Control Theory (035188)
lecture no. 13

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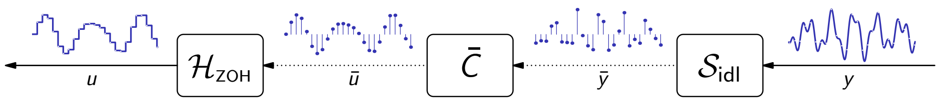
Outline

The controller



We now know that in the frequency domain

- $\mathcal{S}_{\rm idl}$ causes aliasing by folding ultra-$\omega_{\rm N}$ frequencies of $Y(\mathrm{j}\omega)$ to $[-\omega_{\rm N}, \omega_{\rm N}]$ of $\bar{Y}(\mathrm{e}^{\mathrm{j}\omega h})$
- $\bar{C}$ acts as a standard LTI filter, $\bar{U}(\mathrm{e}^{\mathrm{j}\omega h}) = \bar{C}(\mathrm{e}^{\mathrm{j}\omega h})\bar{Y}(\mathrm{e}^{\mathrm{j}\omega h})$
- $\mathcal{H}_{\rm ZOH}$ clones $[-\omega_{\rm N}, \omega_{\rm N}]$ frequency interval of $\bar{U}(\mathrm{e}^{\mathrm{j}\omega h})$ to all $\mathbb{R}$ and filters the result by the low-pass $F_\phi(\mathrm{j}\omega) = (1 - \mathrm{e}^{-\mathrm{j}\omega h})/(\mathrm{j}\omega)$

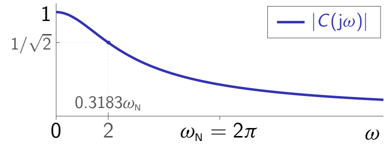
In other words,

$$U(\mathrm{j}\omega) = \frac{1 - \mathrm{e}^{-\mathrm{j}\omega h}}{\mathrm{j}\omega h}\,\bar{C}(\mathrm{e}^{\mathrm{j}\omega h}) \sum_{i\in\mathbb{Z}} Y(\mathrm{j}(\omega + 2\omega_{\rm N} i))$$

Aliasing: example

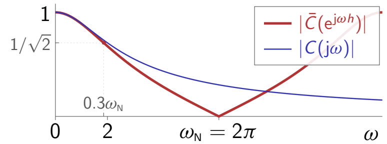
Consider the analog controller

$$C(s) = \frac{2}{s+2}:$$



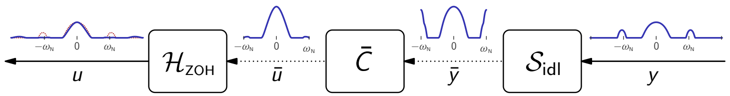
and its discrete Tustin's approximation under $h = 0.5$

$$\bar{C}(z) = \frac{z+1}{3z-1}:$$

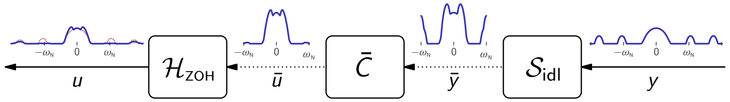


Aliasing: example (contd)

If aliased parts remain qualitatively unchanged, then aliasing is harmless

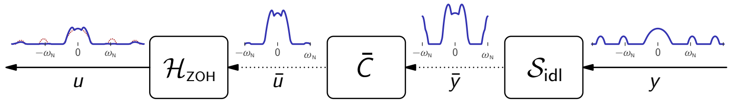


But if they migrate to different frequency bands, then the picture changes



(red dotted lines correspond to the spectrum of Cy).

Moral



Once high-frequency components of y alias as low-frequency ones and blend with low-frequency components of y,

– nothing can be done via a "better" processing by $\bar{C}(z)$.

The only way to cope with this phenomenon is to

– filter out those frequencies in continuous time, before sampling

(kill them while they're young). Low-pass filters doing that are known as

– anti-aliasing filters.

Anti-aliasing filtering: non-control examples

w/o anti-aliasing filter	with anti-aliasing filter
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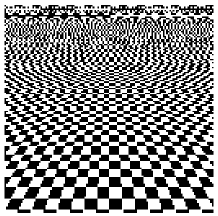
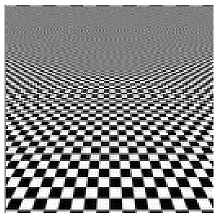
	

where anti-aliasing filters used are

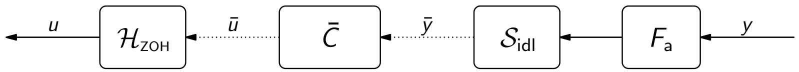
– noncausal low-pass filters with the bandwidth ω_N.

Ideal choice, performance-wise, is

– the ideal low-pass filter with the bandwidth $\omega_b = \omega_N$,

but it's hard to implement.

Anti-aliasing filters in feedback loops



Additional considerations:

– must be causal,

– $|F_a(j\omega)| \ll 1$ for all $\omega \geq \omega_N$,

– avoid adding a substantial phase lag around the crossover.

We already know (Lecture 1) that in finite-dimensional low-pass filters

– the phase lags before the magnitude starts to decay.

