Control Theory (00350188)
lecture no. 5

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Outline

Nobody's perfect

In other words, any
- mathematical model is merely a (more / less accurate) approximation

of the real world.

Modeling uncertainty in control systems

Modeling uncertainty (errors, mismatches) are caused by
- linearization
- unmodeled (high-frequency) dynamics
- parametric drifts
- element failures
- ...

Modeling uncertainty: DC motor

Consider a DC motor, modeled (from input voltage to shaft velocity) as

$$P(s) = \frac{K_{\mathrm{m}}\,\mathrm{e}^{-\tau s}}{(Ls+R)(Js+f)+K_{\mathrm{m}}^2}, \tag{1}$$

where K_{m} is motor constant (= back emf const), R is armature resistance, L is armature inductance, J is load inertia, f is load friction, and the delay τ reflects potential control channel lags (like in digital implementation).

If L and τ are very small, they are neglected and working model becomes

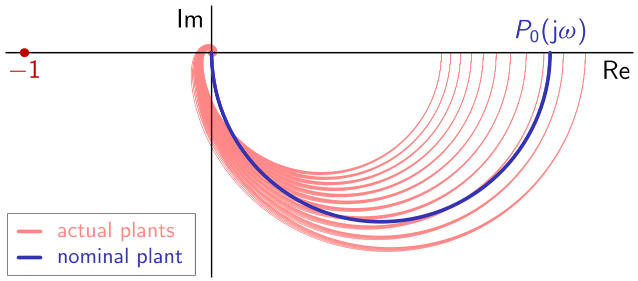
$$P(s) = \frac{K_{\mathrm{m}}}{R(Js+f)+K_{\mathrm{m}}^2}, \tag{2}$$

which is an approximation of (1) (which, in turn, is an approximation of the real DC motor).

Moreover, load inertia J might get changed and resistance R is sensitive to thermal conditions (motor heating) and thus also might get changed.

Modeling uncertainty: DC motor (contd)

Possible frequency responses (for some grid over R and J) look then as



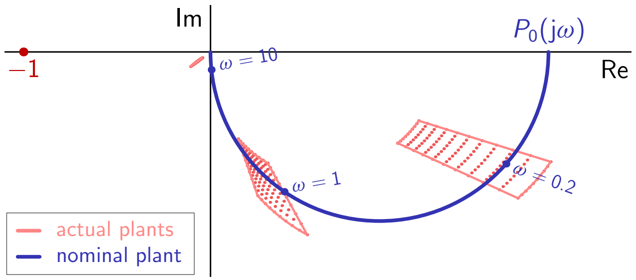
where

- $K_{\mathrm{m}} = 0.0302$, $f = 0.05$, $R_0 = 0.316$, and $J_0 = 0.1$ — nominal values
- $0.9R_0 \le R \le 1.5R_0$ and $0.8J_0 \le J \le 1.2J_0$ — uncertain values
- $L = 8 \cdot 10^{-5}$ and $\tau = 0.1$ — unmodeled dynamics

and the nominal plant $P_0(s) = \dfrac{K_{\mathrm{m}}}{R_0(J_0 s+f)+K_{\mathrm{m}}^2}$.

Modeling uncertainty: DC motor (contd)

Thus, at each frequency, frequency response is a region rather than a point:



Frequency-domain modeling

It then does make sense to describe plant frequency response $P(j\omega)$ at each frequency not as a complex number, but rather as set of its possible values

$$P(j\omega) \in \mathfrak{P}_\omega$$

where $\mathfrak{P}_\omega \subset \mathbb{C}$ is some set for each $\omega \in \mathbb{R}$.

The choice of $\mathfrak{P}_\omega$ is conceptually nontrivial as

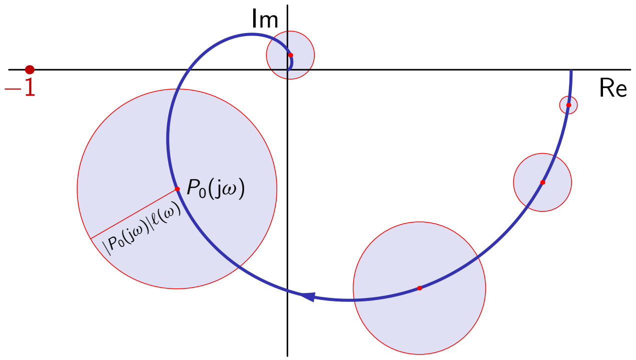
- accurate $\mathfrak{P}_\omega$ are complicated and hard to deal with in control design,
- easily handleable $\mathfrak{P}_\omega$ are typically conservative.

In this course (as frequently in engineering), we

- sacrifice accuracy for simplicity.

Multiplicative unstructured uncertainty

Idea: describe $\mathfrak{P}_\omega$ as disks in the Nyquist plane around some nominal plant



These disks verify $|P(j\omega) - P_0(j\omega)| \le \ell(\omega)|P_0(j\omega)|$, where

- P_0 is nominal plant (our design model) and
- $\ell(\omega) \ge 0$ is multiplicative uncertainty radius.

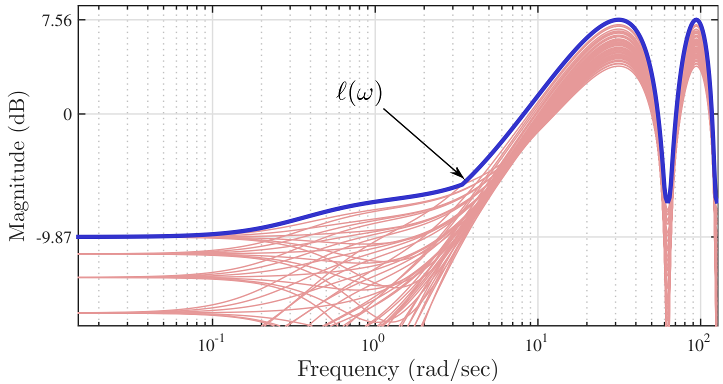
In other words, in this case $\mathfrak{P}_\omega = \left\{ P(j\omega) : \left|\frac{P(j\omega)}{P_0(j\omega)} - 1\right| \le \ell(\omega) \right\}$.

