Control Theory (00350188)
lecture no. 3

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Outline

Naïve shaping of T

Assume that the "ideal" closed-loop system is T_{dream}. We may want to ask

– whether there is C for which $T = T_{\text{dream}}$?

The answer is affirmative,

$$T_{\text{dream}}(s) = \frac{L(s)}{1+L(s)} \iff L(s) = \frac{T_{\text{dream}}(s)}{1-T_{\text{dream}}(s)}.$$

Thus, controller

$$C_{\text{dream}}(s) = \frac{1}{P(s)} \frac{T_{\text{dream}}(s)}{1-T_{\text{dream}}(s)}$$

is exactly what we are looking for.

What's behind C_{dream}

If s_0 is a pole of $P(s)$ and $T_{\text{dream}}(s_0) \neq 1$, then s_0 is a zero of $C_{\text{dream}}(s)$, i.e.

– $C_{\text{dream}}(s)$ tends to cancel poles of $P(s)$.

If s_1 is a zero of $P(s)$ and not that of $T_{\text{dream}}(s)$, then s_1 must be a pole of $C_{\text{dream}}(s)$, i.e.

– $C_{\text{dream}}(s)$ tends to cancel zeros of $P(s)$.

Thus, to achieve arbitrary T_{dream}

– controller should, in general, cancel all poles and zeros of the plant.

Pole-zero cancellations: what's legal & what makes sense

Obviously,

– only stable pole-zero cancellations are legal.

This implies that

– every RHP zero of $P(s)$ must be a zero of $T(s)$ (multiplicity counted)
– at every unstable pole s_p of $P(s)$ the equality

$$T(s_p) = 1$$

must hold (multiple poles impose additional conditions on $\frac{d^i}{ds^i}T(s)$).

Although stable pole-zero cancellations are legal,

– not all stable pole-zero cancellations are welcome.

Canceled poles and zeros are not eliminated from the closed-loop system.

Canceled plant poles

Let s_p ($\operatorname{Re} s_p < 0$) be a pole of $P(s)$ canceled by $C(s)$. In other words, let

$$P(s) = \frac{P_1(s)}{s - s_p} \quad \text{and} \quad C(s) = (s - s_p)C_1(s)$$

for some $P_1(s)$ such that $P_1(s_p) \neq 0$ and $C_1(s)$ such that $|C_1(s_p)| < \infty$. In this case

$$S(s) = \frac{1}{1 + P_1(s)C_1(s)} \quad \text{and} \quad T(s) = 1 - S(s)$$

do not depend on s_p, but

$$T_d(s) = P(s)S(s) = \frac{1}{s - s_p}\,\frac{P_1(s)}{1 + P_1(s)C_1(s)}$$

still has s_p as its pole.

Canceled plant zeros

Let s_z ($\operatorname{Re} s_z < 0$) be a zero of $P(s)$ canceled by $C(s)$. In other words, let

$$P(s) = (s - s_z)P_2(s) \quad \text{and} \quad C(s) = \frac{C_2(s)}{s - s_z}$$

for some $P_2(s)$ such that $|P_2(s_z)| < \infty$ and $C_2(s)$ such that $C_2(s_z) \neq 0$. In this case

$$S(s) = \frac{1}{1 + P_2(s)C_2(s)} \quad \text{and} \quad T(s) = 1 - S(s)$$

do not depend on s_z, but

$$T_c(s) = C(s)S(s) = \frac{1}{s - s_z}\,\frac{C_2(s)}{1 + P_2(s)C_2(s)}$$

has s_z as its pole.

What does it mean?

Thus, for each canceled plant pole s_p and zero s_z we have that

$$T_d(s) = \frac{1}{s - s_p}\,\frac{P_1(s)}{1 + P_1(s)C_1(s)} \quad \text{and} \quad T_c(s) = \frac{1}{s - s_z}\,\frac{C_2(s)}{1 + P_2(s)C_2(s)}$$

Typically,

– If $s_\bullet$ is "fast," its presence in $T_d(s)$ or $T_c(s)$ can in general be ignored
– If $s_\bullet$ is "slow," it slows down disturbance response / control input decay
– If $s_\bullet$ is "oscillatory," oscillations show up in $d \mapsto y$ or $r \mapsto u$

and we have the following rules of thumb:

– Fast well-damped poles/zeros can typially be safely canceled
(though this might produce spikes in the control signal)
– Slow well-damped poles/zeros can be canceled with certain care
(there are pros and cons in canceling slow plant poles vs. shifting them by feedback)
– Lightly-damped poles/zeros shall not be canceled
(unless you reeeally know what you're doing)