Hence,

– the bandwidth ω_b of F_a should be well below ω_N

and, as a result

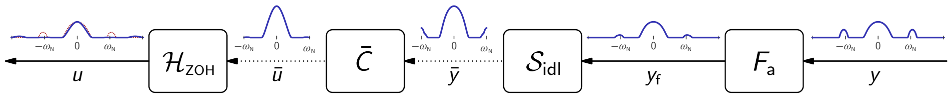
– the choice of the Nyquist frequency should be conservative
(conventional wisdom has it that $\omega_N \geq 10 \div 30\,\omega_c$, where ω_c is the analog crossover)

Aliasing: example (contd)

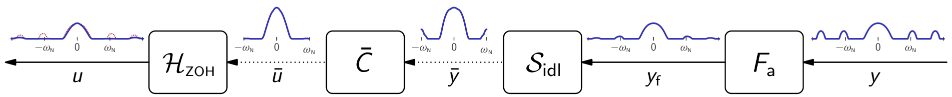
Let

$$F_a(s) = \frac{\omega_b^2}{s^2 + \sqrt{2}\omega_b s + \omega_b^2}, \quad \omega_b = \frac{\omega_N}{5} = 0.4\pi$$

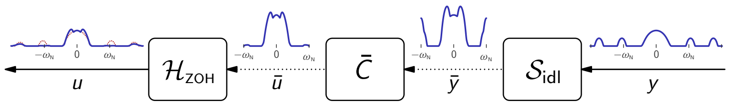
(second-order Butterworth with $|F_a(j\omega)| = 1/\sqrt{1 + (\omega/\omega_b)^4}$). In this case



and



Compare with



Discretizing C: general bilinear transformation

Given $\gamma > 0$, consider the mapping (Tustin corresponds to $\gamma = 2/h$)

$$s \to \gamma \frac{z-1}{z+1} \iff z \to \frac{\gamma + s}{\gamma - s}$$

between s and z complex planes. Every $s = \sigma + j\omega$ is mapped to

$$z = \frac{\gamma + (\sigma + j\omega)}{\gamma - (\sigma + j\omega)} \implies |z|^2 = \frac{(\gamma + \sigma)^2 + \omega^2}{(\gamma - \sigma)^2 + \omega^2}.$$

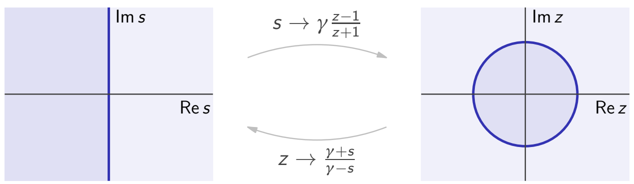
Hence,

$$|z|^2 - 1 = \frac{2\gamma\sigma}{(\gamma - \sigma)^2 + \omega^2}$$

and we end up with the relations

- $|z| < 1 \iff \sigma = \operatorname{Re} s < 0$
- $|z| > 1 \iff \sigma = \operatorname{Re} s > 0$
- $|z| = 1 \iff \sigma = \operatorname{Re} s = 0$

Discretizing C: general bilinear transformation (contd)



Thus,

- any "stable" s is mapped to a "stable" z
- any "unstable" s is mapped to a "unstable" z
- any "borderline" s is mapped to a "borderline" z

Moreover,

- any CT frequency ω is mapped to the DT frequency $\theta = 2\arctan(\omega/\gamma)$ (i.e. bilinear transformations squeeze the whole $j\mathbb{R}$ to $\mathbb{T}$, with no folding effects)
- the lowest $\omega = 0$ is mapped to the lowest $\theta = 0$
- the highest $\omega = \pm\infty$ is mapped to the highest $\theta = \pm\pi$

Discretizing C: more about Tustin

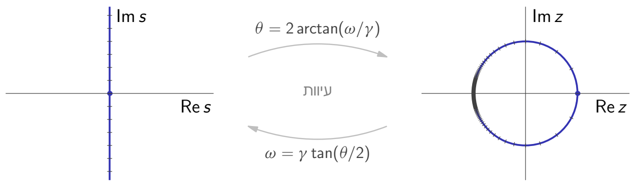
If

$$\bar{C}(z) = C(s)\big|_{s=\frac{2}{h}\frac{z-1}{z+1}},$$

then

- $\bar{C}(z)$ is stable iff $C(s)$ is stable,
- $\bar{C}(z)$ is unstable iff $C(s)$ is unstable,
- the number of integrators in $\bar{C}(z)$ equals that in $C(s)$,
- $\bar{C}(z)$ is bi-proper, unless $C(s)$ has either poles or zeros at $s = 2/h$
- $\bar{C}(z)$ has a zero at $z = -1$ of multiplicity m iff $C(s)$ is strictly proper and its pole excess is m,

Discretizing C: Tustin with pre-warping



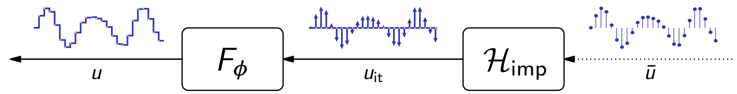
The nonlinear warping of the frequency mapping is not ideal. We would be happier with $\theta = \omega h$ in low frequencies (if folding effects are insignificant). This happens only at $\omega = 0$ and the frequency $\omega_{\text{nowarp}} \in (0, \omega_{\text{N}})$, at which

$$2 \arctan \frac{\omega_{\text{nowarp}}}{\gamma} = \omega_{\text{nowarp}} h \iff \gamma = \omega_{\text{nowarp}} \cot \frac{\omega_{\text{nowarp}} h}{2} \in \left(0, \frac{2}{h}\right).$$

The bilinear transformation with γ as above for a given $\omega_{\text{nowarp}} \in (0, \omega_{\text{N}})$ is known[1] as Tustin with pre-warping. As $\omega_{\text{nowarp}} \to 0$, the ordinary Tustin for $\gamma = 2/h$ is recovered, for which $\mathrm{d}\bar{C}(\mathrm{e}^{\mathrm{j}\theta})/\mathrm{d}\theta|_{\theta=0} = \mathrm{d}C(\omega)/\mathrm{d}\omega|_{\omega=0}$ as well.