DC motor: finding $\ell(\omega)$

To find $\ell(\omega)$, the following steps can be followed:

1. plot $\left|\frac{P(j\omega)}{P_0(j\omega)} - 1\right|$ for different $R \in [0.9R_0, 1.5R_0]$ and $J \in [0.8J_0, 1.2J_0]$;
2. find maximum for every frequency, this is $\ell(\omega)$.

We get:

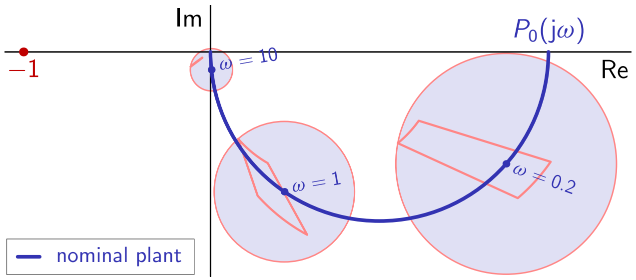


Typically (like in this case) uncertainty radii

- $\ell(\omega)$ are smaller at low frequencies / larger at high frequencies.

DC motor: finding $\ell(\omega)$ (contd)

Now, at each frequency, frequency response is a disk rather than a point:

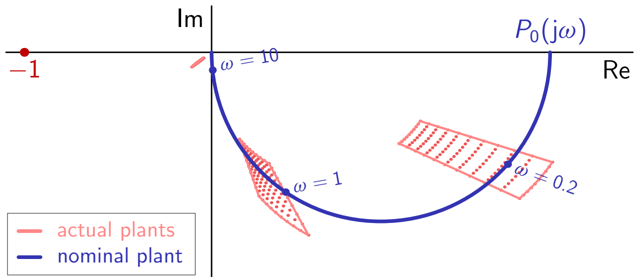


Disks fully cover actual uncertainty regions, hence

- whatever we can guarantee for disks, holds for the actual motor as well
- but not the other way round (conservatism)

Conservatism may be reduced if a "better" nominal plant $P_0(s)$ is chosen.

Choice of nominal plant



Might be highly nontrivial, some possible directions:

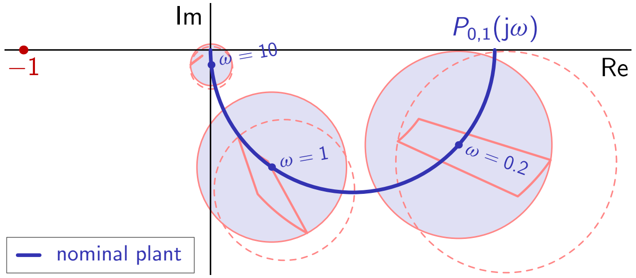
- place $P_0(j\omega)$ at the center of the minimal covering circle at each ω
  (might result in very high-order $P_0(s)$, whose handling is too complicated)
- fixed-order "physical" $P_0(s)$ with parameters in the middle of ranges
  (might not produce the tightest disks, see below)
- fixed-order $P_0(s)$ producing tightest disk
  (immensely complicated, depends on the choice of tightness measure, etc)

DC motor: choice of nominal plant

Let's pick

$$P_0(s) = P_{0,1}(s) := \frac{K_m}{R_1(J_1 s + f) + K_m^2},$$

with $R_1 = 1.2R_0 = 0.3792$ and $J_1 = J_0 = 0.1$ chosen as the median of the corresponding intervals $[0.9R_0, 1.5R_0]$ and $[0.8J_0, 1.2J_0]$. This results in



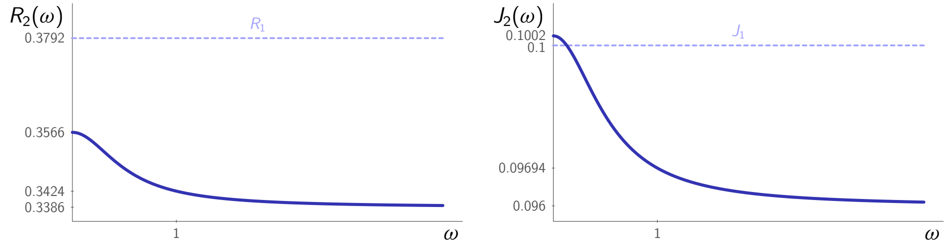
which is not necessarily better than the previous attempt . . .

DC motor: choice of nominal plant (contd)

Consider the following class of nominal plants:

$$P_0(s) = P_{0,2}(s) := \frac{K_m}{R_2(J_2 s + f) + K_m^2}.$$

Let's aim at placing $P_{0,2}(j\omega)$ to the center of the *minimal covering circle* of the uncertainty region of $P(j\omega)$ defined by (2) with interval parameters[1] at each ω. Even in this stripped down setting, solution is frequency dependent:

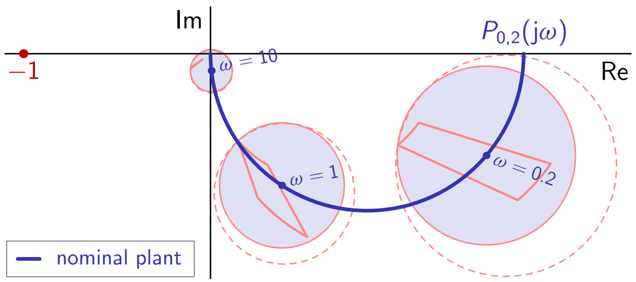


and not always close to the median values $R_1 = 0.3792$ and $J_1 = 0.1$.

[1]Note that this $P(s)$ is not the "real" motor in (1)!

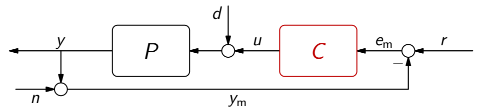
DC motor: choice of nominal plant (contd)

For example, if $R_2 = R_2(1) = 0.3424$ and $J_2 = J_2(1) = 0.09694$, we have:



Note that even for $\omega = 1$ we do not have the *minimal* covering circle. This is due to the addition of $L \neq 0$ and $h \neq 0$ to the "real" model.