Outline

Pole-zero cancellations

"Flexible" loops

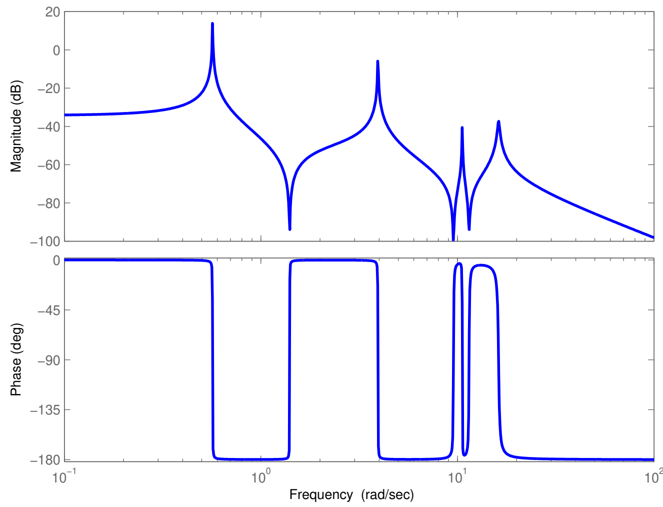
Pendulum on cart

DC motor with flexible transmission

Strong stabilization

What we understand by "flexible" loops

– Loops with one or several resonant frequencies[1]:

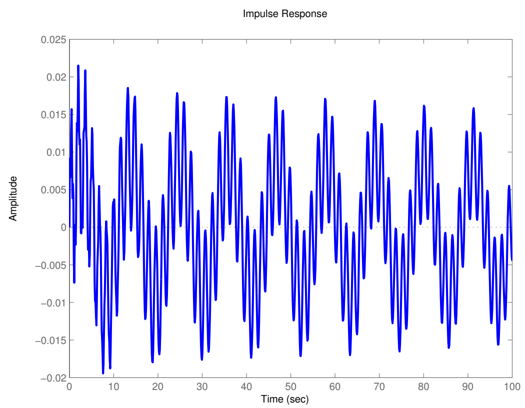


[1]Typical example is flexible mechanical structures.

What is special about flexible systems

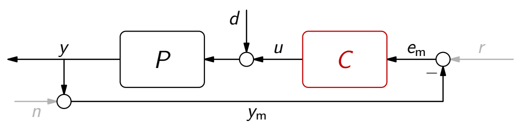
Resonances in frequency domain give rise to

– slowly decaying oscillations, dominating time response:



Ability to cope with oscillations (dampen them) is main leitmotiv in control of flexible systems, frequently more important than high / low gain tradeoff.

What is special about dampening flexible modes



At frequencies around *problematic* resonance peaks (where $|P(j\omega)| \gg 1$),

$$|T_d(j\omega)| = \left|\frac{P(j\omega)}{1 + P(j\omega)C(j\omega)}\right| = \frac{1}{|1/P(j\omega) + C(j\omega)|} \approx \frac{1}{|C(j\omega)|}.$$

This means that it's

– not sufficient to keep $|L(j\omega)| \gg 1$

(this may be achieved with a small $|C(j\omega)|$), but we should endeavor to

– keep $|C(j\omega)|$ high (or, at least, $|C(j\omega)| \not\ll 1$)

at ω's around meaningful resonances. This support the knowledge that

– canceling lightly damped poles of $P(s)$ by $C(s)$ is a bad idea

(as it normally leads to $|C(j\omega)| \ll 1$ at frequencies where $|P(j\omega)| \gg 1$).

What will we learn today

Some tricks about how to

- shape lightly damped loops with high-gain controllers

These tricks are rather non-obvious, sometimes contradicting conventional loop-shaping guidelines.

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Pole-zero cancellations

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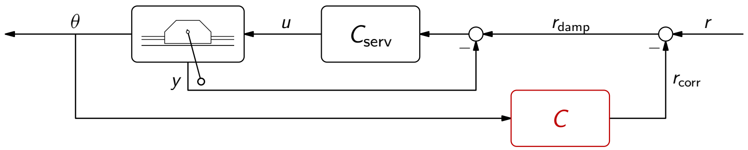
Strong stabilization

Experimental setup



Here pendulum is mounted on a cart, which is controlled by a DC motor. A local servo loop is already closed around the motor, yet it does not dampen oscillations of the pendulum. So, our goal here is to

- close the second loop, dampening pendulum oscillations.

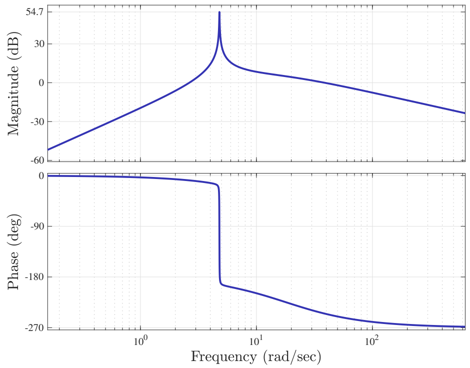
In this case

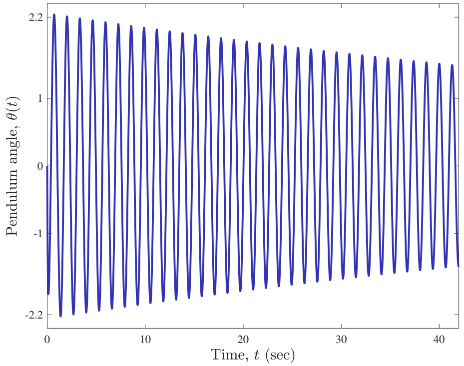
- the system from r_{damp} to θ plays the plant role
- the reference signal r plays the load disturbance role
- the correcting reference r_{corr} plays the control signal role

Plant

After closing the motor position loop, plant becomes of the form:

$$P(s) = \frac{-42s^2}{(s+18)(s^2+0.02s+23)}.$$





These oscillations should be dampened by feedback.