[1]MATLAB: `c2d(C,h,c2dOptions('Method','tustin','PrewarpFrequency',w0))`.

Effects of $\mathcal{H}_{\text{ZOH}}$

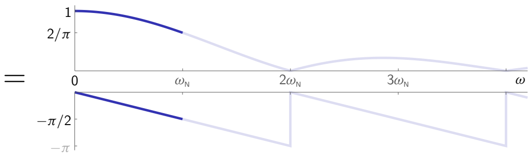


We know that

- $\mathcal{H}_{\text{imp}}$ clones $\bar{U}(\mathrm{e}^{\mathrm{j}\omega h})$ in $[-\omega_{\text{N}} + 2\omega_{\text{N}} i, \omega_{\text{N}} + 2\omega_{\text{N}} i]$ for each $i \in \mathbb{Z}$
- F_ϕ is a low-pass filter, which quite effectively filters out those clones (especially because normally $|\bar{C}(\mathrm{e}^{\mathrm{j}\omega h})| \to 0$ as $\omega \to \omega_{\text{N}}$)

The low-pass F_ϕ introduces a phase lag, linear in ω for $0 \le \omega \le \omega_{\text{N}}$:

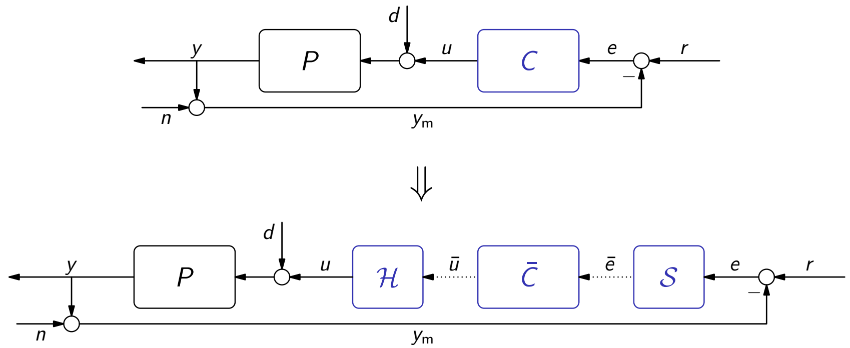
$$\frac{F_\phi(\mathrm{j}\omega)}{h} = \operatorname{sinc} \frac{\omega h}{2} \, \mathrm{e}^{-\omega h/2} =$$



which is inevitable.

Outline

The redesign problem



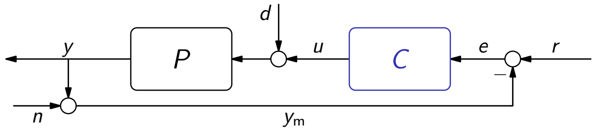
Starting point:

- "good" analog controller C (designed by whatever method)

Goal:

- find $\bar{C}$ such that $\mathcal{H}\bar{C}\mathcal{S} \approx C$.

The redesign problem: a step back



Now we know that

- an anti-aliasing filter F_a must be used, i.e. we shall assume $\mathcal{S} = \mathcal{S}_{idl} F_a$
- $\mathcal{H} = \mathcal{H}_{ZOH}$ contains its own low-pass filter, $F_\phi(s) = (1 - e^{-sh})/s$

both of which add phase lag. It thus makes sense to

- take those low-pass filters into account in the design of C.

If $\omega_N \not\gg 10\omega_c$, a good practice is to

- design C for the plant $F_a P F_\phi$,

where F_a and F_ϕ depend on the intended sampling period h (which, in turn, should be chosen small enough, perhaps with $\omega_N \geq 10\omega_c$).

Accounting for F_ϕ

The transfer function

$$F_\phi(s) = \frac{1 - e^{-sh}}{s}$$

is irrational. This is not a problem in loop shaping, but might be in analytic design methods (like state-space based). As such, it's often approximated:

- $\dfrac{F_\phi(s)}{h} \approx e^{-sh/2}$ approximates phase well
  (classical rule of thumb: sampled-data systems with $h \approx$ delay systems with $e^{-sh/2}$)
- $\dfrac{F_\phi(s)}{h} \approx \dfrac{12}{h^2 s^2 + 6hs + 12}$ is its $[1, 2]$-Padé approximant
  (more accurate than the pure delay and better suited to analytic design methods)

Discretizing C via Tustin with pre-warping

Rule of thumb:

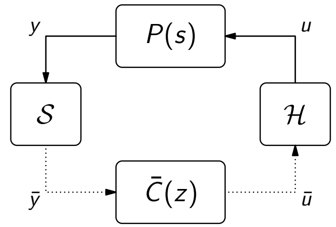
- choose ω_{nowarp} slightly below ω_c of the analog design.

