Control Theory (035188)
lecture no. 9

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Optimization-based design

The underlying idea:

- express requirements in terms of a size of a cost function

to be minimized by the control law.

Important to remember that

- no performance index can reflect all our requirements.

We therefore shall use

- optimization as design tool,

rather than the control goal per se. Every controller is optimal with respect to some cost function. But not every optimal controller makes sense. Thus, optimization methods should be judged by

- simplicity of their solutions
- simplicity of tuning their properties via weighting functions
- byproducts (what do we get for granted)

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Outline

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Mathematical preliminaries

Linear-Quadratic Regulator (LQR) problem: formulation

Linear-Quadratic Regulator (LQR) problem: solution

LQR: solution properties

Positive (semi)definite block matrices

Consider

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}' & A_{21}' \\ A_{12}' & A_{22}' \end{bmatrix} = A'$$

with $A_{22} > 0$ (hence, $\det A_{22} \neq 0$). Given

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we have, exactly like in Lect. 6,

$$\begin{aligned} x'Ax &= x_1'A_{11}x_1 + x_1'A_{12}x_2 + x_2'A_{21}x_1 + x_2'A_{22}x_2 \\ &= x_1'(A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + x_1'A_{12}A_{22}^{-1}A_{21}x_1 + 2x_2'A_{21}x_1 + x_2'A_{22}x_2 \\ &= x_1'(A_{11} - A_{12}A_{22}^{-1}A_{21})x_1 + (x_2' + x_1'A_{12}A_{22}^{-1})A_{22}(x_2 + A_{22}^{-1}A_{21}x_1) \end{aligned}$$

Thus, if $A_{22} > 0$, then $A \geq 0 \iff A_{11} - A_{12}A_{22}^{-1}A_{21} \geq 0$.

CARE

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q = Q' \in \mathbb{R}^{n\times n}$, $R = R' \in \mathbb{R}^{m\times m}$, and $S \in \mathbb{R}^{n\times m}$ be given. The matrix equation

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

is called the continuous-time algebraic Riccati equation (CARE). Its solution $\bar{X} \in \mathbb{R}^{n\times n}$ is said to be stabilizing if the matrix

$$A_K := A - BR^{-1}(S' + B'\bar{X})$$

is Hurwitz[1]. The stabilizing $\bar{X}$, if exists, is unique and satisfies $\bar{X} = \bar{X}'$. We are interested in CAREs for

$$R > 0 \quad \text{and} \quad \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \geq 0$$

(the latter is equivalent to $Q - SR^{-1}S' \geq 0$).

[1] MATLAB: `icare(A,B,Q,R,S)` or `icare(A,B,Q,R)` if $S = 0$.

CARE: example

Let $A = a$, $B = b$, $Q = q \geq 0$, $R = r > 0$, and $S = 0$. The CARE reads

$$2a\bar{X} + q - \frac{b^2}{r}\bar{X}^2 = 0.$$

1. If $b = 0$, then the CARE becomes $2a\bar{X} + q = 0$ and $A_K = a$.
   - if $a \geq 0$, then no stabilizing solution exists
   - if $a < 0$, then $\bar{X} = -q/(2a) \geq 0$ is the stabilizing solution

2. If $b \neq 0$, then the CARE above is quadratic, solvable by

$$\bar{X} = \frac{ar \pm \sqrt{(a^2 r + b^2 q)r}}{b^2} \implies A_K = a - \frac{b^2}{r}\bar{X} = \mp\frac{\sqrt{a^2 r + b^2 q}}{\sqrt{r}}$$

and it is Hurwitz for "+", unless $a = q = 0$. Thus, the stabilizing

$$\bar{X} = \frac{ar + \sqrt{(a^2 r + b^2 q)r}}{b^2} \geq 0.$$

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$$\bar{X} = \frac{ar + \sqrt{(a^2 r + b^2 q) r}}{b^2} > 0.$$

CARE: existence

If

1. (A, B) is stabilizable,
2. $\begin{bmatrix} A - \mathrm{j}\omega I & B \\ Q & S \\ S' & R \end{bmatrix}$ has full column rank $\forall \omega \in \mathbb{R}$,

then the stabilizing solution $\bar{X}$ to

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0$$

exists and is such that $\bar{X} \geq 0$.