Loop shaping: before we start

Some observations are in order:

1. Static gain $P(s)|_{s=0}$ and velocity gain $P(s)/s|_{s=0}$ are both zero.
2. Gain at sub-resonance frequencies ($\omega < 2$) is pretty low.

Thus, plant filters out low-frequency disturbances well (no help required[2]). Also,

3. gain at over-resonance frequencies ($\omega > 50$) is low.

Thus, plant filters out high-frequency noise also well (no help required).

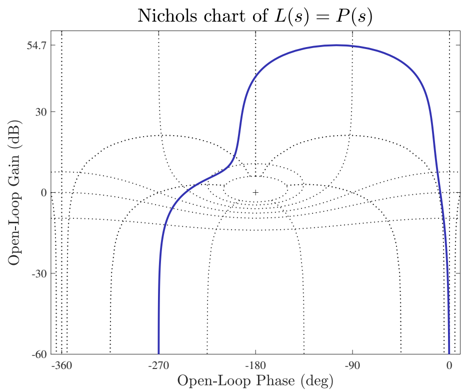
Therefore,

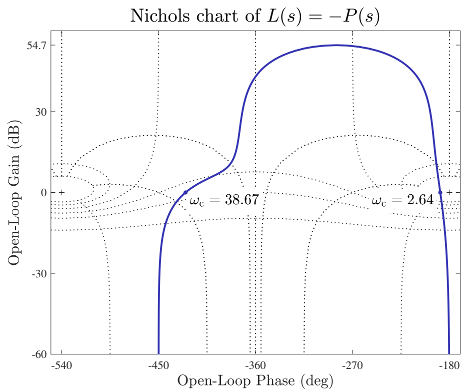
– we only need to interfere around the resonance (i.e. in $2 < \omega < 50$).

[2] In fact, we cannot do much. For example, the static loop gain can only be increased by canceling the zeros at the origin, which is obviously illegal.

Loop shaping: proportional gain

Stabilizing P with negative feedback w/o lowering the resonance too much might not be trivial, even if a dynamics controller is used:





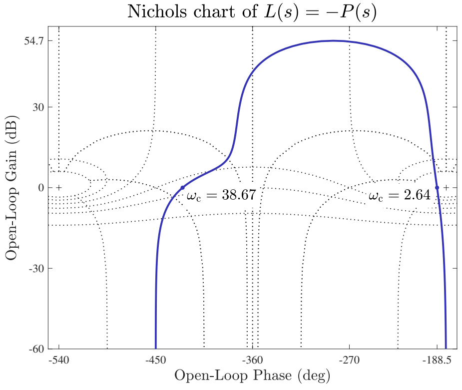
The task is way easier positive feedback is preferable here. Note that

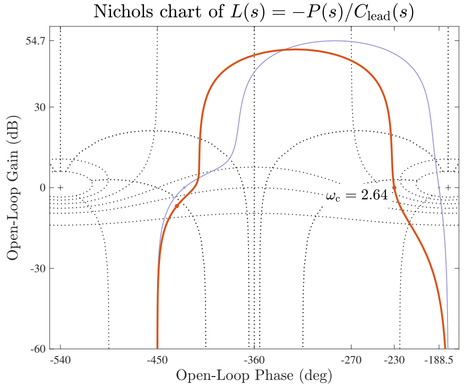
– loop gain has two crossover frequencies in this case,

which means that we have two regions of interest for loop phase ($\arg L(j\omega)$).

Loop shaping: keeping far from the critical point

If we keep plant's first crossover (≈ 2.64) and want phase margin of 50°,





we need to add phase lag around ω_c! This promts the choice

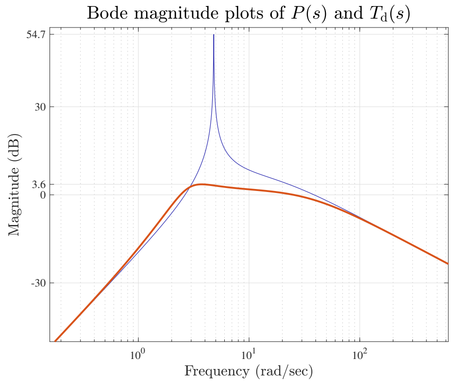
$$C(s) = -\frac{1}{C_{\rm lead}(s)}, \qquad \text{for } \omega_m = \omega_c \text{ and } \phi_m = 41.5°.$$

