# Control Theory (00350188)
# lecture no. 12

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# Outline

Analog redesign

Discrete-time design

Discretized plant and its properties

Classical methods for discrete systems (mostly stability)

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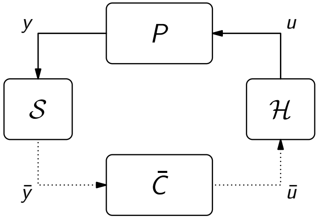
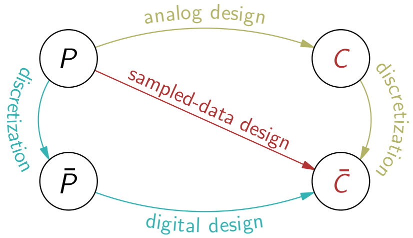
Discrete-time design

Discretized plant and its properties

Classical methods for discrete systems (mostly stability)

# Three approaches to sampled-data control design



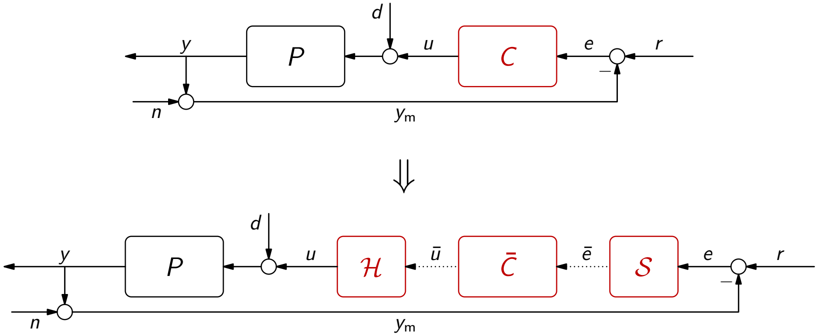




1. Digital redesign of analog controllers
   (do your favorite analog design first, then discretize the resulting controller)

2. Discrete-time design
   (discretize the problem first, then do your favorite discrete design)

3. Direct digital (sampled-data) design
   (design discrete-time controller $\bar{C}(z)$ directly for analog specs)

# The redesign problem




Starting point:

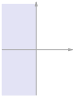
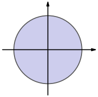
– "good" analog controller $C$ (designed by whatever method)

Goal:

– find $\bar{C}$ such that $\mathcal{H}\bar{C}\mathcal{S} \approx C$

(we consider $\mathcal{S} = \mathcal{S}_{\text{idl}}$, $\mathcal{H} = \mathcal{H}_{\text{ZOH}}$, and periodic sampling with given $h > 0$).

# Discrete transfer functions (from LS)

| continuous-time systems | discrete-time systems |
|---|---|
| differential equations | difference equations |
| Laplace transform | $z$ transform |
| $s$ is the derivative in the time domain | $z$ is the shift in the time domain |
| left-half plane in the $s$-plane :  |  : unit disk in the $z$-plane |
| imaginary axis, $s = \mathrm{j}\omega$ | unit circle, $z = \mathrm{e}^{\mathrm{j}\theta}$ |
| static gain is $G(s)\|_{s=0} = G(0)$ | static gain is $G(z)\|_{z=1} = G(1)$ |
| integral action: pole at $s = 0$ | integral action: pole at $z = 1$ |

# Choice of $\bar{C}$: ad hoc approaches

**Logic:** imitate $C$, often based on approximating derivatives, like

fwd Euler: $\dot{x}(ih) \approx \dfrac{x(ih+h) - x(ih)}{h} \implies s \to \dfrac{z-1}{h}$

bwd Euler: $\dot{x}(ih) \approx \dfrac{x(ih) - x(ih-h)}{h} \implies s \to \dfrac{z-1}{hz}$

Tustin[1]: $\dfrac{\dot{x}(ih+h) + \dot{x}(ih)}{2} \approx \dfrac{x(ih+h) - x(ih)}{h} \implies s \to \dfrac{2}{h}\dfrac{z-1}{z+1}$

making sense if $h$ is “small enough.”

[1] Matlab: `c2d(C,h,'tustin')`, where `C` is a continuous-time system.

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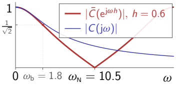
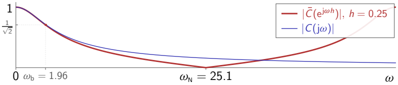
making sense if $h$ is “small enough.”

Example: If $C(s) = 2/(s+2)$, then

$$\bar{C}(z) = C(s)\Big|_{s=\frac{2}{h}\frac{z-1}{z+1}} = \frac{2}{2/h \cdot (z-1)/(z+1) + 2} = \frac{h(z+1)}{(h+1)z + h - 1}.$$

and

$|\bar{C}(e^{j\omega h})| =$ 

or 

# General bilinear transformation

Given $\gamma > 0$, consider the mapping (Tustin corresponds to $\gamma = 2/h$)

$$s \to \gamma \frac{z-1}{z+1} \iff z \to \frac{\gamma + s}{\gamma - s}$$

between $s$ and $z$ complex planes. Every $s = \sigma + \mathrm{j}\omega$ is mapped to

$$z = \frac{\gamma + (\sigma + \mathrm{j}\omega)}{\gamma - (\sigma + \mathrm{j}\omega)} \implies |z|^2 = \frac{(\gamma + \sigma)^2 + \omega^2}{(\gamma - \sigma)^2 + \omega^2}.$$

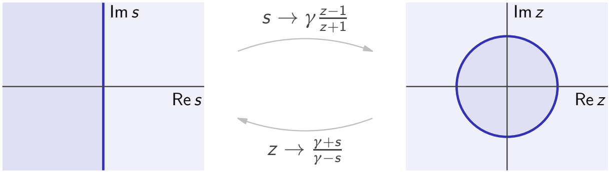
Hence,

$$|z|^2 - 1 = \frac{2\gamma\sigma}{(\gamma - \sigma)^2 + \omega^2}$$

and we end up with the relations

- $|z| < 1 \iff \sigma = \operatorname{Re} s < 0$
- $|z| > 1 \iff \sigma = \operatorname{Re} s > 0$
- $|z| = 1 \iff \sigma = \operatorname{Re} s = 0$

# General bilinear transformation (contd)




Thus,

- any "stable" $s$ is mapped to a "stable" $z$
- any "unstable" $s$ is mapped to a "unstable" $z$
- any "borderline" $s$ is mapped to a "borderline" $z$

Moreover,

- any CT frequency $\omega$ is mapped to the DT frequency $\theta = 2\arctan(\omega/\gamma)$
  (i.e. bilinear transformations squeeze the whole $j\mathbb{R}$ to $\mathbb{T}$, with no folding effects)
- the lowest $\omega = 0$ is mapped to the lowest $\theta = 0$
- the highest $\omega = \pm\infty$ is mapped to the highest $\theta = \pm\pi$