Remark Because

$$\begin{bmatrix} A - \mathrm{j}\omega I & B \\ Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} I & 0 & BR^{-1} \\ 0 & I & SR^{-1} \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A - BR^{-1}S - \mathrm{j}\omega I & 0 \\ Q - SR^{-1}S' & 0 \\ S' & R \end{bmatrix},$$

the second condition holds iff $(A - BR^{-1}S, Q - SR^{-1}S')$ has no pure imaginary unobservable modes. If $S = 0$, unobservable modes of (A, Q) are verified.

Outline

Mathematical preliminaries

Linear-Quadratic Regulator (LQR) problem: formulation

Linear-Quadratic Regulator (LQR) problem: solution

LQR: solution properties

The LQR problem

Given

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

find $u(t)$ that

- stabilizes the systems ($\lim_{t\to\infty} x(t) = 0$),
- minimizes

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t.$$

Assumptions:

$\mathcal{A}_1$: (A, B) is stabilizable

necessary for a stabilizing controller to exist

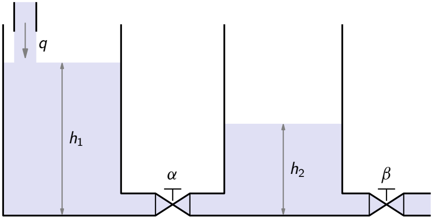
$\mathcal{A}_2$: $R > 0$ and $Q - SR^{-1}S' \geq 0$

guarantees that the optimal $u(t)$ is bounded and that $\mathcal{J} \geq 0$

$\mathcal{A}_3$: $(A - BR^{-1}S, Q - SR^{-1}S')$ has no pure imaginary unobservable modes

guarantees that the optimal $u(t)$ exists and is unique

Example: two-tank system from Lecture 8



With $\alpha = \beta = \sigma = 1$, $h_{\text{eq}} = \left[\begin{smallmatrix} 0.5 \\ 0.25 \end{smallmatrix}\right]$, and $q_{\text{eq}} = 0.5$, the linearized dynamics

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = -\begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$$

where $x_i = h_i - h_{i,\text{eq}}$ and $u = q - q_{\text{eq}}$.

Our goal is to

– regulate x from $x(0)$ to $\lim_{t\to\infty} x(t) = 0$ in a desired matter (regulator problem), which is effectively the set-point tracking of $h(t) \to h_{\text{eq}}$

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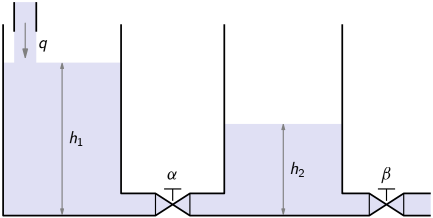

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Example: LQR formulations

We may consider several quantities to penalize:

1. level deviation from the steady state, $\gamma_i := \int_0^\infty x_i^2(t)\,\mathrm{d}t$, for $i = 1, 2$
2. control effort, $\gamma_u := \int_0^\infty u^2(t)\,\mathrm{d}t$
3. rate of level asynchronization, $\gamma_{12} := \int_0^\infty \bigl(\dot{x}_1(t) - 2\dot{x}_2(t)\bigr)^2\,\mathrm{d}t$

These goals are conflicting, e.g. faster response needs higher control effort, so the design is

– the art of tradeoffs,

whose essence is to seek for a right blend of a number of them via weights.

Example: building Q, R, and S

First, express each penalty via LQR quadratic forms. To this end, note that

$$x_i = \begin{bmatrix} C_i & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad u = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \quad \text{and} \quad \dot{x}_1 - 2\dot{x}_2 = C_{12} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

where $C_1 := \begin{bmatrix} 1 & 0 \end{bmatrix}$, $C_2 := \begin{bmatrix} 0 & 1 \end{bmatrix}$, $C_{12} := C_1 - 2C_2 = \begin{bmatrix} 1 & -2 \end{bmatrix}$. Hence,

$$\gamma_i = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} C_i'C_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t,$$

$$\gamma_u = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t,$$

and

$$\gamma_{12} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} A'C_{12}'C_{12}A & A'C_{12}'C_{12}B \\ B'C_{12}'C_{12}A & B'C_{12}'C_{12}B \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t.$$

Example: building Q, R, and S (contd)

Now we could easily mix them. For example, for every $r > 0$

$$\gamma_i + r\gamma_u = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \underbrace{\begin{bmatrix} C_i'C_i & 0 \\ 0 & r \end{bmatrix}}_{\geq 0} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t.$$

which is a standard LQR cost.