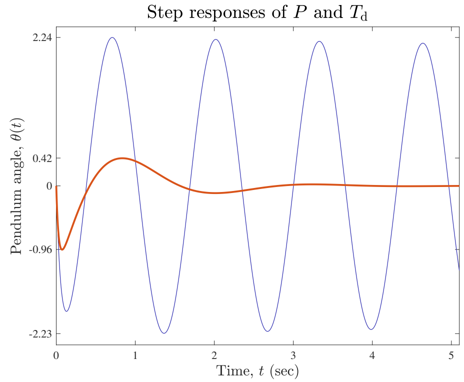
Loop shaping: keeping far from the critical point (contd)

The resulting controller,

$$C(s) = -0.451\,\frac{s + 5.85}{s + 1.188},$$

does its job:





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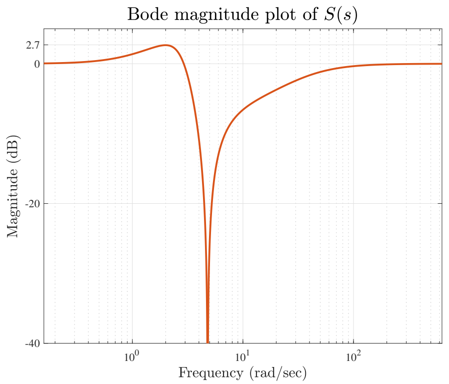
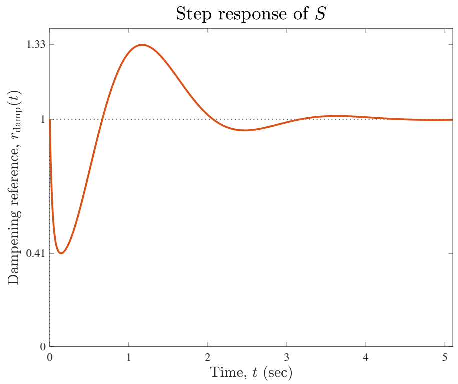
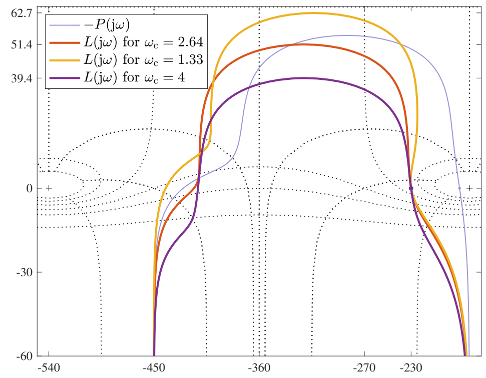




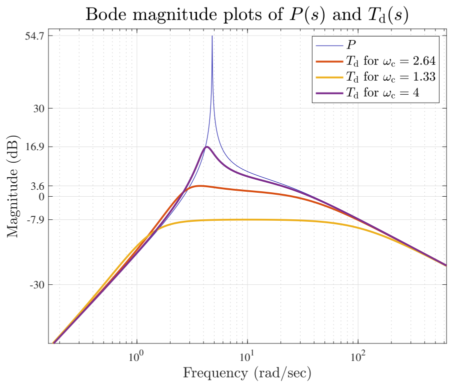
Loop shaping: let's play with ω_c

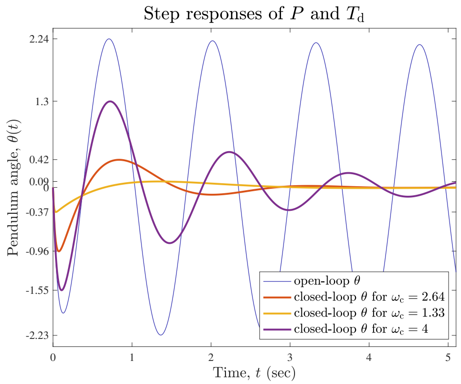
We may want to

- increase damping/accelerate response—by decrease of the first ω_c, or
- decrease control efforts—by increase of the first ω_c.



Loop shaping: let's play with ω_c (contd)





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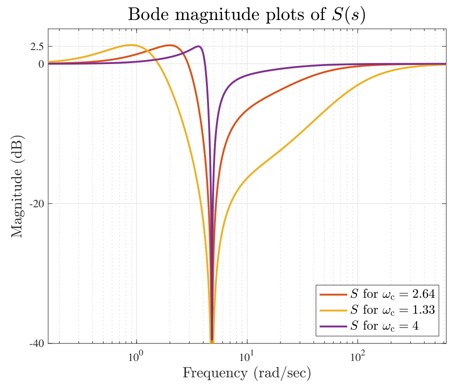
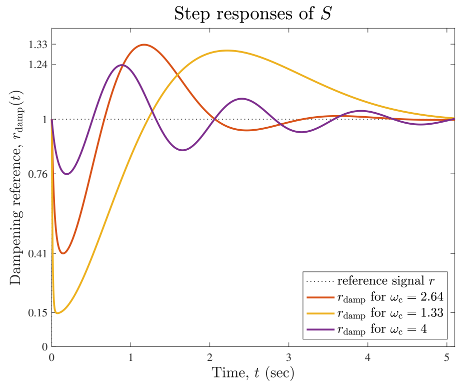




What can we learn from this example?

Shaping flexible loops is characterized by

– over-emphasizing resonance frequencies.

