proper transfer function $G(s)$ whose poles are in $\{s \in \mathbb{C} \mid \mathrm{Re}\, s < 0\}$, except possibly a single pole at the origin, then its response to the harmonic input u having $u(t) = \sin(\omega t + \phi)\mathbb{1}(t)$ for some $\omega > 0$ is

$$y(t) = |G(\mathrm{j}\omega)|\sin\big(\omega t + \phi + \arg G(\mathrm{j}\omega)\big)\mathbb{1}(t) + \Big[\lim_{s \to 0} sG(s)\Big]\frac{\cos\phi}{\omega}\mathbb{1}(t) + y_\mathrm{tr}(t),$$

  where y_tr is a transient component such that $\lim_{t \to \infty} y_\mathrm{tr}(t) = 0$