# Choice of $\bar{C}$: more about Tustin

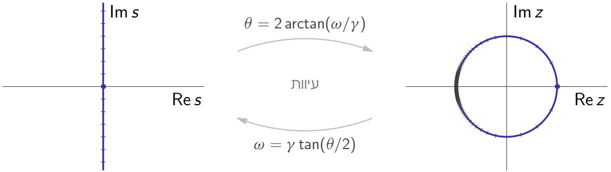
If

$$\bar{C}(z) = C(s)\Big|_{s=\frac{2}{h}\frac{z-1}{z+1}},$$

then

- $\bar{C}$ is stable iff $C$ is stable,
- $\bar{C}$ is unstable iff $C$ is unstable,
- the number of integrators in $\bar{C}(z)$ equals that in $C(s)$,
- $\bar{C}(z)$ is bi-proper, unless $C(s)$ has either poles or zeros at $s = 2/h$
  - zero multiplicity of $C(s)$ at $s = 2/h$ equals the pole excess of $\bar{C}(z)$
  - pole multiplicity of $C(s)$ at $s = 2/h$ equals the zero excess of $\bar{C}(z)$
- $\bar{C}(z)$ has a zero at $z = -1$ of multiplicity $m$ iff $C(s)$ is strictly proper and its pole excess is $m$
- $\bar{C}(z)$ has a pole at $z = -1$ of multiplicity $m$ iff $C(s)$ is non-proper and its zero excess is $m$

# Discretizing $C$: Tustin with pre-warping




The nonlinear warping of the frequency mapping is not ideal. We would be happier with $\theta = \omega h$ in low frequencies (if folding effects are insignificant). This happens only at $\omega = 0$ and the frequency $\omega_{\text{nowarp}} \in (0, \omega_{\text{N}})$, at which

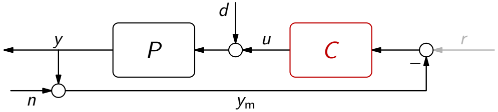
$$2\arctan\frac{\omega_{\text{nowarp}}}{\gamma} = \omega_{\text{nowarp}} h \iff \gamma = \omega_{\text{nowarp}} \cot\frac{\omega_{\text{nowarp}} h}{2} \in \left(0, \frac{2}{h}\right).$$

The bilinear transformation with $\gamma$ as above for a given $\omega_{\text{nowarp}} \in (0, \omega_{\text{N}})$ is known[1] as Tustin with pre-warping. As $\omega_{\text{nowarp}} \to 0$, the ordinary Tustin for $\gamma = 2/h$ is recovered, for which $\mathrm{d}\bar{C}(\mathrm{e}^{\mathrm{j}\theta})/\mathrm{d}\theta|_{\theta=0} = \mathrm{d}C(\omega)/\mathrm{d}\omega|_{\omega=0}$ as well.

[1] Matlab: `c2d(C,h,c2dOptions('Method','tustin','PrewarpFrequency',w0))`.

# Example: DC motor

A DC motor from Lecture 1, controlled in closed loop




Requirements:

– closed-loop stability (of course)
– zero steady-state error for a step in $r$ — always holds
– zero steady-state error for a step in $d$ — integrator in $C(s)$
– good stability margins
– $\omega_c \approx 2$ [rad/sec]

Design:

– LQG loop shaping, with a PI weight $W$ (like in Lecture 11)

# Example: analog design

Weight:

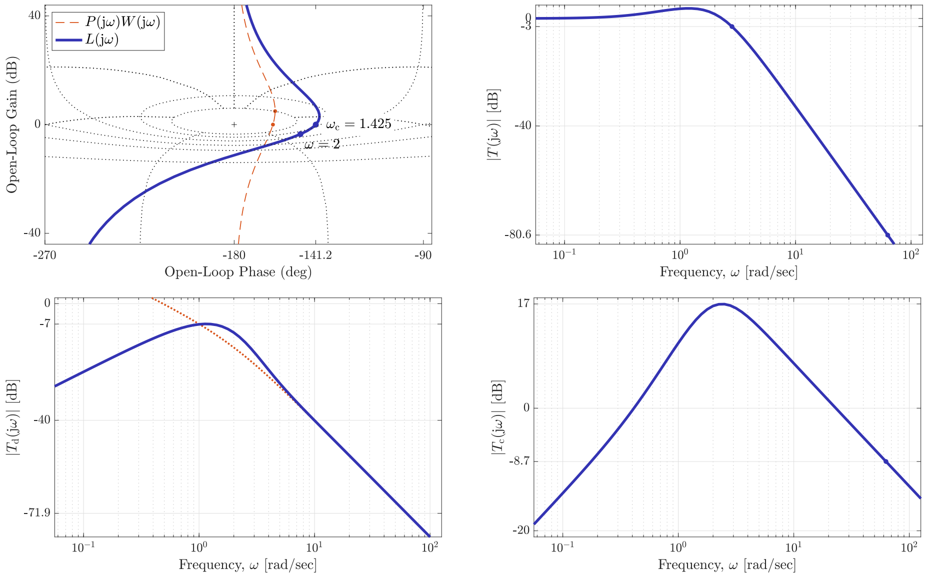
$$W(s) = 5.06\Big(1 + \frac{1}{s}\Big)$$

Controller:

$$C(s) = W(s)C_{\mathrm{a}}(s) = \frac{23.081(s + 2.075)(s + 0.5346)}{s(s^2 + 6.155s + 17.44)}$$

(a pole of $C_{\mathrm{a}}(s)$ cancels the zero of $W(s)$ at $s = 1$). The actual crossover is $\omega_{\mathrm{c}} = 1.4248$ and the closed-loop bandwidth is $\omega_{\mathrm{b}} = 2.8155$.

# Example: analog design (contd)



# Example: controller discretization

Using Tustin, the discretized controllers are

$$h = 0.01: \quad \bar{C}(z) = \frac{0.11337(z+1)(z-0.9795)(z-0.9947)}{(z-1)(z^2-1.939z+0.9403)}$$

$$h = 0.1: \quad \bar{C}(z) = \frac{0.96777(z+1)(z-0.812)(z-0.9479)}{(z-1)(z^2-1.415z+0.5445)}$$

$$h = 0.5: \quad \bar{C}(z) = \frac{2.7375(z+1)(z-0.317)(z-0.7642)}{(z-1)(z^2+0.04973z+0.152)}$$

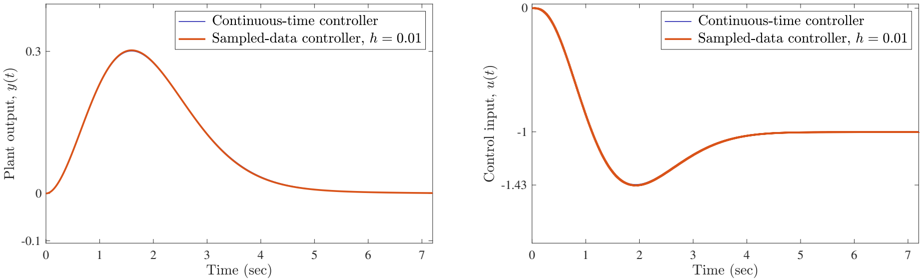
All of them

- preserve integral actions (pole at $s = 0 \to$ pole at $z = 1$)
- have single zeros at $z = -1$ (pole excess of $P(s)$ is 1)
- are bi-proper

which are general properties of the Tustin transformation.