This means that design should not hinge upon increasing gain in the whole region, but rather can

– target narrow frequency bands, around resonances.

Flexible loops typically involve

– multiple crossover regions

with

– alternating regions of high- and low-gains.

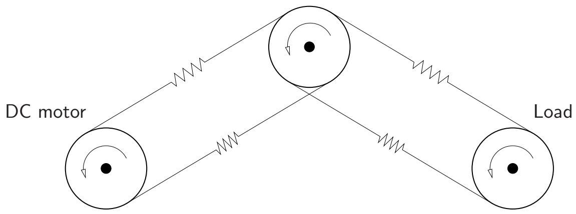
This property is of great importance as it enables us to

– exploit phase lag (even nonminimum-phase or delay) elements

in feedback loops, thus circumventing Bode's gain-phase relation bounds.

Outline

Experimental setup



DC motor is connected with load via flexible transmission. We want:

– complete steady-state rejection of step disturbances,

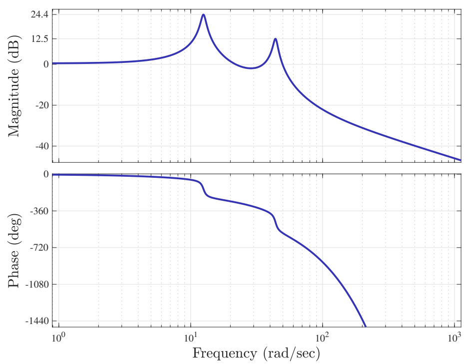
– dampened output response.

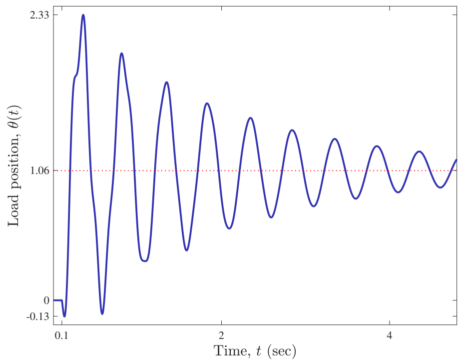
The plant

Plant transfer function (obtained experimentally) is

$$P(s) = \frac{5(-s+40)(s+40)^2}{(s^2+s+156.25)(s^2+3.172s+1936)}\mathrm{e}^{-0.1s},$$

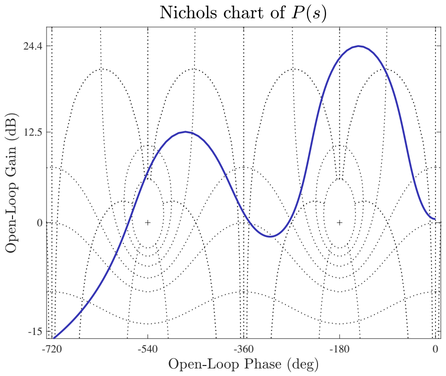
with the following Bode plot and step response:

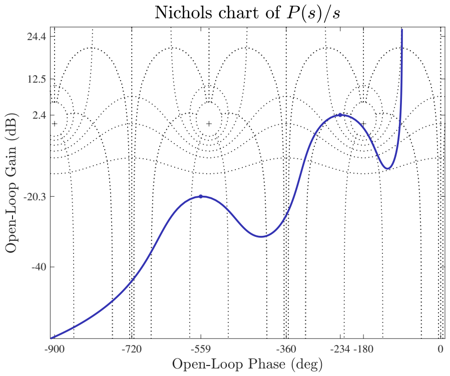




Loop shaping logic

Delay and NMP zero add considerable phase lag:





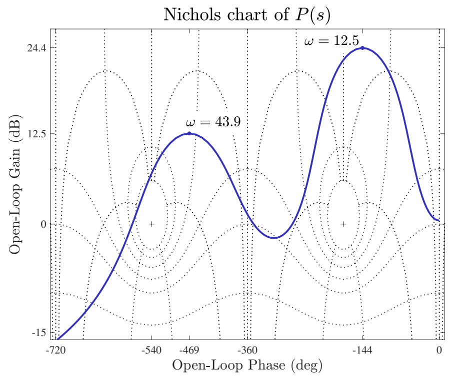
We have

– no hope to squeeze both resonances before the first critical point.

Adding an integral action makes this even clearer, leading to the need to

– add phase lag, again.

What lag do we need?



Endeavoring to locate resonances far from the critical points, we may try to

– move the resonance at $\omega = 12.5$ to $\arg L(\mathrm{j}12.5) \approx -360°$

– move the resonance at $\omega = 43.9$ to $\arg L(\mathrm{j}43.9) \approx -720°$

This requires

– a phase lag of $216°$ at $\omega = 12.5$ and a phase lag of $251°$ at $\omega = 43.9$.

And we also need to have high enough controller gains at those frequencies.