Multiplicative uncertainty and controller



Let $P(j\omega) \in \mathfrak{P}_\omega$, where

$$\mathfrak{P}_\omega = \left\{ P(j\omega) : \left| \frac{P(j\omega)}{P_0(j\omega)} - 1 \right| \le \ell(\omega) \right\}.$$

Then $L(j\omega) = P(j\omega)C(j\omega) \in \mathfrak{L}_\omega$, where

$$\mathfrak{L}_\omega = \left\{ L(j\omega) : \left| \frac{L(j\omega)}{L_0(j\omega)} - 1 \right| \le \ell(\omega) \right\}, \qquad L_0(j\omega) := P_0(j\omega)C(j\omega).$$

Thus,

- loop multiplicative uncertainty radius does not depend on controller.

Outline

Modeling uncertainty

Robust stability

Robust performance

Pole placement

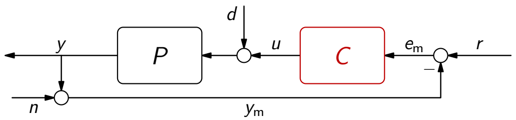
Robustness

The ability of a control system to cope with modeling uncertainty (that is, to preserve required characteristics despite uncertainty) is called robustness.

We may talk about

- robust stability
  (relatively simple problem, we'll discuss it in some technical details)
- robust performance
  (normally, much harder problem, we'll only see a flavor of this kind of problems)

Robust stability

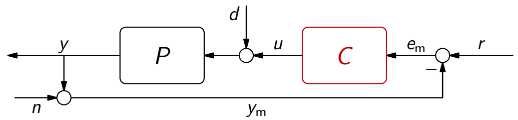


Let P be such that $P(j\omega) \in \mathfrak{P}_\omega$. We say that the

- closed-loop system is robustly stable if it is stable for all $P(j\omega) \in \mathfrak{P}_\omega$.

If the system is robustly stable, we say that C robustly stabilizes it.

Robust stability for multiplicative plant uncertainty



Theorem

Let uncertainty be described as

$$\mathfrak{P}_\omega = \left\{ P(j\omega) : \left| \frac{P(j\omega)}{P_0(j\omega)} - 1 \right| \le \ell(\omega) \right\}$$

and all P in this class share the same unstable poles. A controller C then robustly stabilizes the system iff

1. *C stabilizes nominal plant P_0 and*
2. *$|T_0(j\omega)| < \frac{1}{\ell(\omega)}$, for all ω.*[2]

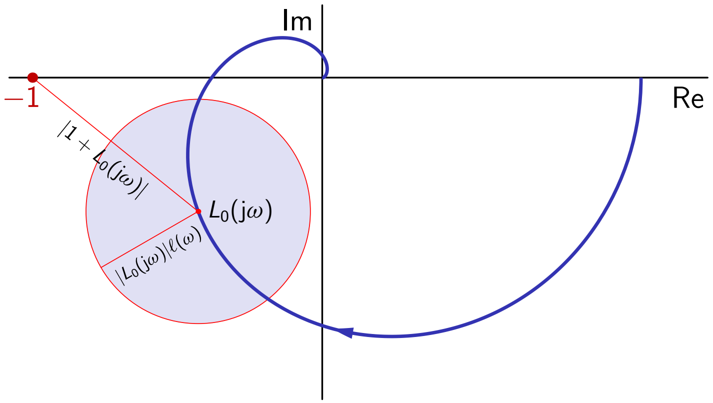
[2] $T_0(s) = L_0(s)/(1 + L_0(s))$ is the nominal complementary sensitivity transfer function.

Robust stability for multiplicative plant uncertainty: proof

When nominal system is stable, we only need to

- ensure that the critical point does not belong to $\mathfrak{L}_\omega$ for all ω.

The result then follows by straightforward geometry:



PI controller design for DC motor

Let's now design a PI controller $C(s) = \frac{k_p(s+k_i)}{s}$ for $P_0(s)$. The closed-loop characteristic polynomial $\chi_{cl}(s) = J_0R_0s^2 + (K_m^2 + K_mk_p + fR_0)s + K_mk_pk_i$, so the system is stable iff

$$k_pk_i > 0 \quad \text{and} \quad k_p > -(K_m + fR_0/K_m).$$

We then choose

- k_i as the maximal gain for which $|T_0(j\omega)|$ monotonically decreases and
- k_p to achieve a given crossover frequency ω_c.

This criterion produces unique coefficients as functions of ω_c:

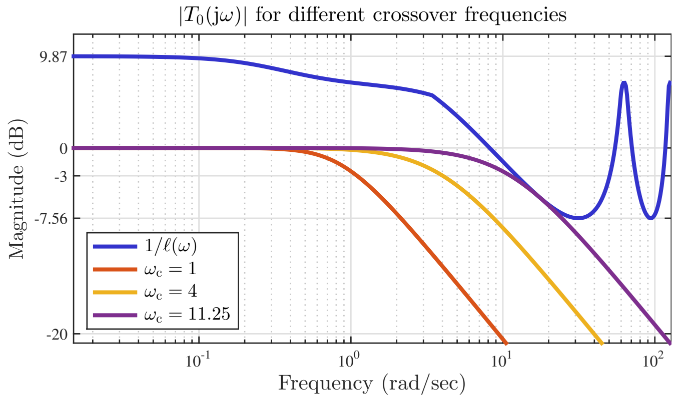
$$k_p = \frac{1.0467\sqrt{(\omega_c^2 + 0.1398)(\omega_c^2 + 0.4195)} - 0.074}{\omega_c^2 + 0.2797},$$

$$k_i = \frac{\omega_c(0.5289\sqrt{(\omega_c^2 + 0.1398)(\omega_c^2 + 0.4195)} + 0.1398\omega_c)}{\omega_c\sqrt{(\omega_c^2 + 0.1398)(\omega_c^2 + 0.4195)} - 0.074}$$

(positive iff $\omega_c > 0.24068$, smaller ω_c's yield undershoot with this strategy).