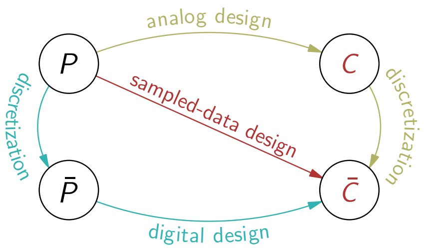
Perhaps the only exception is the integral action, or a PI module, for which the regular Tustin may be preferable to keep its velocity gain unchanged. A possible sequence in this case:

1. split the analog $C = C_{PI} C_{rem}$, where C_{rem} contains no integral actions,
2. approximate C_{PI} by the standard Tustin, to end up with $\bar{C}_{PI}$,
3. approximate C_{rem} by a Tustin with pre-warping, to end up with $\bar{C}_{rem}$,
4. construct $\bar{C} = \bar{C}_{PI} \bar{C}_{rem}$.

Outline

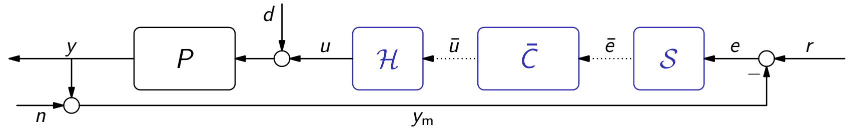
Three approaches to sampled-data control design





1. Digital redesign of analog controllers
   (do your favorite analog design first, then discretize the resulting controller)
2. Discrete-time design
   (discretize the problem first, then do your favorite discrete design)
3. Direct digital (sampled-data) design
   (design discrete-time controller $\bar{C}(z)$ directly for analog specs)

What does $\bar{C}$ see?



Input:

$$\bar{e} = \mathcal{S}r - \mathcal{S}n - \mathcal{S}Pd - \mathcal{S}P\mathcal{H}\bar{u},$$

where $\bar{u}$ is its output (cf. the analog $e = r - n - Pd - Pu$). Observations:

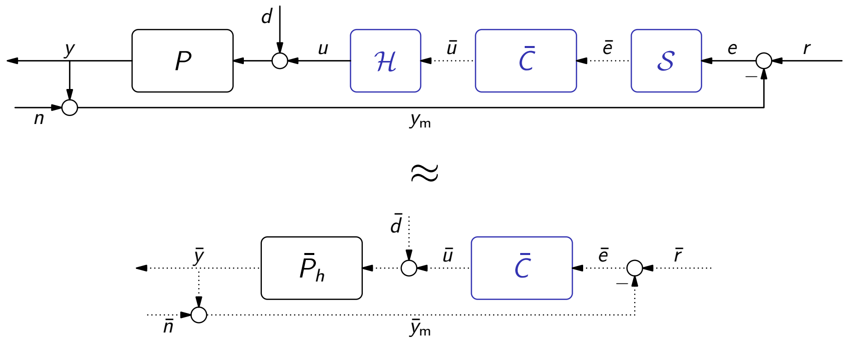
- the discrete $\bar{P}_h := \mathcal{S}P\mathcal{H} : \bar{u} \mapsto \bar{y}$ is the plant from the viewpoint of $\bar{C}$,
- sampled reference signal $\bar{r} := \mathcal{S}r$ replaces r,
- sampled noise signal $\bar{n} := \mathcal{S}n$ replaces n,
- $\mathcal{S}Pd$ doesn't fit, unless we assume that $d \approx \mathcal{H}\bar{d}$ for some $\bar{d}$

In other words,

$$\bar{e} \approx \bar{r} - \bar{n} - \mathcal{S}P\mathcal{H}\bar{d} - \mathcal{S}P\mathcal{H}\bar{u} = \bar{r} - \bar{n} - \bar{P}_h\bar{d} - \bar{P}_h\bar{u}$$

(if d can be viewed as piecewise constant, like u, if $\mathcal{H} = \mathcal{H}_{\mathrm{ZOH}}$).

What does $\bar{C}$ see? (contd)



meaning that $\bar{C}$ lives in a pure discrete (stroboscopic) world and this world approximates the reality well if

- disturbance d may be approximated by a piecewise-constant $\mathcal{H}_{\mathrm{ZOH}}\bar{d}$
- sampler $\mathcal{S} = \mathcal{S}_{\mathrm{idl}}F_{\mathrm{a}}$ and F_{a} filters out ultra-Nyquist frequencies of n

Outline

Sampled-data controllers

Analog redesign: Part II

Discrete-time design

Discretized plant and its properties

Classical methods for discrete systems (mostly stability)

Discretization

Our task is to find

$$\bar{P}_h = \mathcal{S}P\mathcal{H}_{\text{ZOH}} = \mathcal{S}_{\text{idl}}\overbrace{F_\text{a}P}^{P_\text{a}}\mathcal{H}_{\text{ZOH}}$$

for given LTI P and F_a. Let

$$P_\text{a}: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = 0 \\ y(t) = Cx(t) \end{cases}$$

($P_\text{a}(s)$ is always strictly proper, for so is $F_\text{a}(s)$). Because

- having $\mathcal{H}_{\text{ZOH}}$ at the input implies that $u = \mathcal{H}_{\text{ZOH}}\bar{u}$ for some discrete $\bar{u}$,
- having $\mathcal{S}_{\text{idl}}$ at the output implies that only $\bar{y}[i] = y(ih)$ is of interest,

finding $\bar{P}_h$ is

- equivalent to finding the mapping $\bar{u} \mapsto \bar{y}$.