Tool: nonminimum-phase PID

This can be achieved by a PID controller with RHP zeros. Consider

$$C_{\mathrm{PID}}(s) = k\Big(-1 + \frac{1}{\tau_{\mathrm{i}} s} + \tau_{\mathrm{d}} s\Big) \implies C_{\mathrm{PID}}(\mathrm{j}\omega) = -k + \mathrm{j}k\Big(\tau_{\mathrm{d}}\omega - \frac{1}{\tau_{\mathrm{i}}\omega}\Big).$$

If we need phase to be in $(-270°, -90°)$, then $k > 0$ and

$$\arg C_{\mathrm{PID}}(\mathrm{j}\omega) = -180° - \arctan\frac{\tau_{\mathrm{d}}\omega^2 - 1/\tau_{\mathrm{i}}}{\omega} \quad \text{[in degrees]}.$$

Given $\omega_2 > \omega_1 > 0$, the equations

$$\arg C_{\mathrm{PID}}(\mathrm{j}\omega_i) = \phi_i \in (-270°, -90°), \quad i = 1, 2$$

are solved by

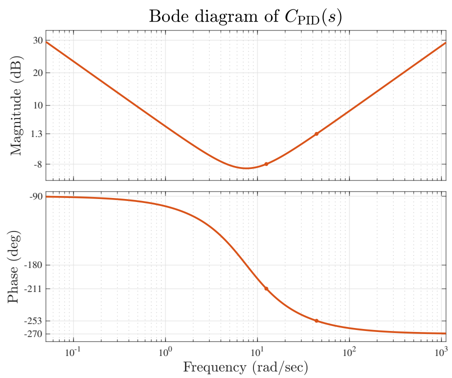
$$\tau_{\mathrm{i}} = \frac{\omega_2^2 - \omega_1^2}{\omega_1\omega_2(\omega_2\tan\phi_1 - \omega_1\tan\phi_2)} \quad \text{and} \quad \tau_{\mathrm{d}} = \frac{\omega_1\tan\phi_1 - \omega_2\tan\phi_2}{\omega_2^2 - \omega_1^2}.$$

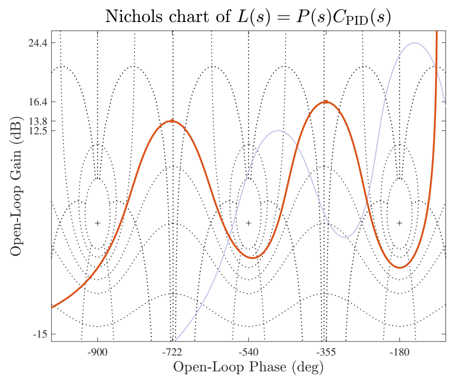
The controller

For our data ($\phi_1 = -211°$, $\phi_2 = -253°$, tuned manually) we end up with

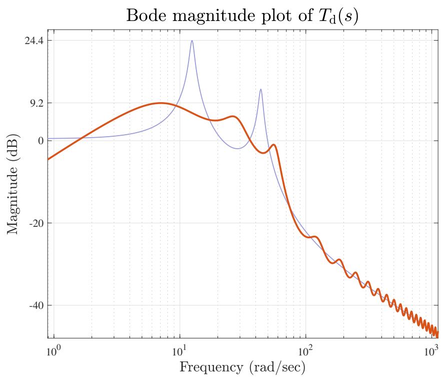
$$C_{\mathrm{PID}}(s) = 0.343\Big(-1 + \frac{1}{0.229s} + 0.0757s\Big) = \frac{0.026(s^2 - 13.21s + 57.68)}{s}$$

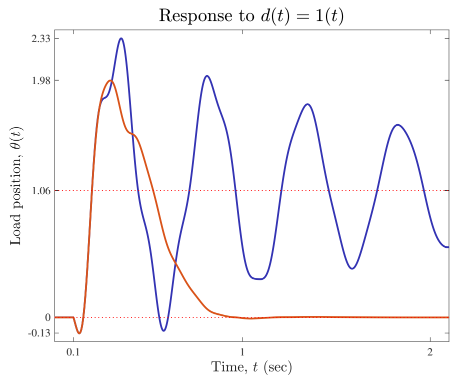
(k was also tuned manually), for which





Closed-loop disturbance response





show substantially better dampening. But

– the PID controller transfer function is non-proper.

Still, there are simple methods to have a proper controller.

Proper D-part

The non-proper D-part, $\tau_d s$, is normally implemented as

$$\frac{\tau_d s}{\alpha \tau_d s + 1} \quad \text{for } 0.05 \le \alpha \le 0.3.$$

Let's choose α to render $|C(\infty)| = 10$ [dB]. To this end,

$$C(s) = k\Big(-1 + \frac{1}{\tau_i s} + \frac{\tau_d s}{\alpha \tau_d s + 1}\Big) \implies C(\infty) = k\frac{1-\alpha}{\alpha} > 0,$$

so that

$$\alpha = \frac{k}{k + C(\infty)}.$$

Thus, we need

$$\alpha = \frac{k}{k + 3.1623} = 0.0979 \approx 0.1,$$

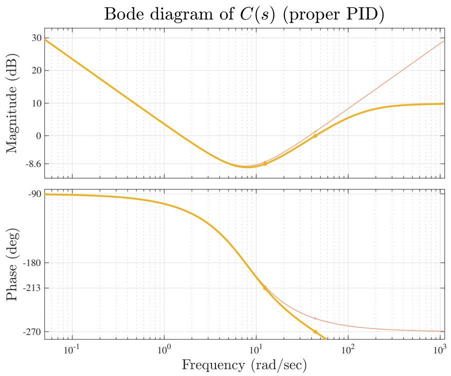
which is well within conventional bounds.