# Example: $d(t) = \mathbb{1}(t)$ and $n(t) = 0$

Responses with $h = 0.01$:



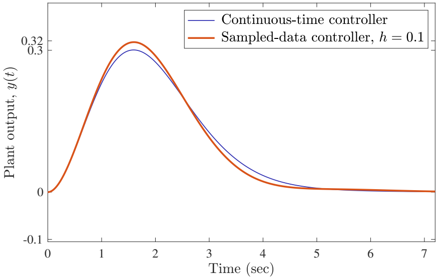

sampled-data response $\approx$ analog response

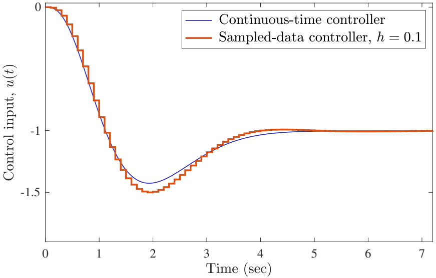
$\Downarrow$

adequate sampling rate

# Example: $d(t) = \mathbb{1}(t)$ and $n(t) = 0$

The same with $h = 0.1$:



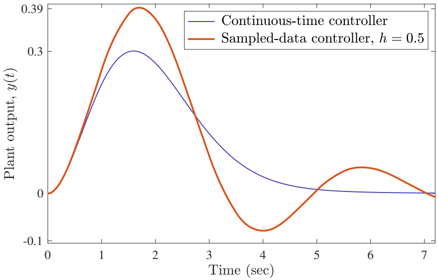

sampled-data response starts getting worse than analog response

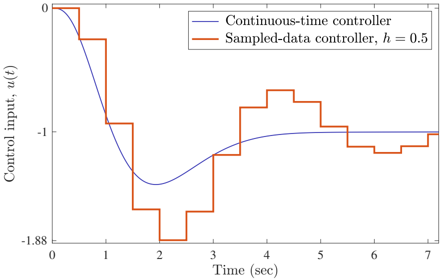
$\Downarrow$

sampling rate starts to become problematic

# Example: $d(t) = \mathbb{1}(t)$ and $n(t) = 0$

And now the same with $h = 0.5$:




sampled-data response is substantially worse than analog response

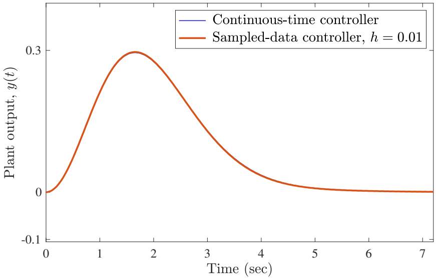
$$\Downarrow$$

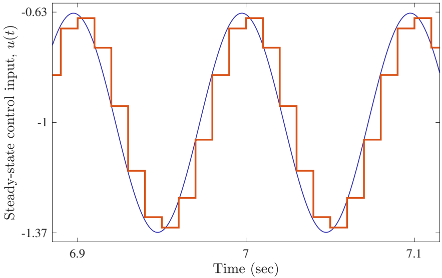
sampling rate becomes inadequate

(further increase of $h$ eventually results in an unstable closed-loop system).

# Example: $d(t) = \mathbb{1}(t)$ and $n(t) = \sin(20\pi t + 0.1)$

Responses with $h = 0.01$:




sampled-data response $\approx$ analog response

$$\Downarrow$$

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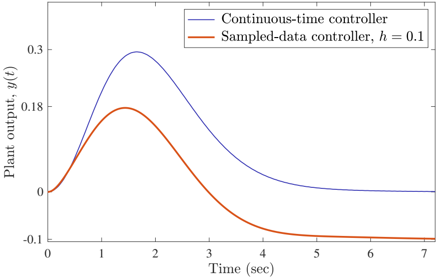


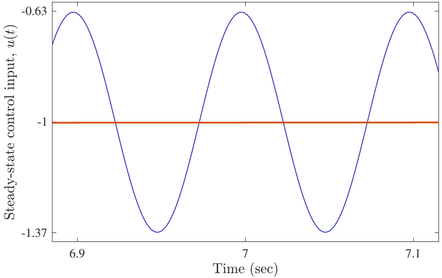


Oops,

- sampled-data response is qualitatively different from analog response (steady-state error is nonzero, the harmonic of measurement noise disappears)

Why?

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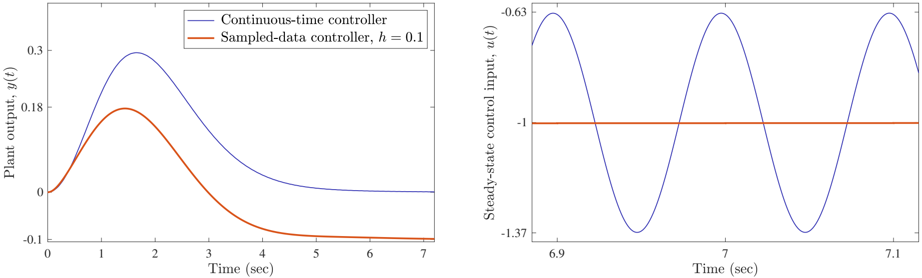


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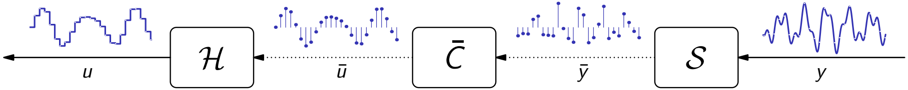
# The sampled-data controller




We now know that in the frequency domain

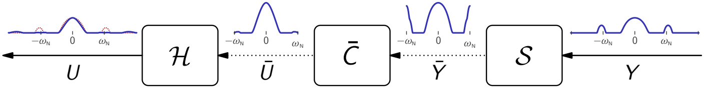
- $\mathcal{S}$ causes aliasing by folding ultra-$\omega_N$ frequencies of $Y(j\omega)$ to $[-\omega_N, \omega_N]$ of $\bar{Y}(e^{j\omega h})$
- $\bar{C}$ acts as a standard LTI filter, $\bar{U}(e^{j\omega h}) = \bar{C}(e^{j\omega h})\bar{Y}(e^{j\omega h})$
- $\mathcal{H}$ clones $[-\omega_N, \omega_N]$ frequency interval of $\bar{U}(e^{j\omega h})$ to all $\mathbb{R}$ and filters the result by the low-pass $hF_\phi$, where $F_\phi(j\omega) = (1 - e^{-j\omega h})/(j\omega h)$

In other words,

$$U(j\omega) = \frac{1 - e^{-j\omega h}}{j\omega h}\,\bar{C}(e^{j\omega h}) \sum_{i\in\mathbb{Z}} Y(j(\omega + 2\omega_N i))$$

# Effects of aliasing

If aliased parts remain qualitatively unchanged, then aliasing is harmless




But if they migrate to different frequency bands, then the picture changes

(red dotted lines correspond to the spectrum of $Cy$).

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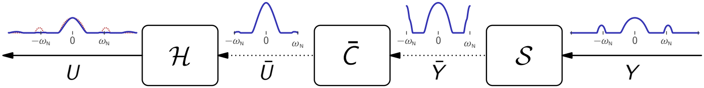


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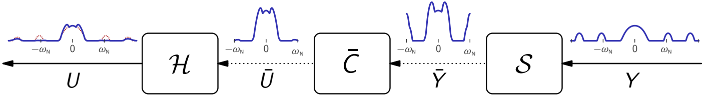


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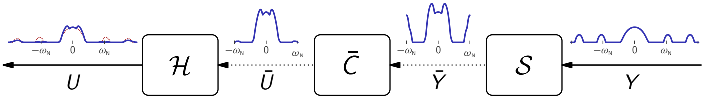
# Moral




Once high-frequency components of $y$ alias as low-frequency ones and blend with low-frequency components of $y$,

- nothing can be done via a “better” processing by $\bar{C}(z)$.