Robust stability of PI controlled DC motor

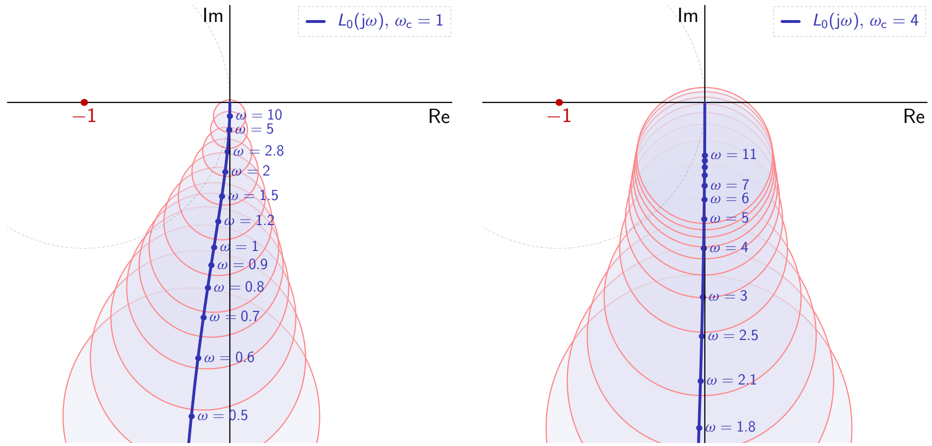
Comparing $|T_0(j\omega)|$ with $1/\ell(\omega)$ for different ω_c, we get:



Thus, the system is robustly stable only if $\omega_c < 11.25$.

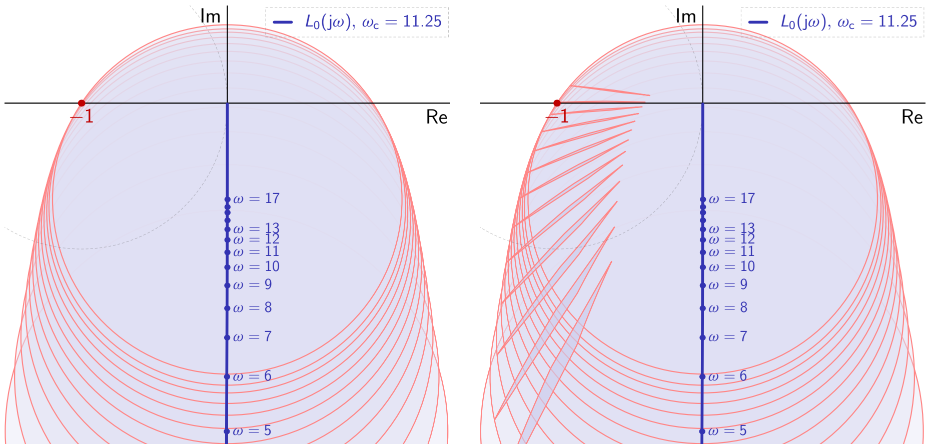
Robust stability of PI controlled DC motor (contd)

The same can be seen through uncertainty disks in the Nyquist plane:



Robust stability of PI controlled DC motor (contd)

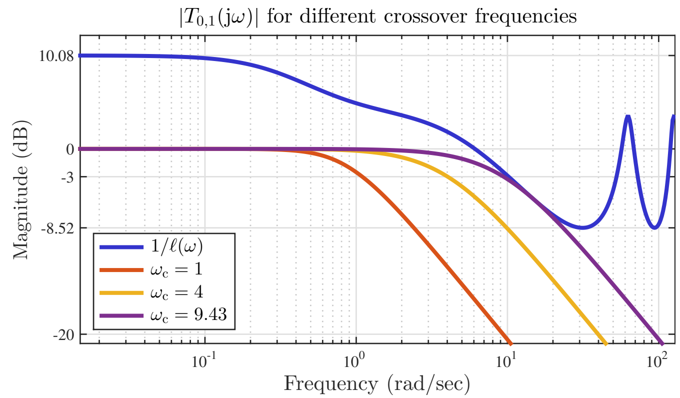
With the borderline ω_c the disk touches the critical point (at $\omega = 17.05$)



We see that the actual uncertainty areas (on the right) are also very close to the critical point. This means that the obtained bound on ω_c is virtually non-conservative.

Robust stability of PI controlled DC motor (contd)

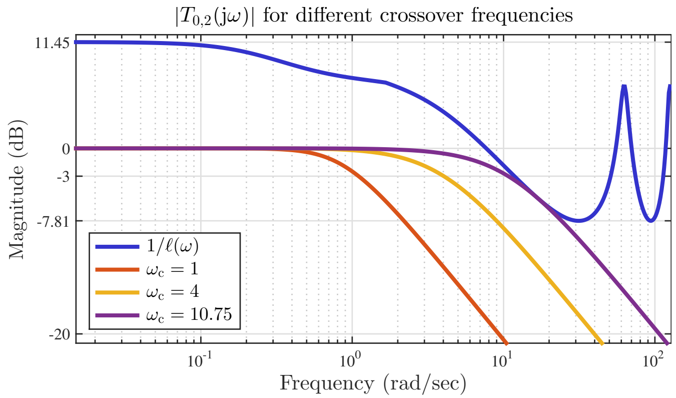
With $P_{0,1}$ as the nominal plan (median nominal R and J), the result is



The largest attainable ω_c is less than 84% of what we obtained with T_0.

Robust stability of PI controlled DC motor (contd)

With $P_{0,2}$ as the nominal plan (best fit for $\omega = 1$), the result is



The largest attainable ω_c is less than 96% of what we obtained with T_0.

These two examples illustrate the fact that the

- choice of "the best" nominal model (design model) is highly nontrivial

Bandwidth limitations due to robust stability

Since $\ell(\omega)$ is typically larger at high frequencies, the condition

$$|T_0(j\omega)| < \frac{1}{\ell(\omega)}$$

imposes limitations on the achievable closed-loop bandwidth[3] ω_b.

[3] And, consequently, on the loop crossover frequency ω_c.