Discretization (contd)

Define $\bar{x}[i] := x(ih)$. For a given $\bar{x}[i]$,

$$\bar{x}[i+1] = \text{e}^{Ah}\bar{x}[i] + \int_{ih}^{(i+1)h} \text{e}^{A(ih+h-s)}Bu(s)\text{d}s$$

Because $u(t) = \bar{u}[i]$ for all $t \in (ih, (i+1)h]$, we have that

$$\bar{x}[i+1] = \text{e}^{Ah}\bar{x}[i] + \int_{ih}^{(i+1)h} \text{e}^{A(ih+h-s)}\text{d}sB\bar{u}[i] = \text{e}^{Ah}\bar{x}[i] + \int_0^h \text{e}^{As}\text{d}sB\bar{u}[i]$$

Because $\bar{y}[i] = y(ih) = C\bar{x}[i]$, the mapping $\bar{u} \mapsto \bar{y}$ satisfies the relation

$$\bar{P}_h: \begin{cases} \bar{x}[t+1] = \bar{A}\bar{x}[t] + \bar{B}\bar{u}[t], & \bar{x}[0] = 0 \\ \bar{y}[t] = C\bar{x}[t] \end{cases}$$

where $\bar{A} := \text{e}^{Ah}$ and $\bar{B} := \int_0^h \text{e}^{As}\text{d}sB$.

Discretization (contd)

The dynamics

$$\bar{P}_h: \begin{cases} \bar{x}[t+1] = \bar{A}\bar{x}[t] + \bar{B}\bar{u}[t], & \bar{x}[0] = 0 \\ \bar{y}[t] = C\bar{x}[t] \end{cases}$$

is a standard LTI discrete system in state space. Its transfer function[2],

$$\bar{P}_h(z) = C(zI - \bar{A})^{-1}\bar{B}$$

is always strictly proper, for $\bar{P}_h(\infty) = 0$.

[2]MATLAB: `Ph=c2d(P,h)` or `[Ad,Bd]=c2d(A,B,h)`.

Discretization: example 1

If

$$P_\text{a}(s) = \frac{b}{s+a}$$

then $A = -a$, $B = b$, and $C = 1$, so that

$$\bar{A} = \text{e}^{-ah} \quad \text{and} \quad \bar{B} = \int_0^h \text{e}^{-as}\text{d}sb = \frac{1 - \text{e}^{-ah}}{a}b$$

(with well defined $\lim_{a\to 0} \bar{B} = hb$). As a result,

$$\bar{P}_h(z) = C(zI - \bar{A})^{-1}\bar{B} = \frac{(1 - \text{e}^{-ah})b/a}{z - \text{e}^{-ah}}$$

It has

- one pole, at e^{-ah}, and
- no zeros,

similarly to the continuous-time $P_\text{a}(s)$.

Discretization: example 2

If

$$P_a(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

then by the linearity of the discretization procedure

$$\bar{P}_h(z) = \frac{h}{z-1} - \frac{1-e^{-h}}{z-e^{-h}} = \frac{(h+e^{-h}-1)z + 1 - (1+h)e^{-h}}{(z-1)(z-e^{-h})}$$

This transfer function

- has two poles, at $e^{0h} = 1$ and e^{-h} and
- one zero, at $-(1-(1+h)e^{-h})/(h+e^{-h}-1) \in (-1, 0)$

While poles are still exponents of those of $P_a(s)$, the zero is an artefact.

Discretization: example 3

If

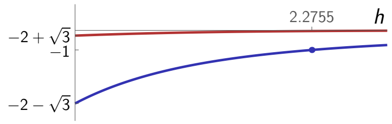
$$P_a(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

then by the linearity of the discretization procedure

$$\begin{aligned}\bar{P}_h(z) &= \frac{h}{z-1} - \frac{2(1-e^{-h})}{z-e^{-h}} + \frac{1-e^{-2h}}{z-e^{-2h}} \\ &= \frac{2h-3+4e^{-h}-e^{-2h}}{2} \frac{(z-z_{h,1})(z-z_{h,2})}{(z-1)(z-e^{-h})(z-e^{-2h})}\end{aligned}$$

where

$$\begin{bmatrix} z_{h,1} \\ z_{h,2} \end{bmatrix} =$$



Poles follow the already familiar pattern, but now we have

- two zeros, one of which is nonminimum-phase for $h < 2.2755$

Poles and zeros of $\bar{P}_h(z)$

Poles of $\bar{P}_h(z)$ are simple. If $P_a(s)$ has a pole at $s = p_i$, then

- $\bar{P}_h(z)$ has a pole at $z = e^{p_i h}$ $\quad |e^{p_i h}| < 1\ (=1) \iff \operatorname{Re} p_i < 0\ (=0)$

Zeros of $\bar{P}_h(z)$ are a mess. We only know that

- the number of finite zeros of $\bar{P}_h(z)$ is $n-1$ for almost all $h > 0$
- if $P_a(s)$ has m finite zeros $(m < n)$ at $s = z_i$, then as $h \downarrow 0$
  - m zeros of $\bar{P}_h(z)$ approach $e^{z_i h}$,
  - the remaining $n-m-1$ zeros, aka sampling zeros, approach the roots of Euler–Frobenius polynomials $Q_{n-m-1}(z)$, independent of $P_a(s)$:

$n-m$	$Q_{n-m-1}(z)$
2	$z+1$
3	z^2+4z+1
4	$z^3+11z^2+11z+1$
5	$z^4+26z^3+66z^2+26z+1$

As $Q_k(z) = z^k Q_k(1/z)$ and $Q_k(0) \neq 0$, $\quad Q_k(z_0) = 0 \iff Q_k(1/z_0) = 0$.
Therefore, $Q_k(z)$ has root(s) outside the closed unit disk for all $k \geq 2$.