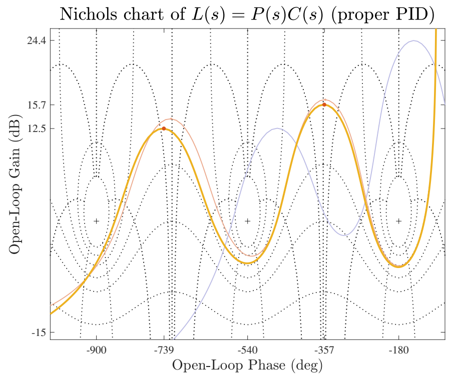
The proper controller

Thus, we have

$$C(s) = 0.343\Big(-1 + \frac{1}{0.229s} + \frac{0.0757s}{0.0074s + 1}\Big) \approx \frac{3.1(s^2 - 14s + 64)}{s(s + 132.1)},$$

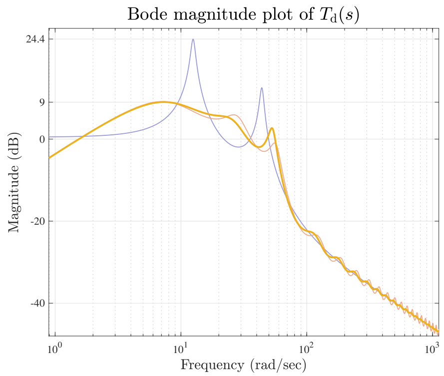
for which

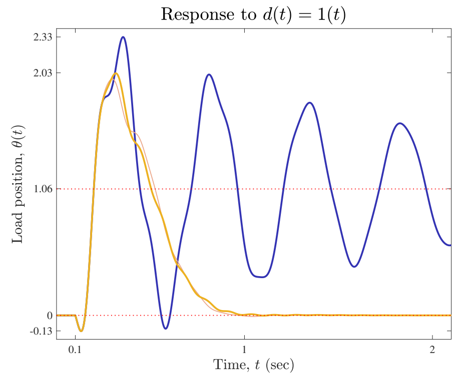




with a slight deterioration of the last phase margin and controller gain.

Closed-loop disturbance response (contd)





have comparable dampening.

Remark 1 There are other approaches to render $C(s)$ proper. For example, we may add a low-pass filter $F_{lp}(s)$ to the plant, design a PID $C_{PID}(s)$ for $P(s)F_{lp}(s)$, and then implement the proper $C(s) = F_{lp}(s)C_{PID}(s)$. Try it with $F_{lp}(s) = 1/(s/175 + 1)$ and $\phi_1 = 197^\circ$ and $\phi_2 = 242^\circ$ at the resonances.

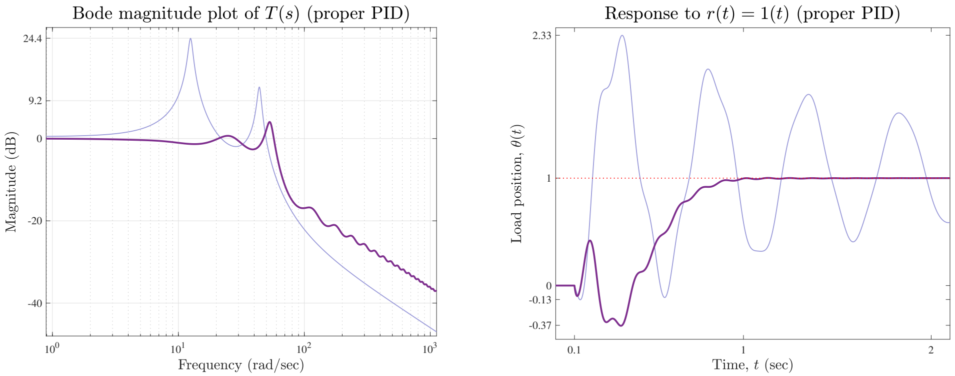
Remark 2 Small fast oscillations are the result of getting closer to the last critical point. To get rid of them, we may use a (complex) lead around the last crossover. Try it.

Command response

The complementary sensitivity transfer function in this case is

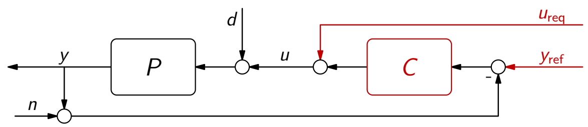
$$T(s) = \frac{15.5\,(-s+40)(s+40)^2(s^2-14s+64)}{\chi_{\mathrm{cl}}(s)}\,\mathrm{e}^{-0.1s}$$

for some Hurwitz $\chi_{\mathrm{cl}}(s)$ and its step response is not quite satisfactory:



Let's improve that without altering C ...