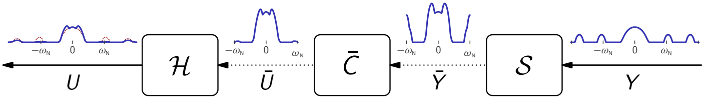
The only way to cope with this phenomenon is to

- filter out those frequencies in continuous time, before sampling

(nip them in the bud). Low-pass filters doing that are known as

- anti-aliasing filters.

# Moral

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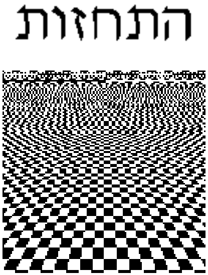
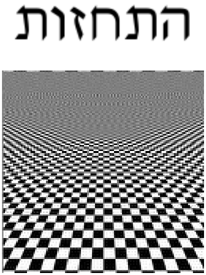
# Anti-aliasing filtering: non-control examples

| w/o anti-aliasing filter | with anti-aliasing filter |
|---|---|
|   |   |

where anti-aliasing filters used are

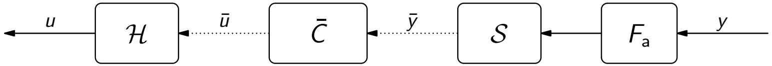
– noncausal low-pass filters with the bandwidth $\omega_N$.

Best choice, performance-wise, is

– the ideal low-pass filter $F_{ilp}$ with the bandwidth $\omega_b = \omega_N$,

whose impulse response $f_{ilp}(t) = \operatorname{sinc}(\omega_N t)$, but it is hard to implement.

# Anti-aliasing filters in feedback loops




Additional considerations:

- must be causal,
- $|F_a(j\omega)| \ll 1$ for all $\omega \geq \omega_N$,
- avoid adding a substantial phase lag around the crossover.

We already know (Lecture 1) that finite-dimensional low-pass filters

- introduce phase lags before the magnitude starts to decay.

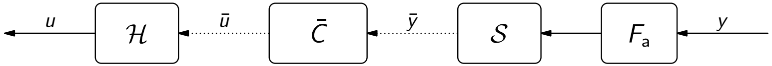
Hence,

- the bandwidth $\omega_b$ of $F_a$ should be well below $\omega_N$

and, as a result

- the choice of the Nyquist frequency should be conservative (conventional wisdom has it that $\omega_N \geq 10 \div 30\,\omega_c$, where $\omega_c$ is the analog crossover)

# Anti-aliasing filters in feedback loops

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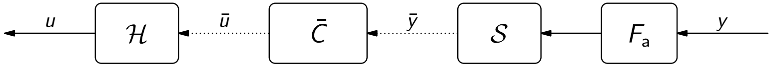


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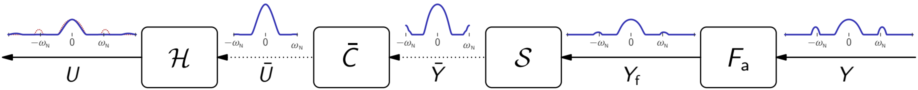
# Aliasing: example (contd)

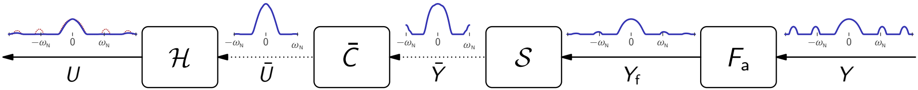
Let

$$F_a(s) = \frac{\omega_b^2}{s^2 + \sqrt{2}\omega_b s + \omega_b^2}, \quad \omega_b = \frac{\omega_N}{5} = 0.4\pi$$

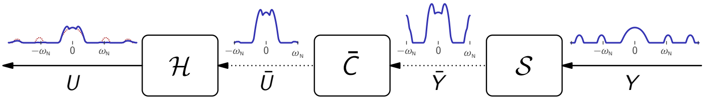
(second-order Butterworth with $|F_a(j\omega)| = 1/\sqrt{1 + (\omega/\omega_b)^4}$). In this case



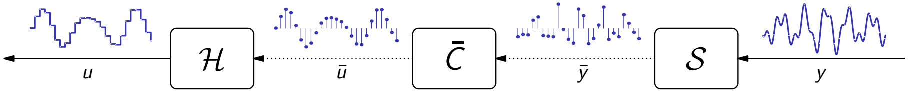

and




Compare with



# Effects of $\mathcal{H}$




With

$$U(\mathrm{j}\omega) = h\,\frac{1-\mathrm{e}^{-\mathrm{j}\omega h}}{\mathrm{j}\omega h}\,\bar{U}(\mathrm{e}^{\mathrm{j}\omega h})$$

we effectively (factor $h$ is offset by $1/h$ of the sampler) have the FIR

$$F_\phi(s) = \frac{1-\mathrm{e}^{-sh}}{sh}$$

in the loop. This is a low-pass filter, whose frequency response

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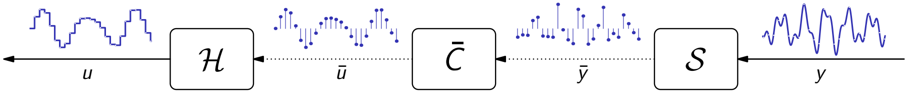
This is a low-pass $F_\phi$, having a phase lag $\omega h/2$ for $0 \le \omega \le \omega_N$.

# Effects of $\mathcal{H}$

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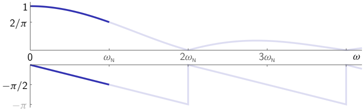


This is a low-pass $F_\phi$, having a phase lag $\omega h/2$ for $0 \le \omega \le \omega_{\mathrm{N}}$.

# Effects on the design of $C$

Things to remember:

- sample sufficiently fast
- account for an anti-aliasing filter $F_\mathrm{a}$ unless sensor is digital
- account for low-pass in the ZOH, $F_\phi$

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Advised to

- design $C$ for the augmented $F_{\mathrm{a}} P F_{\phi}$.

Remark Because $F_{\phi}(s) = (1 - \mathrm{e}^{-sh})/(sh)$ is infinite dimensional, its approximations

$$F_{\phi}(s) \approx F_{\phi,\mathrm{del}} := \mathrm{e}^{-sh/2} \quad \text{or} \quad F_{\phi}(s) \approx F_{\phi,2} := \frac{12}{h^2 s^2 + 6hs + 12}$$

can be used in analytic design methods, such as state space. Note that $F_{\phi,2}(s)$ is the $[1,2]$-Padé approximant of $F_{\phi}(s)$, whose bandwidth $\omega_{\mathrm{b}} \approx 2.7233/h < \omega_{\mathrm{N}}$ and frequency response in $\omega \in [0, \omega_{\mathrm{N}}]$ is not far from that of $F_{\phi}$, e.g.

$$F_{\phi,2}(\mathrm{j}\omega_{\mathrm{N}}) \approx 0.6326\mathrm{e}^{-\mathrm{j}1.4583} \quad \text{vs.} \quad F_{\phi}(\mathrm{j}\omega_{\mathrm{N}}) = \frac{2}{\pi}\mathrm{e}^{-\mathrm{j}\pi/2} \approx 0.6366\mathrm{e}^{-\mathrm{j}1.5708}.$$