Discretization: example 4

If

$$P_a(s) = \frac{\omega_n^2}{s^2+\omega_n^2} = \frac{j\omega_n/2}{s+j\omega_n} - \frac{j\omega_n/2}{s-j\omega_n}$$

then

$$\bar{P}_h(z) = \frac{1}{2}\left(\frac{1-e^{-j\omega_n h}}{z-e^{-j\omega_n h}} + \frac{1-e^{j\omega_n h}}{z-e^{j\omega_n h}}\right) = \frac{(1-\cos(\omega_n h))(z+1)}{z^2-2\cos(\omega_n h)z+1}.$$

If

- $\cos(\omega_n h) \neq \pm 1$, then $\bar{P}_h(z)$ has two poles at $e^{\pm j\omega_n h}$ and a zero at -1,
- $\cos(\omega_n h) = 1$, then $\bar{P}_h(z) = 0$,
- $\cos(\omega_n h) = -1$, then $\bar{P}_h(z) = 2/(z+1)$.

Thus, even the order of $P_c(s)$ is not always preserved under discretization.

When order drops?

Consider

$$\bar{P}_h(z) = \sum_{i=1}^{n} \frac{\bar{b}_i}{z - \bar{a}_i} \quad \text{where } \bar{a}_i := e^{a_i h} \text{ and } \bar{b}_i := \frac{e^{a_i h} - 1}{a_i} b_i.$$

Two pathological cases, where the order of $\bar{P}_h(z)$ is smaller than n:

1. $\bar{a}_i = \bar{a}_j$, although $a_i \neq a_j$, which is equivalent to

$$e^{a_i h} = e^{a_j h} \iff a_i h = a_j h + j2\pi k \text{ for some } k \in \mathbb{Z} \setminus \{0\}$$

   or $a_i - a_j = j2\omega_N k$.
2. $\bar{b}_i = 0$, although $b_i \neq 0$, which is equivalent to

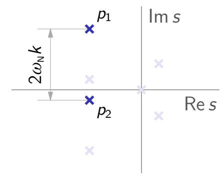
$$(e^{a_i h} = 1) \wedge (a_i \neq 0) \iff a_i h = j2\pi k \text{ for some } k \in \mathbb{Z} \setminus \{0\}$$

   or $a_i = j2\omega_N k$. But if the latter condition holds, then $\exists j \neq i$ such that $a_j = -j2\omega_N k$. Hence, $a_i - a_j = j2\omega_N(2k)$ and this case is covered by 1.

Pathological sampling

We say that sampling is pathological with respect to P_a if there are at least 2 poles of $P_a(s)$, say p_1 and p_2, such that

$$p_1 - p_2 = j\frac{2\pi}{h}k = j2\omega_N k \quad \Longleftarrow$$



for some $k \in \mathbb{Z} \setminus \{0\}$. If sampling is pathological, then

- some parts of dynamics of P are not visible by the *discrete* controller.

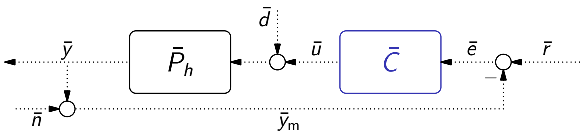
But these parts don't disappear, they are just in the blind spot of $\bar{C}$, which cannot counteract anything caused by them (e.g. instability or oscillations).

As the minimum distance between poles for h being pathological is $2\omega_N$,

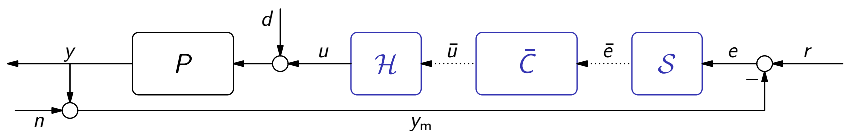
- "sufficiently fast" sampling is never pathological.

Fundamental stability result

If sampling is pathological with respect to no unstable poles of $P_a(s)$, then $\bar{C}$ stabilizes



iff $\bar{C}$ stabilizes



Outline

Sampled-data controllers

Analog redesign: Part II

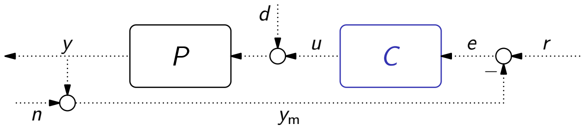
Discrete-time design

Discretized plant and its properties

Classical methods for discrete systems (mostly stability)

Discrete unity feedback

We may now drop all signs of discretization and consider a discrete system,

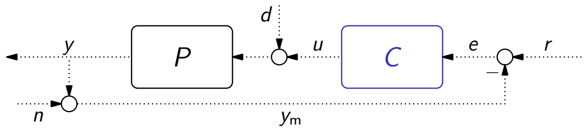


for a given

$$P(z) = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \frac{N_P(z)}{D_P(z)}$$

with $b_m \neq 0$ and $m \leq n$ (typically, $m = n - 1$).

Internal stability



The closed-loop system is said to be

– internally stable if all Gang of Four transfer functions

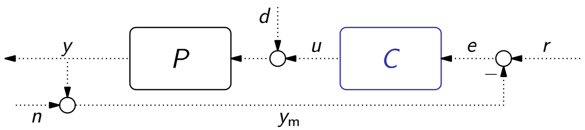
$$\begin{bmatrix} S(z) & T_{\mathrm{d}}(z) \\ T_{\mathrm{c}}(z) & T(z) \end{bmatrix} := \frac{1}{1 + P(z)C(z)} \begin{bmatrix} 1 \\ C(z) \end{bmatrix} \begin{bmatrix} 1 & P(z) \end{bmatrix}$$

are stable,

i.e. the corresponding transfer function is proper and has no poles outside the open unit disk $\mathbb{D}$.