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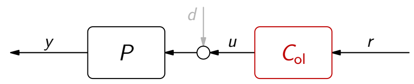
Open-loop control



Let y_0 be the response of the nominal plant P_0. Then

$$y_0 = P_0 C_{ol} r = T_{ref} r \quad \text{and hence} \quad y = P C_{ol} r = \frac{P}{P_0} T_{ref} r$$

Thus, the normalized control mismatch

$$\frac{|Y(j\omega) - Y_0(j\omega)|}{|R(j\omega)|} = \left| \frac{P(j\omega)}{P_0(j\omega)} - 1 \right| |T_{ref}(j\omega)| \le \ell(\omega) \, |T_{ref}(j\omega)|$$

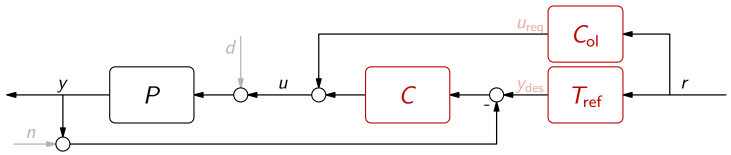
and in the frequency range where $T_{ref}(j\omega) \approx 1$ (good tracking performance)

– control mismatch equals the uncertainty radius of the plant.

In other words,

– open-loop control has no effect on uncertainty.

2DOF control: reference response



With $C_{ol} = T_{ref}/P_0$,

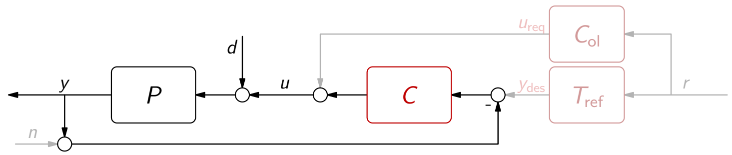
$$y_0 = T_{ref} r \quad \text{and} \quad y = \frac{P}{1+PC} \frac{T_{ref}}{P_0} r + \frac{PC}{1+PC} T_{ref} r = \frac{T}{T_0} T_{ref} r$$

and the normalized control mismatch

$$\frac{|Y(j\omega) - Y_0(j\omega)|}{|R(j\omega)|} = \left| \frac{T(j\omega)}{T_0(j\omega)} - 1 \right| |T_{ref}(j\omega)|$$

depends now upon the uncertainty radius of the complementary sensitivity transfer function (rather than of the plant itself).

2DOF control: disturbance response



In this case,

$$y_0 = T_{d,0} d \quad \text{and} \quad y = T_d d = \frac{T}{T_0} T_{d,0} d,$$

where $T_{d,0} = P_0/(1 + P_0 C)$ is the nominal disturbance sensitivity. Then the normalized control mismatch

$$\frac{|Y(j\omega) - Y_0(j\omega)|}{|D(j\omega)|} = \left| \frac{T(j\omega)}{T_0(j\omega)} - 1 \right| |T_{d,0}(j\omega)|$$

also depends upon the uncertainty radius of the complementary sensitivity transfer function. What can we say about it ?

Disks mapping under feedback

It can be shown that if the robust stability condition $|T_0(j\omega)| < \frac{1}{\ell(\omega)}$ holds,

$$\left|\frac{T(j\omega)}{T_0(j\omega)} - 1\right| \le \ell_{T_0}(\omega) := \frac{\ell(\omega)}{|1+L_0(j\omega)| - \ell(\omega)|L_0(j\omega)|} = \frac{\ell(\omega)|S_0(j\omega)|}{1-\ell(\omega)|T_0(j\omega)|}$$

where $S_0(s) = 1 - T_0(s)$ is the nominal sensitivity function.

Remark: As a matter of fact, a disk in the L-plane with the center at L_0 is transformed into a T-plane disk, whose center is not T_0, but rather

$$T_1(j\omega) = \frac{|1-\ell^2(\omega)T_0(j\omega)|^2}{1-\ell^2(\omega)|T_0(j\omega)|^2}\frac{T_0(j\omega)}{1-\ell^2(\omega)T_0(j\omega)}, \quad \text{with } \ell_{T_1}(\omega) = \frac{\ell(\omega)|S_0(j\omega)|}{|1-\ell^2(\omega)T_0(j\omega)|}$$

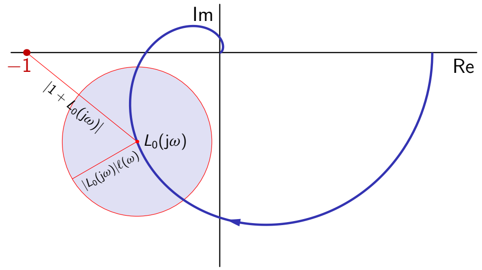
(normalized radius). This disk is always *contained* in the disk around T_0 defined above and its non-normalized radius is

$$\ell_{T_1}(\omega)|T_1(j\omega)| = \frac{\ell(\omega)|S_0(j\omega)|}{1-\ell(\omega)|T_0(j\omega)|}\frac{|T_0(j\omega)|}{1+\ell(\omega)|T_0(j\omega)|} \le \ell_{T_0}(\omega)|T_0(j\omega)|.$$

But the use of T_1 as the nominal T for controller design might not be easy (complexity).

Disks mapping by feedback: what does it mean?

Remember this:



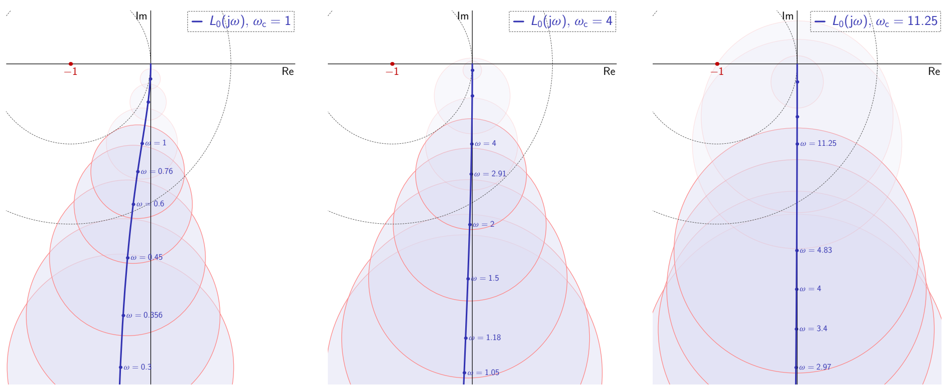
The relation $\ell_{T_0}(\omega) = \frac{\ell(\omega)}{|1+L_0(j\omega)|-\ell(\omega)|L_0(j\omega)|}$ effectively says then that