# Outline

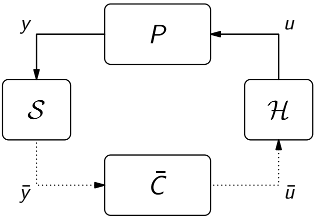
Analog redesign

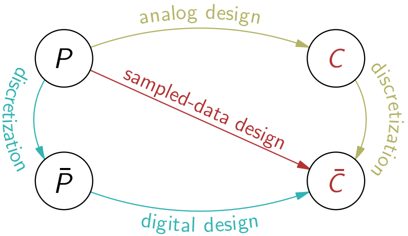
Discrete-time design

Discretized plant and its properties

Classical methods for discrete systems (mostly stability)

# Three approaches to sampled-data control design



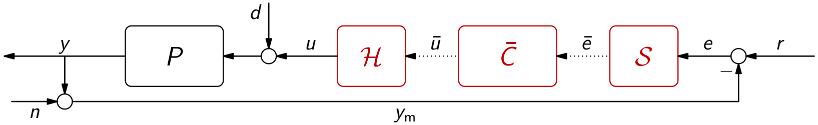




1. Digital redesign of analog controllers
   (do your favorite analog design first, then discretize the resulting controller)

2. Discrete-time design
   (discretize the problem first, then do your favorite discrete design)

3. Direct digital (sampled-data) design
   (design discrete-time controller $\bar{C}(z)$ directly for analog specs)

# What does $\bar{C}$ see ?




Input:

$$\bar{e} = \mathcal{S}r - \mathcal{S}n - \mathcal{S}Pd - \mathcal{S}P\mathcal{H}\bar{u},$$

where $\bar{u}$ is its output (cf. the analog $e = r - n - Pd - Pu$). Observations:

- the discrete $\bar{P}_h := \mathcal{S}P\mathcal{H} : \bar{u} \mapsto \bar{y}$ is the plant from the viewpoint of $\bar{C}$,
- sampled reference signal $\bar{r} := \mathcal{S}r$ replaces $r$,
- sampled noise signal $\bar{n} := \mathcal{S}n$ replaces $n$,
- $\mathcal{S}Pd$ doesn't fit, unless we assume that $d \approx \mathcal{H}\bar{d}$ for some $\bar{d}$

In other words,

$$\bar{e} \approx \bar{r} - \bar{n} - \mathcal{S}P\mathcal{H}\bar{d} - \mathcal{S}P\mathcal{H}\bar{u} = \bar{r} - \bar{n} - \bar{P}_h\bar{d} - \bar{P}_h\bar{u}$$

(if $d$ can be viewed as piecewise constant, like $u$, in the case of $\mathcal{H} = \mathcal{H}_{\mathrm{ZOH}}$)

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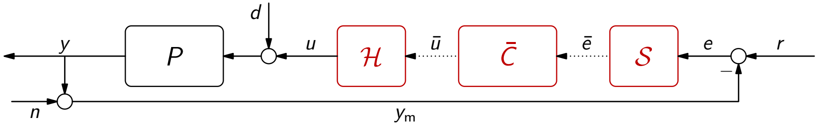


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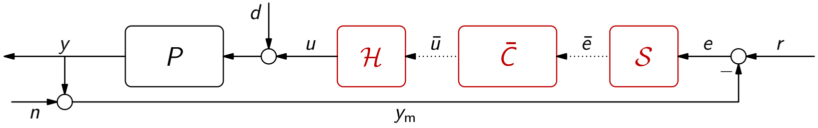


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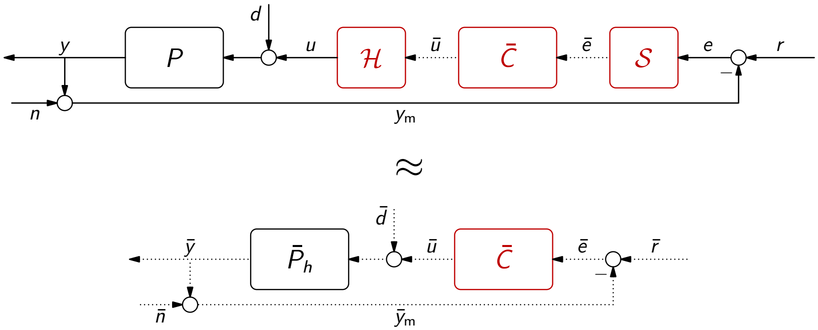
# What does $\bar{C}$ see? (contd)




meaning that $\bar{C}$ lives in a pure discrete (stroboscopic) world and this world approximates the reality well if

- disturbance $d$ may be approximated by a piecewise-constant $\mathcal{H}\bar{d}$
  it is still assumed that $\mathcal{H} = \mathcal{H}_{\text{ZOH}}$
- sampler $\mathcal{S} = \mathcal{S}_{\text{idl}} F_a$ and $F_a$ filters out ultra-Nyquist frequencies of $n$
  or the sensor is digital, like an encoder, in which case noise is digital by nature

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# Discretization

Our task is to find

$$\bar{P}_h = \mathcal{S}P\mathcal{H} = \mathcal{S}_{\text{idl}}\overbrace{F_\text{a}P}^{P_\text{a}}\mathcal{H}_{\text{ZOH}}$$

for given LTI $P$ and $F_\text{a}$. Let

$$P_\text{a}: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0 \\ y(t) = Cx(t) \end{cases}$$

($P_\text{a}(s)$ is always strictly proper, for so is $F_\text{a}(s)$). Because

- having $\mathcal{H}$ at the input implies that $u = \mathcal{H}_{\text{ZOH}}\bar{u}$ for some discrete $\bar{u}$,
- having $\mathcal{S}_{\text{idl}}$ at the output implies that only $\bar{y}[i] = y(ih)$ is of interest,

finding $\bar{P}_h$ is

- equivalent to finding the mapping $\bar{u} \mapsto \bar{y}$.

# Discretization (contd)

Define $\bar{x}[i] := x(ih)$. For a given $\bar{x}[i]$,

$$\bar{x}[i+1] = \mathrm{e}^{Ah}\bar{x}[i] + \int_{ih}^{(i+1)h} \mathrm{e}^{A(ih+h-s)}Bu(s)\mathrm{d}s$$

Because $u(t) = \bar{u}[i]$ for all $t \in (ih, (i+1)h]$, we have that

$$\bar{x}[i+1] = \mathrm{e}^{Ah}\bar{x}[i] + \int_{ih}^{(i+1)h} \mathrm{e}^{A(ih+h-s)}\mathrm{d}sB\bar{u}[i] = \mathrm{e}^{Ah}\bar{x}[i] + \int_{0}^{h} \mathrm{e}^{As}\mathrm{d}sB\bar{u}[i]$$

Because $\bar{y}[i] = y(ih) = C\bar{x}[i]$, the mapping $\bar{u} \mapsto \bar{y}$ satisfies the relation

$$\bar{P}_h : \begin{cases} \bar{x}[t+1] = \bar{A}\bar{x}[t] + \bar{B}\bar{u}[t], \quad \bar{x}[0] = 0 \\ \bar{y}[t] = C\bar{x}[t] \end{cases}$$

where $\bar{A} := \mathrm{e}^{Ah}$ and $\bar{B} := \int_{0}^{h} \mathrm{e}^{As}\mathrm{d}sB$.