Internal stability is the formalism helping to avoid unstable cancellations.

Characteristic polynomial



If $C(z) = N_C(z)/D_C(z)$ is proper, then the closed-loop system is internally stable iff its characteristic polynomial

$$\chi_{\mathrm{cl}}(z) = N_P(z) N_C(z) + D_P(z) D_C(z)$$

has all roots in $\mathbb{D}$ (such polynomials are known as Schur).

Root locus

The technique is exactly as in the continuous-time case. Start with writing

$$\chi_{\mathrm{cl}}(z) = 0 \quad \Longleftrightarrow \quad -\frac{1}{k} = G_k(z),$$

where k is a parameter to change, in $(0, \infty)$, and $G_k(z)$ is a proper transfer function. This representation is termed the root-locus form. All rules, which we know from the continuous-time analysis, apply then literally.

What changes is the meaning of the results, because

– stability / performance areas become different.

For example, no asymptote remains in the stability area ($\mathbb{D}$), which implies that we can afford

– no high-gain feedback in discrete setting if $P(z)$ is strictly proper,

which is normally the case.

Root locus: example

Consider again

$$P(z) = \frac{\overbrace{(h + e^{-h} - 1)}^{>0} z + \overbrace{1 - (1 + h)e^{-h}}^{>0}}{(z-1)(z-e^{-h})},$$

which is the discretization of $P(s) = 1/[s(s+1)]$, and the "P" $C(z) = k$.

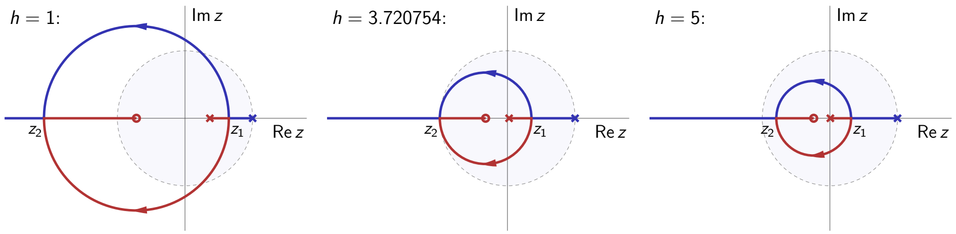
- start: $z = 1$ and $z = e^{-h}$ (poles of $G_k(z) = P(z)$);
- end: $z = -\frac{1-(1+h)e^{-h}}{h+e^{-h}-1} \in (-1, 0)$ and $z \to -\infty + j0$, as the pole excess is 1 (one asymptote, with the angle 180°);
- real axis: between the poles and to the left of the zero
- breakaway / break-in: by $dP(z)/dz = 0$ for real z,

$$z_{1,2} = e^{-h} + \frac{(1-e^{-h})\sqrt{1-e^{-h}}}{\sqrt{1-e^{-h}} \pm \sqrt{h}} = 1 \mp \frac{(1-e^{-h})\sqrt{h}}{\sqrt{1-e^{-h}} \pm \sqrt{h}}$$

  with $e^{-h} < z_1 < 1$ (breakaway) and z_2 to the left of the zero (break-in) and $z_2 \le -1$ if $0 < h < 3.720754$ and $-1 < z_1 < 0$ if $h \ge 3.720754$.

Root locus: example (contd)

For various sampling periods,

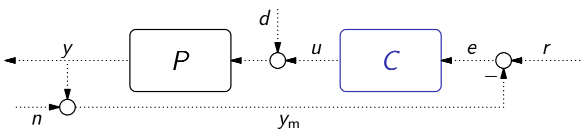


In all cases the system is stable only if k is sufficiently small. In fact, for

$$0 < k < \frac{1 - e^{-h}}{1 - (h+1)e^{-h}}$$

which can be derived by the Jury stability criterion (discrete counterpart of the Routh criterion).

Nyquist criterion



The same logic, as in the continuous-time case. The return difference

$$1 + L(z) = 1 + P(z)C(z) = \frac{\chi_{cl}(z)}{\chi_{ol}(z)}$$

still has open-loop poles as its poles and closed-loop poles as its zeros. The line of reasonings is then

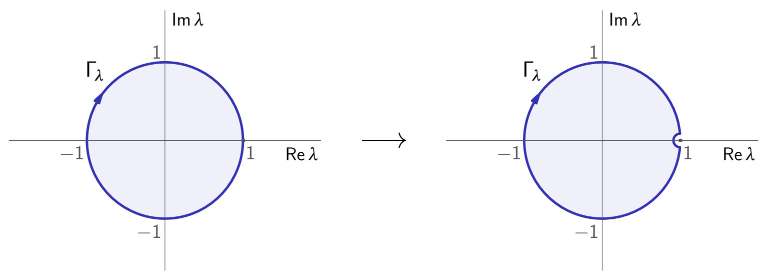
1. define simple closed contour Γ_z containing all $\mathbb{C} \setminus \bar{\mathbb{D}}_1$;
2. determine the mapping Γ_L of Γ_z by the loop gain $L(z)$;
3. count the number ν of *clockwise* encirclings of $(-1, 0)$ by Γ_L.