Likewise, for every $\lambda \in (0, 1)$

$$\lambda\gamma_i + (1-\lambda)\gamma_{12} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t,$$

where

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} \lambda C_i'C_i + (1-\lambda)A'C_{12}'C_{12}A & (1-\lambda)A'C_{12}'C_{12}B \\ (1-\lambda)B'C_{12}'C_{12}A & (1-\lambda)B'C_{12}'C_{12}B \end{bmatrix} \geq 0$$

because $Q - SR^{-1}S = \lambda C_i'C_i \geq 0$ (note that $R = C_{12}B = 1 \neq 0$).

Outline

Solution

Theorem

If $\mathcal{A}_{1\text{–}3}$ hold, then the unique stabilizing controller that minimizes

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t$$

is

$$u(t) = -R^{-1}(S' + B'\bar{X})x(t),$$

where $\bar{X} = \bar{X}' \geq 0$ is the stabilizing solution to the CARE

$$A'\bar{X} + \bar{X}A + Q - (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = 0.$$

The optimal cost is then

$$\mathcal{J}_{opt} = x_0'\bar{X}x_0.$$

Remark The optimal control law is a state feedback, with the gain $K = -R^{-1}(S' + B'\bar{X})$. The closed-loop "$A$" matrix $A + BK$ is Hurwitz because $\bar{X}$ is the stabilizing solution.

Proof

Denote $K := -R^{-1}(S' + B'\bar{X})$ and define

$$
\begin{aligned}
M_X &:= \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} + \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix} \\
&= \begin{bmatrix} Q + A'\bar{X} + \bar{X}A & S + \bar{X}B \\ S' + B'\bar{X} & R \end{bmatrix} = \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix}
\end{aligned}
$$

because

$$
Q + A'\bar{X} + \bar{X}A = (S + \bar{X}B)R^{-1}(S' + B'\bar{X}) = K'RK \quad \& \quad S' + B'\bar{X} = -RK
$$

for every Riccati solution $\bar{X}$. Thus,

$$
\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix} - \begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix} \qquad (\star)
$$

which is a key relation.

Proof (contd)

Now, the quadratic form in the cost $\mathcal{J}$ is

$$
\begin{aligned}
&\begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
&\quad = \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} -K' \\ I \end{bmatrix} R \begin{bmatrix} -K & I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
&\qquad\qquad - \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \left(\begin{bmatrix} \bar{X} \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + \begin{bmatrix} A' \\ B' \end{bmatrix} \begin{bmatrix} \bar{X} & 0 \end{bmatrix} \right) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\
&\quad = \big(u(t) - Kx(t)\big)' R \big(u(t) - Kx(t)\big) - x'(t)\bar{X}\dot{x}(t) - \dot{x}'(t)\bar{X}x(t) \\
&\quad = \big(u(t) - Kx(t)\big)' R \big(u(t) - Kx(t)\big) - \frac{\mathrm{d}}{\mathrm{d}t}\big(x'(t)\bar{X}x(t)\big)
\end{aligned}
$$

Proof (contd)

Thus,

$$\begin{aligned}\mathcal{J} &= \int_0^\infty \Big(\big(u(t) - Kx(t)\big)' R \big(u(t) - Kx(t)\big) - \frac{\mathrm{d}}{\mathrm{d}t}\big(x'(t)\bar{X}x(t)\big) \Big) \mathrm{d}t \\ &= \int_0^\infty \big(u(t) - Kx(t)\big)' R \big(u(t) - Kx(t)\big) \mathrm{d}t + x_0'\bar{X}x_0 - \lim_{t\to\infty} x'(t)\bar{X}x(t).\end{aligned}$$