# Discretization (contd)

The dynamics

$$\bar{P}_h : \begin{cases} \bar{x}[t+1] = \bar{A}\bar{x}[t] + \bar{B}\bar{u}[t], \quad \bar{x}[0] = 0 \\ \bar{y}[t] = C\bar{x}[t] \end{cases}$$

is a standard LTI discrete system in state space. Its transfer function[2],

$$\bar{P}_h(z) = C(zI - \bar{A})^{-1}\bar{B}$$

is always strictly proper, for $\bar{P}_h(\infty) = 0$.

Note that

- $\lambda \in \operatorname{spec}(A) \subset \mathbb{C} \implies e^{\lambda h} \in \operatorname{spec}(\bar{A}) \subset \mathbb{C}$
- $\bar{\lambda} \in \operatorname{spec}(\bar{A}) \subset \mathbb{C} \implies \exists \lambda \in \operatorname{spec}(A) \subset \mathbb{C}$ such that $e^{\lambda h} = \bar{\lambda}$

[2]Matlab: `Ph=c2d(P,h)` or `[Ad,Bd]=c2d(A,B,h)`.

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# Discretization: example 1

If

$$P_{\mathrm{a}}(s) = \frac{b}{s+a}$$

then $A = -a$, $B = b$, and $C = 1$, so that

$$\bar{A} = \mathrm{e}^{-ah} \quad \text{and} \quad \bar{B} = \int_0^h \mathrm{e}^{-as} \mathrm{d}s b = \frac{1 - \mathrm{e}^{-ah}}{a} b$$

(with well defined $\lim_{a \to 0} \bar{B} = hb$). As a result,

$$\bar{P}_h(z) = C(zI - \bar{A})^{-1} \bar{B} = \frac{(1 - \mathrm{e}^{-ah}) b/a}{z - \mathrm{e}^{-ah}}$$

It has

- one pole, at $\mathrm{e}^{-ah}$, and
- no zeros,

similarly to the continuous-time $P_{\mathrm{a}}(s)$.

# Discretization: example 2

If

$$P_{\mathrm{a}}(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

then by the linearity of the discretization procedure

$$\bar{P}_h(z) = \frac{h}{z-1} - \frac{1-\mathrm{e}^{-h}}{z-\mathrm{e}^{-h}} = \frac{(h+\mathrm{e}^{-h}-1)z + 1 - (1+h)\mathrm{e}^{-h}}{(z-1)(z-\mathrm{e}^{-h})}$$

This transfer function

- has two poles, at $\mathrm{e}^{0h} = 1$ and $\mathrm{e}^{-h}$ and
- one zero, at $-(1-(1+h)\mathrm{e}^{-h})/(h+\mathrm{e}^{-h}-1) \in (-1,0)$

While poles are still exponents of those of $P_{\mathrm{a}}(s)$, the zero is an artefact.

# Discretization: example 3

If

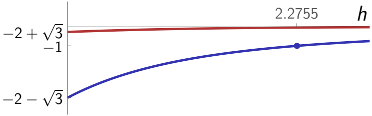
$$P_{\mathrm{a}}(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

then by the linearity of the discretization procedure

$$\begin{aligned}\bar{P}_h(z) &= \frac{h}{z-1} - \frac{2(1-\mathrm{e}^{-h})}{z-\mathrm{e}^{-h}} + \frac{1-\mathrm{e}^{-2h}}{z-\mathrm{e}^{-2h}} \\ &= \frac{2h-3+4\mathrm{e}^{-h}-\mathrm{e}^{-2h}}{2}\,\frac{(z-z_{h,1})(z-z_{h,2})}{(z-1)(z-\mathrm{e}^{-h})(z-\mathrm{e}^{-2h})}\end{aligned}$$

where

$$\begin{bmatrix} z_{h,1} \\ z_{h,2} \end{bmatrix} =$$




Poles follow the already familiar pattern, but now we have

- two zeros, one of which is nonminimum-phase for $h < 2.2755$

# Poles and zeros of $\bar{P}_h(z)$

Poles of $\bar{P}_h(z)$ are simple. If $P_a(s)$ has a pole at $s = p_i$, then

- $\bar{P}_h(z)$ has a pole at $z = e^{p_i h}$ $\quad |e^{p_i h}| < 1\ (=1) \iff \operatorname{Re} p_i < 0\ (=0)$

Zeros of $\bar{P}_h(z)$ are a mess. We only know that

- the number of finite zeros of $\bar{P}_h(z)$ is $n-1$ for almost all $h > 0$
- if $P_a(s)$ has $m$ finite zeros ($m < n$) at $s = z_i$, then as $h \downarrow 0$
  - $m$ zeros of $\bar{P}_h(z)$ approach $e^{z_i h}$,
  - the remaining $n-m-1$ zeros, aka sampling zeros, approach the roots of Euler–Frobenius polynomials $Q_{n-m-1}(z)$, independent of $P_a(s)$:

| $n-m$ | $Q_{n-m-1}(z)$ |
| --- | --- |
| 2 | $z+1$ |
| 3 | $z^2+4z+1$ |
| 4 | $z^3+11z^2+11z+1$ |
| 5 | $z^4+26z^3+66z^2+26z+1$ |

As $Q_k(z) = z^k Q_k(1/z)$ and $Q_k(0) \neq 0$, $Q_k(z_0) = 0 \iff Q_k(1/z_0) = 0$. Therefore, $Q_k(z)$ has root(s) outside the closed unit disk for all $k \geq 2$.

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# Discretization: example 4

If

$$P_a(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} = \frac{j\omega_n/2}{s + j\omega_n} - \frac{j\omega_n/2}{s - j\omega_n}$$

then

$$\bar{P}_h(z) = \frac{1}{2}\left(\frac{1 - e^{-j\omega_n h}}{z - e^{-j\omega_n h}} + \frac{1 - e^{j\omega_n h}}{z - e^{j\omega_n h}}\right) = \frac{(1 - \cos(\omega_n h))(z+1)}{z^2 - 2\cos(\omega_n h)z + 1}.$$

If

- $\cos(\omega_n h) \neq \pm 1$, then $\bar{P}_h(z)$ has two poles at $e^{\pm j\omega_n h}$ and a zero at $-1$,
- $\cos(\omega_n h) = 1$, then $\bar{P}_h(z) = 0$,
- $\cos(\omega_n h) = -1$, then $\bar{P}_h(z) = 2/(z+1)$.

Thus, even the order of $P_c(s)$ is not always preserved under discretization.

# When order drops?

Consider

$$\bar{P}_h(z) = \sum_{i=1}^{n} \frac{\bar{b}_i}{z - \bar{a}_i} \quad \text{where } \bar{a}_i := e^{a_i h} \text{ and } \bar{b}_i := \frac{e^{a_i h} - 1}{a_i} b_i.$$

Two pathological cases, where the order of $\bar{P}_h(z)$ is smaller than $n$:

1. $\bar{a}_i = \bar{a}_j$, although $a_i \neq a_j$, which is equivalent to

$$e^{a_i h} = e^{a_j h} \iff a_i h = a_j h + j2\pi k \text{ for some } k \in \mathbb{Z} \setminus \{0\}$$

or $a_i - a_j = j2\omega_N k$.