By the argument principle, $\nu = \#_{\text{clsd-loop unstable poles}} - \#_{\text{opn-loop unstable poles}}$.

Nyquist contour

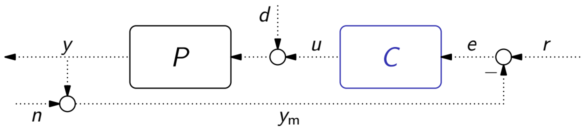
The contour encircling the unstable region $\mathbb{C} \setminus \bar{\mathbb{D}}_1$ is cumbersome. A simple workaround is to redefine $z \to 1/\lambda$. The unstable region in terms of λ is $\mathbb{D}_1$ and the contour around it is the unit circle, $\Gamma_\lambda = \mathbb{T}$. Some observations:

- the (clockwise) Γ_λ is mapped by $L(\lambda)$ as the frequency response $L(e^{j\theta})$ under increasing θ (the frequency for λ is $-\theta$);
- if $L(\lambda)$, equivalently $L(z)$, has poles at $\mathbb{T}$, the contour is altered as



  with the same completion rules as in the continuous-time case.

Steady-state performance



Nothing changes vis-à-vis the continuous-time case, except replacing $s = 0$ with $z = 1$. For example, if $d[t] = \mathbb{1}[t]$, then by the Final Value Theorem

$$y_{ss} := \lim_{t\to\infty} y[t] = \lim_{z\to 1}(z-1)T_d(z)D(z) = \lim_{z\to 1}(z-1)T_d(z)\frac{z}{z-1} = T_d(1),$$

which is the static gain of (stable) T_d. Moreover,

$$y_{ss} = 0 \quad\Longleftrightarrow\quad (P(1) = 0) \vee (|C(1)| = \infty),$$

where the latter condition requires an integral action in C.

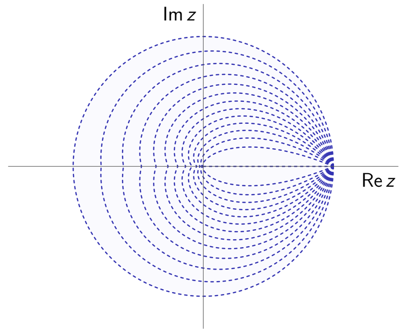
Transient performance and poles

Messier, e.g. discrete 1-order systems can exhibit oscillations and the role of zeros is not clear. So normally understood via discretized models.

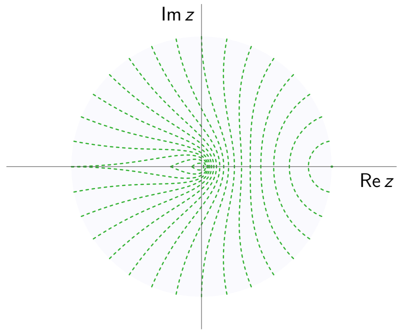
Because $\mathbb{1} = \mathcal{H}_{ZOH}\bar{\mathbb{1}}$, we have $\mathcal{S}_{idl} G \mathbb{1} = \bar{G}_h \bar{\mathbb{1}}$, i.e. the

– step response of the discrete $\bar{G}_h$ is the sampled version of that of G.

If $G(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$ for $\zeta \in [0,1]$, then $\bar{G}_h(z)$ has its poles at $z = e^{-\zeta\omega_n h} e^{\pm j\sqrt{1-\zeta^2}\omega_n h}$. Constant ζ and $\omega_n h$ contours are



and



Deadbeat control

Given n-order $P(z)$ and n_c-order $C(z)$. If the attained

$$\chi_{cl}(z) = z^{n+n_c}$$

(it is Schur), we say that the response is deadbeat. In this case we have

– finite duration of transients, of at most $n + n_c$ steps,

We know it as the FIR (finite impulse response) property, impossible in the finite-dimensional continuous-time LTI case. For example, consider

$$S(z) = \frac{1}{1 + P(z)C(z)} = \frac{b_{n+n_c} z^{n+n_c} + b_{n+n_c-1} z^{n+n_c-1} + \cdots + b_1 z + b_0}{\chi_{cl}(z)}$$
$$= b_{n+n_c} + b_{n+n_c-1} z^{-1} + \cdots + b_1 z^{1-n-n_c} + b_0 z^{-n-n_c}$$

Its impulse response

$$s[t] = b_{n+n_c}\delta[t] + \cdots + b_1\delta[t-n-n_c+1] + b_0\delta[t-n-n_c]$$

indeed ends after at most $n + n_c$ steps.

Deadbeat control: example

Consider

$$P(z) = \frac{h^2}{2}\frac{z+1}{(z-1)^2}$$

which is the discretized $1/s^2$. With $\chi_{cl}(z) = z^3$ we have (see Lecture 2)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & h^2/2 & 0 \\ 1 & -2 & h^2/2 & h^2/2 \\ 0 & 1 & 0 & h^2/2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/4 \\ 5/(2h^2) \\ -3/(2h^2) \end{bmatrix}$$

so that

$$C(z) = \frac{2}{h^2}\frac{5z-3}{4z+3}$$

In this case

$$S(z) = \frac{(z-1)^2(4z+3)}{4z^3} \implies e[t] = \delta[t] - \frac{1}{4}\delta[t-1] - \frac{3}{4}\delta[t-2]$$

with $r[t] = \mathbb{1}[t]$ (for which $R(z) = \frac{z}{z-1}$ and $S(z)R(z) = 1 - \frac{1}{4}z^{-1} - \frac{3}{4}z^{-2}$).