- feedback reduces uncertainty level at frequencies ω, where the disk $\mathfrak{L}_\omega$ is at a distance of at least 1 from the critical point and that
- the further $\mathfrak{L}_\omega$ from the critical point $-1+j0$, the lower the uncertainty level in $T(j\omega)$ is (provided we pick T_0 as the nominal T, of course)

Also, the relation $\ell_{T_0}(\omega) = \frac{\ell(\omega)|S_0(j\omega)|}{1-\ell(\omega)|T_0(j\omega)|}$ implies that

- uncertainty is always aggravated by feedback at ω's where $|S_0(j\omega)| > 1$

Robust performance of PI controlled DC motor



$\ell_{T_0}(\omega) < \ell(\omega)$, $\forall\omega < 0.76$
$\ell_{T_0}(\omega) < \frac{\ell(\omega)}{2}$, $\forall\omega < 0.356$

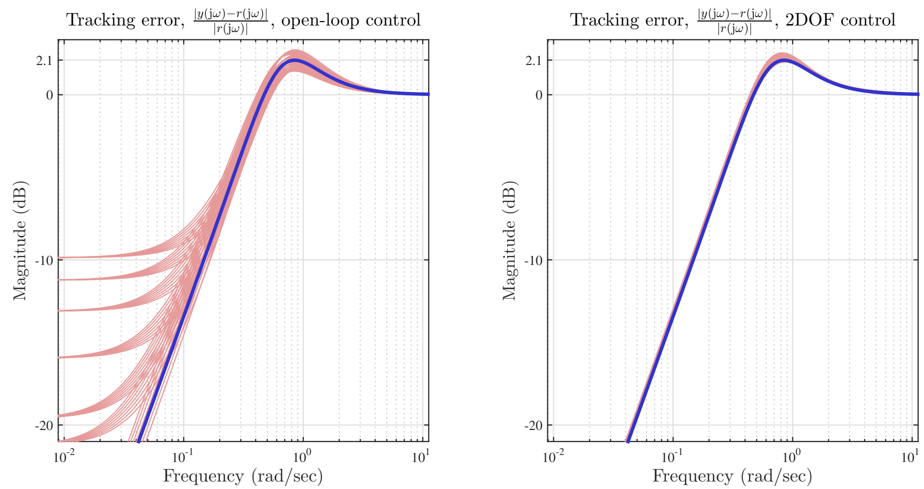
$\ell_{T_0}(\omega) < \ell(\omega)$, $\forall\omega < 2.91$
$\ell_{T_0}(\omega) < \frac{\ell(\omega)}{2}$, $\forall\omega < 1.18$

$\ell_{T_0}(\omega) < \ell(\omega)$, $\forall\omega < 4.83$
$\ell_{T_0}(\omega) < \frac{\ell(\omega)}{2}$, $\forall\omega < 2.97$
$\ell_{T_0}(\omega) \to \infty$ at $\omega \approx 17.046$

In all 3 cases $\ell_{T_0}(0) = 0$, which is the result of the use of an integral action in the controller (as then $S_0(0) = 0$, while $\ell(0)|T_0(0)| = 0.2676 < 1$).

Robust performance: DC motor comparison

Let us choose $T_{\rm ref}(s) = \frac{\omega_N^2}{s^2+\sqrt{2}\omega_N s+\omega_N^2}$ with $\omega_N = \frac{2}{3}$ and compare 2 strategies discussed in the beginning of this section. The feedback controller is the PI discussed above with $\omega_c = 4$. Advantages of feedback are clear:



Outline

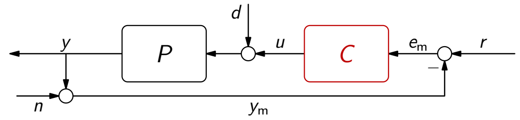
Modeling uncertainty

Robust stability

Robust performance

Pole placement

Modal analysis: idea



Express closed-loop performance requirements

– in terms of the location of closed-loop poles

which are roots of the characteristic polynomial

$$\chi_{cl}(s) := N_P(s)N_C(s) + D_P(s)D_C(s),$$

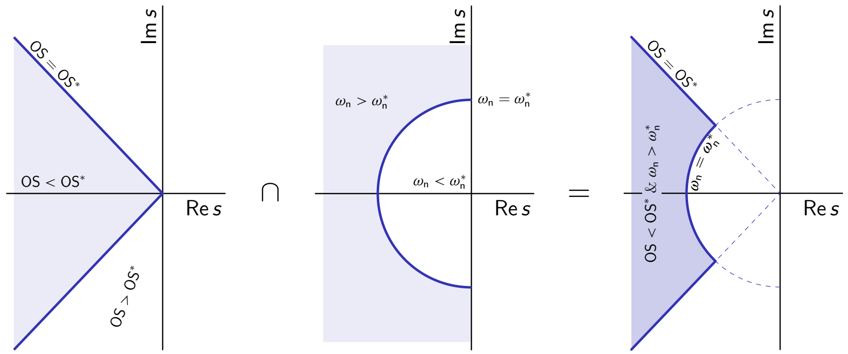
where

$$P(s) = \frac{N_P(s)}{D_P(s)} \qquad \text{and} \qquad C(s) = \frac{N_C(s)}{D_C(s)}$$

and $\deg \chi_{cl}(s) = \deg D_P(s) + \deg D_C(s)$ (assuming that $P(s)$ and $C(s)$ are proper and there are no pole / zero cancellations between $P(s)$ and $C(s)$).