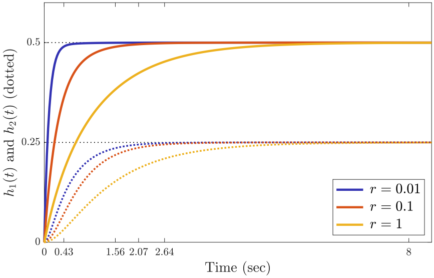
If $u(t)$ is stabilizing, then $\lim_{t\to\infty} x'(t)\bar{X}x(t) = 0$. Because $R > 0$,

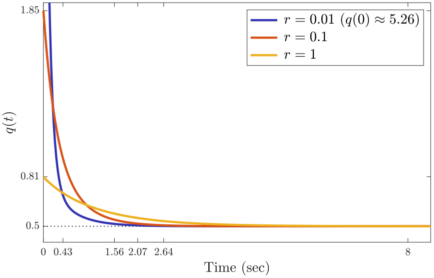
$$\mathcal{J} = \int_0^\infty \big(u(t) - Kx(t)\big)' R \big(u(t) - Kx(t)\big) \mathrm{d}t + x_0'\bar{X}x_0 \geq x_0'\bar{X}x_0$$

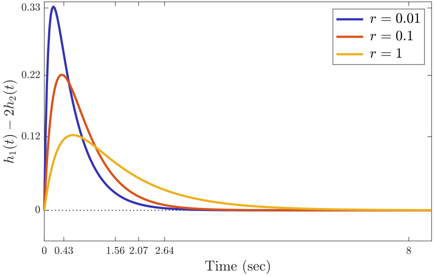
whenever the system is asymptotically stable and the equality is attained by $u(t) = Kx(t)$, which *is* stabilizing ($\bar{X}$ is the stabilizing solution). $\square$

Example: design 1

Consider $\mathcal{J} = \gamma_1 + r\gamma_u$ for various control energy weights $r > 0$. Results:







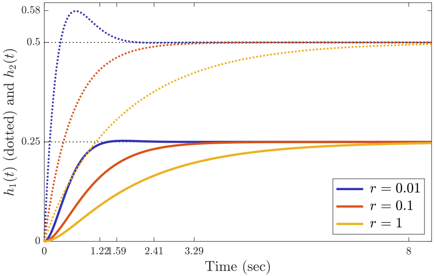
The gains and closed-loop eigenvalues are

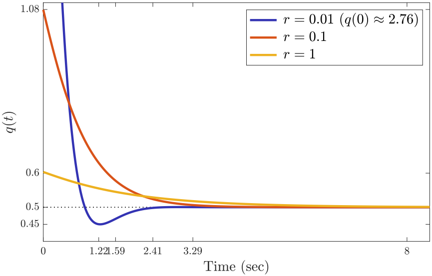
$$K = -\begin{bmatrix} 9.1264 & 0.7722 \\ \hdashline 2.4595 & 0.4841 \\ \hdashline 0.5316 & 0.1729 \end{bmatrix}$$

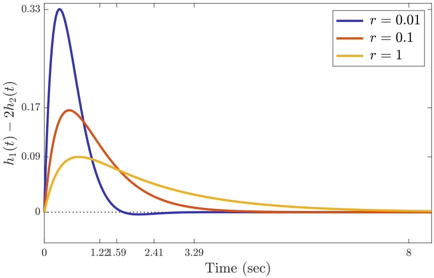
and $\{-10.15, -1.97\}$, $\{-3.75, -1.71\}$, and $\{-2.7, -0.83\}$.

Example: design 2

Consider $\mathcal{J} = \gamma_2 + r\gamma_u$ for various control energy weights $r > 0$. Results:







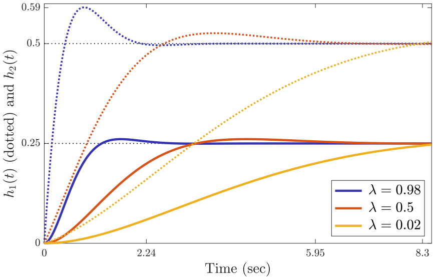
The gains and closed-loop eigenvalues are