2. $\bar{b}_i = 0$, although $b_i \neq 0$, which is equivalent to

$$(e^{a_i h} = 1) \wedge (a_i \neq 0) \iff a_i h = j2\pi k \text{ for some } k \in \mathbb{Z} \setminus \{0\}$$

or $a_i = j2\omega_N k$. But if the latter condition holds, then $\exists j \neq i$ such that $a_j = -j2\omega_N k$. Hence, $a_i - a_j = j2\omega_N(2k)$ and this case is covered by 1.

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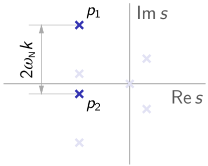
   or $a_i = \mathrm{j}2\omega_{\mathrm{N}} k$. But if the latter condition holds, then $\exists j \neq i$ such that $a_j = -\mathrm{j}2\omega_{\mathrm{N}} k$. Hence, $a_i - a_j = \mathrm{j}2\omega_{\mathrm{N}}(2k)$ and this case is covered by 1.

# Pathological sampling

We say that sampling is pathological with respect to $P_a$ if there are at least 2 poles of $P_a(s)$, say $p_1$ and $p_2$, such that

$$p_1 - p_2 = \mathrm{j}\frac{2\pi}{h}k = \mathrm{j}2\omega_N k \quad \Longleftarrow$$




for some $k \in \mathbb{Z} \setminus \{0\}$. If sampling is pathological, then

— some parts of dynamics of $P$ are not visible by the *discrete* controller.

But these parts don't disappear, they are just in the blind spot of $\bar{C}$, which cannot counteract anything caused by them (e.g. instability or oscillations).

As the minimum distance between poles for $h$ being pathological is $2\omega_N$,

— "sufficiently fast" sampling is never pathological.

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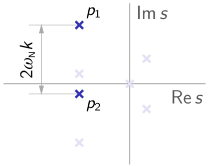


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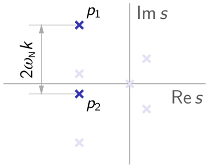


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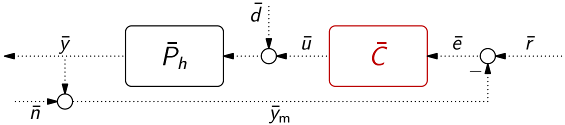
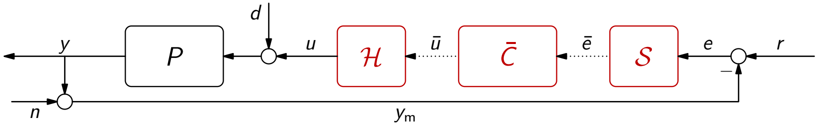
# Fundamental stability result

If sampling is pathological with respect to no unstable poles of $P_a(s)$, then $\bar{C}$ stabilizes




iff $\bar{C}$ stabilizes



# Outline

Analog redesign

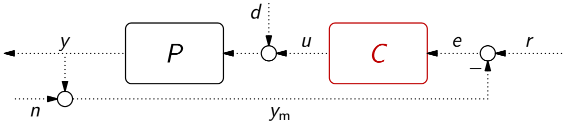
Discrete-time design

Discretized plant and its properties

Classical methods for discrete systems (mostly stability)

# Discrete unity feedback

We may now drop all signs of discretization and consider a discrete system,

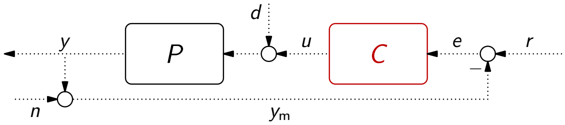



for a given

$$P(z) = \frac{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \frac{N_P(z)}{D_P(z)}$$

with $b_m \neq 0$ and $m \leq n$ (typically, $m = n - 1$).

# Internal stability




The closed-loop system is said to be

– internally stable if all Gang of Four transfer functions

$$\begin{bmatrix} S(z) & T_{\mathrm{d}}(z) \\ T_{\mathrm{c}}(z) & T(z) \end{bmatrix} := \frac{1}{1+P(z)C(z)} \begin{bmatrix} 1 \\ C(z) \end{bmatrix} \begin{bmatrix} 1 & P(z) \end{bmatrix}$$

are stable,

i.e. the corresponding transfer function is proper and has no poles outside the open unit disk $\mathbb{D}$.

Internal stability is the formalism helping to avoid unstable cancellations.

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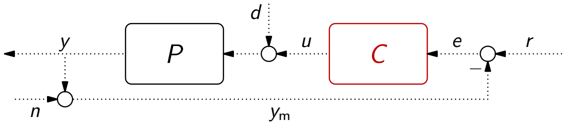


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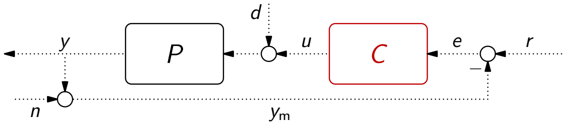
# Characteristic polynomial




If $C(z) = N_C(z)/D_C(z)$ is proper, then the closed-loop system is internally stable iff its characteristic polynomial

$$\chi_{cl}(z) = N_P(z)N_C(z) + D_P(z)D_C(z)$$

has all roots in $\mathbb{D}$ (such polynomials are known as Schur).

# Root locus

The technique is exactly as in the continuous-time case. Start with writing

$$\chi_{\text{cl}}(z) = 0 \quad \iff \quad -\frac{1}{k} = G_k(z),$$

where $k$ is a parameter to change, in $(0, \infty)$, and $G_k(z)$ is a proper transfer function. This representation is termed the root-locus form. All rules, which we know from the continuous-time analysis, apply then literally.

What changes is the meaning of the results, because

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For example, no asymptote remains in the stability area ($\mathbb{D}$), which implies that we can afford

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# Root locus: example

Consider again

$$P(z) = \frac{\overbrace{(h + \mathrm{e}^{-h} - 1)}^{>0} z + \overbrace{1 - (1 + h)\mathrm{e}^{-h}}^{>0}}{(z - 1)(z - \mathrm{e}^{-h})},$$

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$$z_{1,2} = e^{-h} + \frac{(1-e^{-h})\sqrt{1-e^{-h}}}{\sqrt{1-e^{-h}} \pm \sqrt{h}} = 1 \mp \frac{(1-e^{-h})\sqrt{h}}{\sqrt{1-e^{-h}} \pm \sqrt{h}}$$

with $e^{-h} < z_1 < 1$ (breakaway) and $z_2$ to the left of the zero (break-in) and $z_2 \le -1$ if $0 < h < 3.720754$ and $-1 < z_1 < 0$ if $h \ge 3.720754$.

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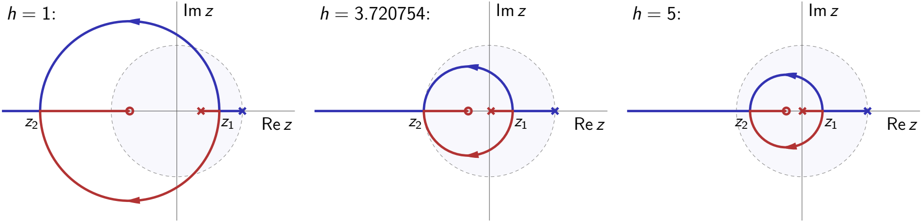
# Root locus: example (contd)

For various sampling periods,

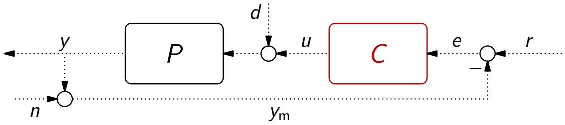



In all cases the system is stable only if $k$ is sufficiently small. In fact, for

$$0 < k < \frac{1 - e^{-h}}{1 - (h+1)e^{-h}}$$

which can be derived by the Jury stability criterion (discrete counterpart of the Routh criterion).