Modal analysis: idea (contd)

Example:



small overshoot + fast transients = "good" performance

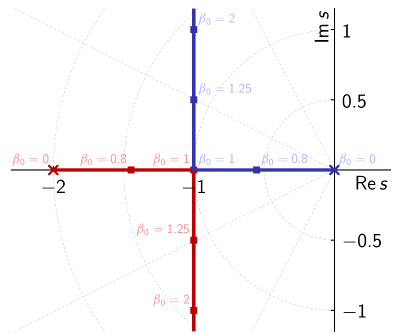
– precise for 2-order systems w/o zeros
– justified for systems with 2-order dominant dynamics

Example: static controller

Let $P(s) = 1/(s^2 + 2s)$ and controller is of the form $C(s) = \beta_0$. Then

$$\chi_{cl}(s) = s^2 + 2s + \beta_0.$$

Closed-loop poles can only be placed to points on root-locus branches:



Example: 1-order strictly proper controller

Let $P(s) = 1/(s^2 + 2s)$ and controller is of the form $C(s) = \frac{\beta_0}{\alpha_1 s + \alpha_0}$. Then

$$\begin{aligned} \chi_{\text{cl}}(s) &= \alpha_1 s^3 + (\alpha_0 + 2\alpha_1)s^2 + 2\alpha_0 s + \beta_0 \\ &= \chi_3 s^3 + \chi_2 s^2 + \chi_1 s + \chi_0. \end{aligned}$$

Still constrained: $\chi_1 - 2\chi_2 + 4\chi_3 = 0$. Alternative form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \chi_3 \\ \chi_2 \\ \chi_1 \\ \chi_0 \end{bmatrix},$$

which cannot be solved for arbitrarily χ_i.

Example: 1-order bi-proper controller

Let $P(s) = 1/(s^2 + 2s)$ and controller is of the form $C(s) = \frac{\beta_1 s + \beta_0}{\alpha_1 s + \alpha_0}$. Then

$$\begin{aligned} \chi_{\text{cl}}(s) &= \alpha_1 s^3 + (\alpha_0 + 2\alpha_1)s^2 + (\beta_1 + 2\alpha_0)s + \beta_0 \\ &= \chi_3 s^3 + \chi_2 s^2 + \chi_1 s + \chi_0. \end{aligned}$$

Unconstrained, χ_i can be arbitrary. Alternative form:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_S} \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \chi_3 \\ \chi_2 \\ \chi_1 \\ \chi_0 \end{bmatrix},$$

which can be solved for arbitrarily χ_i as $\det M_S = 1 \neq 0$.

Example: 2-order strictly proper controller

Let $P(s) = 1/(s^2 + 2s)$ and controller is of the form $C(s) = \frac{\beta_1 s + \beta_0}{\alpha_2 s^2 + \alpha_1 s + \alpha_0}$. Then

$$\begin{aligned} \chi_{\text{cl}}(s) &= \alpha_2 s^4 + (\alpha_1 + 2\alpha_2)s^3 + (\alpha_0 + 2\alpha_1)s^2 + (\beta_1 + 2\alpha_0)s + \beta_0 \\ &= \chi_4 s^4 + \chi_3 s^3 + \chi_2 s^2 + \chi_1 s + \chi_0. \end{aligned}$$

Unconstrained, χ_i can be arbitrary, which is seen from

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{M_S} \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \chi_4 \\ \chi_3 \\ \chi_2 \\ \chi_1 \\ \chi_0 \end{bmatrix}$$

(always solvable in χ_i as $\det M_S = 1 \neq 0$).

Example: what can we learn from it

- controlers of sufficient high order needed for arbitrary pole placement
- polynomial equations reduce to linear equations

Preliminary: multiplication of polynomials

Let $A(s) = a_n s^n + \cdots + a_1 s + a_0$ and $B(s) = b_m s^m + \cdots + b_1 s + b_0$ with $n \geq m$, so that

$$C(s) := A(s)B(s) = c_{n+m} s^{n+m} + c_{n+m-1} s^{n+m-1} + \cdots + c_1 s + c_0.$$

The coefficients of $C(s)$ can be calculated from the table

	$a_n s^n$	$a_{n-1} s^{n-1}$	$\cdots$	$a_1 s$	a_0
$b_m s^m$	$a_n b_m s^{n+m}$	$a_{n-1} b_m s^{n+m-1}$	$\cdots$	$a_1 b_m s^{m+1}$	$a_0 b_m s^m$
$b_{m-1} s^{m-1}$	$a_n b_{m-1} s^{n+m-1}$	$a_{n-1} b_{m-1} s^{n+m-2}$	$\cdots$	$a_1 b_{m-1} s^m$	$a_0 b_{m-1} s^{m-1}$
$b_{m-2} s^{m-2}$	$a_n b_{m-2} s^{n+m-2}$	$a_{n-1} b_{m-2} s^{n+m-3}$	$\cdots$	$a_1 b_{m-2} s^{m-1}$	$a_0 b_{m-2} s^{m-2}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$b_2 s^2$	$a_n b_2 s^{n+2}$	$a_{n-1} b_2 s^{n+1}$	$\cdots$	$a_1 b_2 s^3$	$a_0 b_2 s^2$
$b_1 s$	$a_n b_1 s^{n+1}$	$a_{n-1} b_1 s^n$	$\cdots$	$a_1 b_1 s^2$	$a_0 b_1 s$
b_0	$a_n b_0 s^n$	$a_{n-1} b_0 s^{n-1}$	$\cdots$	$a_1 b_0 s$	$a_0 b_0$

by summing up elements on each anti-diagonal.

Preliminary: multiplication of polynomials (contd)

This results in the following formula for coefficients of $C(s)$:

$$\begin{bmatrix} c_{n+m} \\ c_{n+m-1} \\ \vdots \\ c_n \\ c_{n-1} \\ \vdots \\ c_m \\ c_{m-1} \\ \vdots \\ c_0 \end{bmatrix} = \underbrace{\begin{bmatrix} a_n & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-m} & a_{n-m+1} & \cdots & a_n \\ a_{n-m-1} & a_{n-m} & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_m \\ 0 & a_0 & \cdots & a_{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{bmatrix}}_{M_{a,m} \in \mathbb{R}^{(n+m+1)\times(m+1)}} \begin{bmatrix} b_m \\ b_{m-1} \\ \vdots \\ b_0 \end{bmatrix}$$

Sylvester matrix

Let (here $a_n \neq 0$)

$$D_P(s) = a_n s^n + \cdots + a_1 s + a_0 \quad \text{and} \quad N_P(s) = b_n s^n + \cdots + b_1 s + b_0.$$

The $(2n+1) \times (2n+2)$ matrix

$$M_S := \begin{bmatrix} M_{a,n} & M_{b,n} \end{bmatrix} = \begin{bmatrix} a_n & 0 & \cdots & 0 & b_n & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 & b_{n-1} & b_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_n & b_0 & b_1 & \cdots & b_n \\ 0 & a_0 & \cdots & a_{n-1} & 0 & b_0 & \cdots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 & 0 & 0 & \cdots & b_0 \end{bmatrix}$$

called Sylvester matrix, associated with $D_P(s)$ and $N_P(s)$.