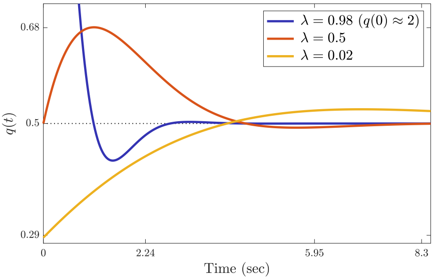
$$K = -\begin{bmatrix} 2.2057 & 4.6384 \\ \hdashline 0.6923 & 0.9320 \\ \hdashline 0.1350 & 0.1441 \end{bmatrix}$$

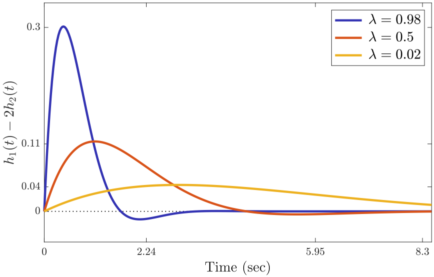
and $\{-2.6 \pm \mathrm{j}1.81\}$, $\{-2.15, -1.54\}$, and $\{-2.59, -0.55\}$.

Example: design 3

Consider $\mathcal{J} = \lambda\gamma_2 + (1-\lambda)\gamma_{12}$ for various weights $\lambda \in (0,1)$. Results:







The gains and closed-loop eigenvalues are

$$K = -\begin{bmatrix} 0.7417 & 4.5167 \\ \hdashline -1.5858 & 3.1716 \\ \hdashline -2.4655 & 4.0738 \end{bmatrix}$$

and $\{-1.87 \pm j1.87\}$, $\{-0.71 \pm j0.71\}$, and $\{-0.27 \pm j0.27\}$.

Guaranteed exponential decay

Consider now the cost

$$\mathcal{J}_\alpha = \int_0^\infty \mathrm{e}^{2\alpha t} \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t$$

for some $\alpha \geq 0$. To render $\mathcal{J}_\alpha$ finite,

- $x(t)$ and $u(t)$ must decay faster than $\mathrm{e}^{-\alpha t}$.

The variables $x_\alpha(t) := \mathrm{e}^{\alpha t} x(t)$ and $u_\alpha(t) := \mathrm{e}^{\alpha t} u(t)$ satisfy

$$\dot{x}_\alpha(t) = \alpha \mathrm{e}^{\alpha t} x(t) + \mathrm{e}^{\alpha t} \dot{x}(t) = (\alpha I + A) x_\alpha(t) + B u_\alpha(t)$$

and the cost in terms of them reads

$$\mathcal{J}_\alpha = \int_0^\infty \begin{bmatrix} x_\alpha'(t) & u_\alpha'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x_\alpha(t) \\ u_\alpha(t) \end{bmatrix} \mathrm{d}t.$$

This is the original LQR, modulo the substitution $A \to \alpha I + A$ in the CARE.

Guaranteed exponential decay

Consider now the cost

$$\mathcal{J}_\alpha = \int_0^\infty \mathrm{e}^{2\alpha t} \begin{bmatrix} x'(t) & u'(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t$$

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This is the original LQR, modulo the substitution $A \to \alpha I + A$ in the CARE.

Guaranteed exponential decay (contd)

Thus, the control law minimizing $\mathcal{J}_\alpha$ is

$$u(t) = K_\alpha x(t),$$

where $K_\alpha := -R^{-1}(S' + B'\bar{X}_\alpha)$ and $\bar{X}_\alpha = \bar{X}_\alpha' \geq 0$ is the stabilizing solution (i.e. such that $\alpha I + A + BK_\alpha$ is Hurwitz) to

$$(\alpha I + A)'\bar{X}_\alpha + \bar{X}_\alpha(\alpha I + A) + Q - (S + \bar{X}_\alpha B)R^{-1}(S' + B'\bar{X}_\alpha) = 0.$$

Moreover, because $\lambda \in \operatorname{spec}(\alpha I + A) \iff \lambda - \alpha \in \operatorname{spec}(A)$,