# Nyquist criterion




The same logic, as in the continuous-time case. The return difference

$$1 + L(z) = 1 + P(z)C(z) = \frac{\chi_{\text{cl}}(z)}{\chi_{\text{ol}}(z)}$$

still has open-loop poles as its poles and closed-loop poles as its zeros. The line of reasonings is then

1. define simple closed contour $\Gamma_z$ containing all $\mathbb{C} \setminus \bar{\mathbb{D}}_1$;
2. determine the mapping $\Gamma_L$ of $\Gamma_z$ by the loop gain $L(z)$;
3. count the number $\nu$ of *clockwise* encirclings of $(-1, 0)$ by $\Gamma_L$.

By the argument principle, $\nu = \#_{\text{clsd-loop unstable poles}} - \#_{\text{opn-loop unstable poles}}$.

# Nyquist contour

The contour encircling the unstable region $\mathbb{C} \setminus \bar{\mathbb{D}}_1$ is cumbersome. A simple workaround is to redefine $z \to 1/\lambda$. The unstable region in terms of $\lambda$ is $\mathbb{D}_1$ and the contour around it is the unit circle, $\Gamma_\lambda = \mathbb{T}$. Some observations:

- the (clockwise) $\Gamma_\lambda$ is mapped by $L(\lambda)$ as the frequency response $L(\mathrm{e}^{\mathrm{j}\theta})$ under increasing $\theta$ (the frequency for $\lambda$ is $-\theta$);
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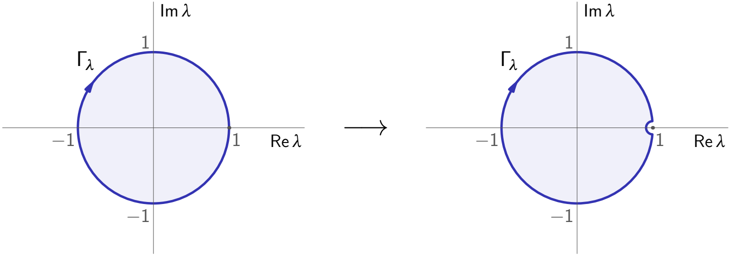


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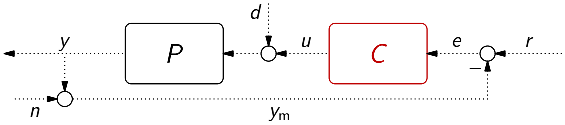
# Steady-state performance




Nothing changes vis-à-vis the continuous-time case, except replacing $s = 0$ with $z = 1$. For example, if $d[t] = \mathbb{1}[t]$, then by the Final Value Theorem

$$y_{ss} := \lim_{t\to\infty} y[t] = \lim_{z\to 1}(z-1)\,T_d(z)D(z) = \lim_{z\to 1}(z-1)\,T_d(z)\frac{z}{z-1} = T_d(1),$$

which is the static gain of (stable) $T_d$. Moreover,

$$y_{ss} = 0 \quad\iff\quad (P(1) = 0) \vee (|C(1)| = \infty),$$

where the latter condition requires an integral action in $C$.

# Transient performance and poles

Messier, e.g. discrete 1-order systems can exhibit oscillations and the role of zeros is not clear. So normally understood via discretized models.

Because $\mathbb{1} = \mathcal{H}_{\mathrm{ZOH}}\bar{\mathbb{1}}$, we have $\mathcal{S}_{\mathrm{idl}} G \mathbb{1} = \bar{G}_h \bar{\mathbb{1}}$, i.e. the

- step response of the discrete $\bar{G}_h$ is the sampled version of that of $G$.

If $G(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$ for $\zeta \in [0, 1]$, then $\bar{G}_h(z)$ has its poles at $z = e^{-\zeta\omega_n h} e^{\pm j\sqrt{1-\zeta^2}\omega_n h}$. Constant $\zeta$ and $\omega_n h$ contours are

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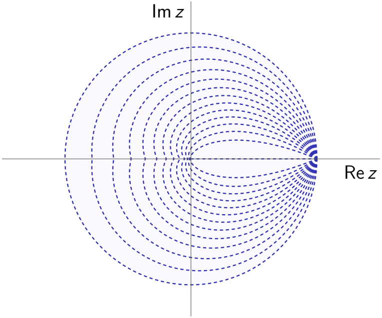


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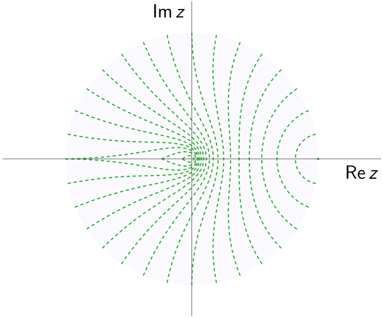

# Deadbeat control

Given $n$-order $P(z)$ and $n_c$-order $C(z)$. If the attained

$$\chi_{cl}(z) = z^{n+n_c}$$

(it is Schur), we say that the response is deadbeat. In this case we have

- finite duration of transients, of at most $n + n_c$ steps.

We know it as the FIR (finite impulse response) property, impossible in the finite-dimensional continuous-time LTI case. For example, consider

$$S(z) = \frac{1}{1 + P(z)C(z)} = \frac{b_{n+n_c} z^{n+n_c} + b_{n+n_c-1} z^{n+n_c-1} + \cdots + b_1 z + b_0}{\chi_{cl}(z)}$$
$$= b_{n+n_c} + b_{n+n_c-1} z^{-1} + \cdots + b_1 z^{1-n-n_c} + b_0 z^{-n-n_c}$$

Its impulse response

$$s[t] = b_{n+n_c}\delta[t] + \cdots + b_1 \delta[t-n-n_c+1] + b_0 \delta[t-n-n_c]$$

indeed ends after at most $n + n_c$ steps.

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# Deadbeat control: example

Consider

$$P(z) = \frac{h^2}{2}\frac{z+1}{(z-1)^2}$$

which is the discretized $1/s^2$. With $\chi_{\mathrm{cl}}(z) = z^3$ we have (see Lecture 5)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & h^2/2 & 0 \\ 1 & -2 & h^2/2 & h^2/2 \\ 0 & 1 & 0 & h^2/2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/4 \\ 5/(2h^2) \\ -3/(2h^2) \end{bmatrix}$$

so that

$$C(z) = \frac{2}{h^2}\frac{5z-3}{4z+3}$$

In this case

$$S(z) = \frac{(z-1)^2(4z+3)}{4z^3} \implies e[t] = \delta[t] - \frac{1}{4}\delta[t-1] - \frac{3}{4}\delta[t-2]$$

with $r = \mathbb{1}$ (for which $R(z) = \frac{z}{z-1}$ and $S(z)R(z) = 1 - \frac{1}{4}z^{-1} - \frac{3}{4}z^{-2}$).