Sylvester matrix (contd)

We need also some sub-matrices of M_S:

M_{S1} is the $(2n+1) \times (2n+1)$ matrix obtained from M_S by eliminating its $(n+2)$th column

M_{S2} is the $2n \times 2n$ matrix obtained from M_S by eliminating its 1st row and 1st and $(n+2)$th columns

That is:

$$M_{S1} := \begin{bmatrix} a_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 & b_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_n & b_1 & \cdots & b_n \\ 0 & a_0 & \cdots & a_{n-1} & b_0 & \cdots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 & 0 & \cdots & b_0 \end{bmatrix}$$

and M_{S2} is in green.

Sylvester's theorem

Theorem

Polynomials $D_P(s)$ and $N_P(s)$ relatively prime iff the associated Sylvester matrix M_S has full (row) rank.

Corollary

$D_P(s)$ and $N_P(s)$ relatively prime iff $\det M_{S1} \neq 0$ (or $\det M_{S2} \neq 0$).

Example: Let $D_P(s) = s(s+2)$ and $N_P(s) = s + 2$. Then

$$M_{S1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

is indeed singular (and so is M_{S2}) as its 3rd and 4th columns coincide.

Pole placement: n-order controller

Let $P(s)$ have (irreducible) order n and consider n-order controller:

$$C(s) = \frac{\beta_n s^n + \cdots + \beta_1 s + \beta_0}{\alpha_n s^n + \cdots + \alpha_1 s + \alpha_0}$$

This yields $2n$-order $\chi_{\mathrm{cl}}(s) = \chi_{2n} s^{2n} + \cdots \chi_1 s + \chi_0$ satisfying

$$\underbrace{\begin{bmatrix} a_n & 0 & \cdots & 0 & b_n & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 & b_{n-1} & b_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_n & b_0 & b_1 & \cdots & b_n \\ 0 & a_0 & \cdots & a_{n-1} & 0 & b_0 & \cdots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 & 0 & 0 & \cdots & b_0 \end{bmatrix}}_{M_S} \begin{bmatrix} \alpha_n \\ \alpha_{n-1} \\ \vdots \\ \alpha_0 \\ \beta_n \\ \beta_{n-1} \\ \vdots \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \chi_{2n} \\ \chi_{2n-1} \\ \vdots \\ \chi_{n+1} \\ \chi_n \\ \vdots \\ \chi_0 \end{bmatrix}$$

- $2n+1$ equations, $2n+2$ variables, full-rank $M_S \implies \infty$ many solutions

Pole placement: $(n-1)$-order controller

Let's try to reduce the order of the controller to $n-1$. This implies:

$$\alpha_n = \beta_n = \chi_{2n} = 0$$

and then:

$$\underbrace{\begin{bmatrix} a_n & \cdots & 0 & b_n & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_1 & \cdots & a_n & b_1 & \cdots & b_n \\ a_0 & \cdots & a_{n-1} & b_0 & \cdots & b_{n-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_0 & 0 & \cdots & b_0 \end{bmatrix}}_{M_{S2}} \begin{bmatrix} \alpha_{n-1} \\ \vdots \\ \alpha_0 \\ \beta_{n-1} \\ \vdots \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \chi_{2n-1} \\ \vdots \\ \chi_{n+1} \\ \chi_n \\ \vdots \\ \chi_0 \end{bmatrix}$$

- $2n$ equations, $2n$ variables, $\det M_S \neq 0 \implies$ unique solution
- any further reduction impossible (more equations than variables)

n-order controller: exploiting freedom we have

We have one "spare" variable in this case, which can be exploited to

- bring about additional properties to the controller.

For example, we may enforce $\beta_n = 0$ (strictly proper controller). Then:

$$\underbrace{\begin{bmatrix} a_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 & b_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_n & b_1 & \cdots & b_n \\ 0 & a_0 & \cdots & a_{n-1} & b_0 & \cdots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 & 0 & \cdots & b_0 \end{bmatrix}}_{M_{S1}} \begin{bmatrix} \alpha_n \\ \alpha_{n-1} \\ \vdots \\ \alpha_0 \\ \beta_{n-1} \\ \vdots \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \chi_{2n} \\ \chi_{2n-1} \\ \vdots \\ \chi_{n+1} \\ \chi_n \\ \vdots \\ \chi_0 \end{bmatrix}$$

- $2n+1$ equations, $2n+1$ variables, $\det M_{S1} \neq 0 \implies$ unique solution

n-order controller: exploiting freedom we have (contd)

Another possibility is to enforce $\alpha_0 = 0$ (integral action). Then:

$$\underbrace{\begin{bmatrix} a_n & 0 & \cdots & 0 & b_n & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 & b_{n-1} & b_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_{n-1} & b_0 & b_1 & \cdots & b_n \\ 0 & a_0 & \cdots & a_{n-2} & 0 & b_0 & \cdots & b_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & b_0 \end{bmatrix}}_{M_{S3}} \begin{bmatrix} \alpha_n \\ \alpha_{n-1} \\ \vdots \\ \alpha_1 \\ \beta_n \\ \beta_{n-1} \\ \vdots \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \chi_{2n} \\ \chi_{2n-1} \\ \vdots \\ \chi_{n+1} \\ \chi_n \\ \vdots \\ \chi_0 \end{bmatrix}$$

– $2n+1$ equations, $2n+1$ variables, $\det M_{S3} \neq 0 \implies$ unique solution (the non-singularity of M_{S3} can be proved under condition that $b_0 \neq 0$).

Pole-placement as a design tool

Pros:

- ☺ arbitrary pole placement
- ☺ easily computable

Cons:

- ☹ (almost) no control over controller poles
- ☹ no control over controller zeros
- ☹ no dominance guarantees