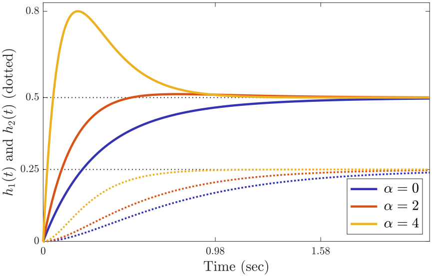
– $\alpha I + A + BK_\alpha$ is Hurwitz iff $\operatorname{spec}(A + BK_\alpha) \in \{s \in \mathbb{C} \mid \operatorname{Re} s < -\alpha\}$

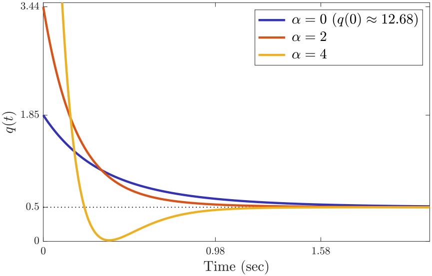
and we end up with the property that

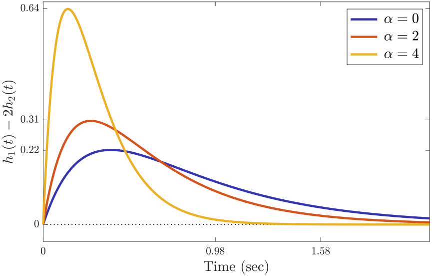
– minimizing $\mathcal{J}_\alpha$ ensures that closed-loop eigenvalues have $\operatorname{Re} \lambda_i < -\alpha$.

Example: design 1 with various α's

Consider $\mathcal{J} = \gamma_1 + r\gamma_u$ for various decay guarantees $\alpha \geq 0$. Results:







The gains and closed-loop eigenvalues are

$$K = -\begin{bmatrix} 2.4595 & 0.4841 \\ \hdashline 4.8730 & 2 \\ \hdashline 11.4128 & 25.8879 \end{bmatrix}$$

and $\{-3.75, -1.71\}$, $\{-5.59, -2.28\}$, and $\{-8.6, -5.72\}$, to the left of $-\alpha$, indeed.

So beware, "more stable" $\not\Longrightarrow$ "better".

Outline

Scaling the cost $\mathcal{J}$

Optimal solutions for

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \quad \text{and} \quad \gamma \begin{bmatrix} Q & S \\ S' & R \end{bmatrix}$$

coincide. Indeed, the CARE for the latter reads

$$A'\bar{X}_\gamma + \bar{X}_\gamma A + \gamma Q - (\gamma S + \bar{X}_\gamma B)(\gamma R)^{-1}(\gamma S' + B'\bar{X}_\gamma) = 0$$

or, equivalently,

$$A'(\gamma^{-1}\bar{X}_\gamma) + (\gamma^{-1}\bar{X}_\gamma)A + Q - (S + (\gamma^{-1}\bar{X}_\gamma)B)R^{-1}(S' + B'(\gamma^{-1}\bar{X}_\gamma)) = 0$$

Hence, $\bar{X}_\gamma = \gamma\bar{X}$ and

$$K_\gamma = -(\gamma R)^{-1}(\gamma S' + B'\bar{X}_\gamma) = -R^{-1}(S' + B'\bar{X})$$

is independent of γ. but the optimal cost is proportional to γ

CARE manipulations

By $(\star)$,

$$
\begin{aligned}
\begin{bmatrix} B'(-sI-A')^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix}
&= (I - B'(-sI-A')^{-1}K')R(I - K(sI-A)^{-1}B) \\
&\quad + B'(-sI-A')^{-1}\bar{X}\bigl(A(sI-A)^{-1} + I\bigr)B \\
&\quad + B'\bigl((-sI-A')^{-1}A' + I\bigr)\bar{X}(sI-A)^{-1}B
\end{aligned}
$$

The last two terms above cancel each other, because (see Lect. 6, Slide 24) $A(sI-A)^{-1} + I = s(sI-A)^{-1}$ and $(-sI-A')^{-1}A' + I = -s(-sI-A')^{-1}$. Thus, we end up with the relation

$$
\begin{bmatrix} B'(-sI-A')^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} (sI-A)^{-1}B \\ I \end{bmatrix}
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$$

CARE manipulations: $m = 1$ and $S = 0$

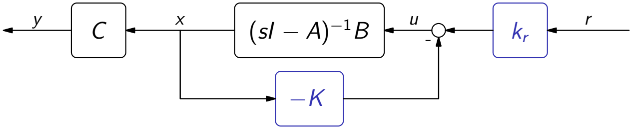
Denote $r = R$ (to emphasize that it is scalar) and rewrite the relation as

$$(1 + L_{\text{sf}}(-s))(1 + L_{\text{sf}}(s)) = 1 + \frac{1}{r} B'(-sI - A')^{-1} Q (sI - A)^{-1} B, \qquad (\star\star)$$

where

$$L_{\text{sf}}(s) := -K(sI - A)^{-1}B$$

is the loop transfer function in



Return difference equality ($m = 1$ and $S = 0$)

If $s = \mathrm{j}\omega$, then $L_{\mathsf{sf}}(-\mathrm{j}\omega) = \overline{L_{\mathsf{sf}}(-\mathrm{j}\omega)}$ and Eqn. ($\star\star$) reads

$$|1 + L_{\mathsf{sf}}(\mathrm{j}\omega)|^2 = 1 + \frac{1}{r} B'(-\mathrm{j}\omega I - A')^{-1} Q(\mathrm{j}\omega I - A)^{-1} B$$

known as the return difference equality.

Note that $Q \geq 0$ implies that

$$B'(-\mathrm{j}\omega I - A')^{-1} Q(\mathrm{j}\omega I - A)^{-1} B = [(\mathrm{j}\omega I - A)^{-1} B]' Q [(\mathrm{j}\omega I - A)^{-1} B] \geq 0$$

Hence, the return difference equality ensures that

$$|1 + L_{\mathsf{sf}}(\mathrm{j}\omega)|^2 \geq 1, \quad \forall \omega \in \mathbb{R}.$$

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known as[2] the return difference equality.

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Hence, the return difference equality ensures that

$$|1 + L_{\mathrm{sf}}(\mathrm{j}\omega)|^2 \geq 1, \quad \forall \omega \in \mathbb{R}.$$

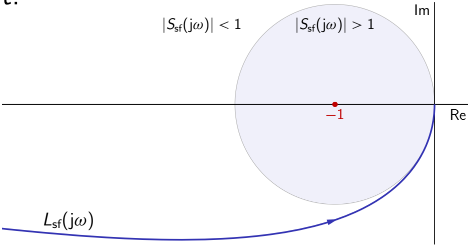
[2] $1 + L_{\mathrm{sf}}(s)$ is the return-difference t.f., remember it from the Nyquist criterion proof.

Return difference equality: implication

The inequality

$$|1 + L_{\mathrm{sf}}(\mathrm{j}\omega)|^2 \geq 1 \iff |S_{\mathrm{sf}}(\mathrm{j}\omega)| \leq 1, \quad \forall \omega \in \mathbb{R}$$

implies that the polar plot of $L_{\mathrm{sf}}(\mathrm{j}\omega)$ is outside the open unit disk centered at the critical point:

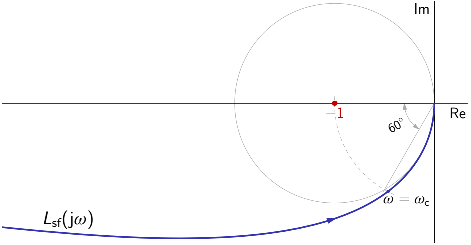


Remark There is no conflict with Bode's sensitivity integral

$$\int_0^\infty \ln|S_{\mathrm{sf}}(\mathrm{j}\omega)| \mathrm{d}\omega = \pi \sum_{i=1}^{m} \mathrm{Re}\, p_i.$$

It just means that the pole excess of $L_{\mathrm{sf}}(s)$ is exactly 1 ($L_{\mathrm{sf}}(s)$ is strictly proper).

Stability margins of LQR ($m = 1$ and $S = 0$)



An immediate consequence is that LQR optimal loop guarantees

- gain margin $\mu_g = \infty$ $\quad$ $L_{sf}(j\omega)$ does not cross the real axis in $[-1, 0)$
- phase margin $\mu_{ph} \geq 60°$ $\quad$ $L_{sf}(j\omega)$ is further from -1 than $-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

whenever the LQR parameter $S = 0$.