2DOF architecture



Signals of interest (as long as $y_{\mathrm{ref}} = Pu_{\mathrm{req}}$):

$$y = y_{\mathrm{ref}} + T_{\mathrm{d}}\,d - T\,n \quad \text{and} \quad u = u_{\mathrm{req}} - T\,d - T_{\mathrm{c}}\,n.$$

Let's use

$$y_{\mathrm{ref}} = T_{\mathrm{ref}}\mathbb{1} \quad \text{and} \quad u_{\mathrm{req}} = C_{\mathrm{ol}}\mathbb{1} = \frac{T_{\mathrm{ref}}}{P}\mathbb{1}$$

for some reference model such that

- all nonminimum-phase zeros, and the delay, of $P(s)$ are those of $T_{\mathrm{ref}}(s)$
- pole excess of $T_{\mathrm{ref}}(s) \geq$ poles excess of $P(s)$
- $T_{\mathrm{ref}}(0) = 1$ (zero steady-state error)
- T_{ref} has smooth and sufficiently fast transients

Example 2: 2DOF design

With

$$P(s) = \frac{5(-s+40)(s+40)^2}{(s^2+s+12.5^2)(s^2+3.172s+44^2)}\,\mathrm{e}^{-0.1s},$$

the reference model T_{ref} has three stability-related constraints:

1. it must have a zero at $s = 40$,
2. it must have a delay of 0.1 sec,
3. its pole excess must be at least 1.

With the requirement $T_{\mathrm{ref}}(0) = 1$ we may pick

$$T_{\mathrm{ref}}(s) = \frac{-s+40}{40(\tau s+1)^2}\,\mathrm{e}^{-0.1s}$$

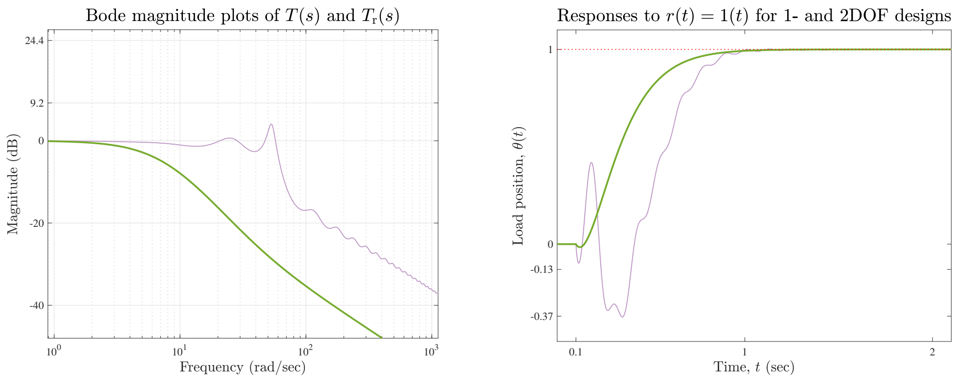
and tune $\tau > 0$ to have a desired settling time of its step response. Then

$$C_{\mathrm{ol}}(s) = \frac{T_{\mathrm{ref}}(s)}{P(s)} = \frac{(s^2+s+12.5^2)(s^2+3.172s+44^2)}{200(\tau s+1)^2(s+40)^2}$$

is proper and has all its poles in the OLHP, hence stable as well.

Example 2: 2DOF design (contd)

For $\tau = 1/8$ (picked to remain with about the same settling time) we have:



which is a much better[3] transient response.

[3] Virtually no effect of the zero at $s = 40$, the poles of $T_{\mathrm{ref}}(s)$ at $s = -8$ are dominant.

Outline

Pole-zero cancellations

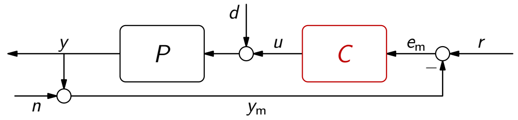
"Flexible" loops

Pendulum on cart

DC motor with flexible transmission

Strong stabilization

Stabilization with stable controllers



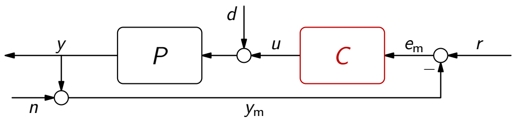
Stable controllers, especially for stable plants, are preferable since we want to maintain stability during

- sensor / actuator failures
- sensor / actuator saturation

We say that

- P is strongly stabilizable if it can be stabilized by a stable controller.

Parity interlacing property



P is strongly stabilizable iff its transfer function has

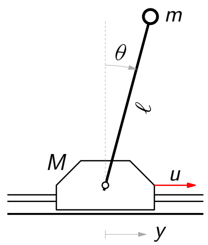
- even number of real poles between every pair of real zeros in RHP

(including $+\infty$). This property called the parity interlacing property.

Example 1: Let $P(s) = \frac{s-1}{s(s-2)}$. It has 2 RHP zeros at $\{1, \infty\}$ and between them one pole at 2. Hence it is not strongly stabilizable.

Example 2: Let $P(s) = \frac{(s-1)^2(s^2-s+1)}{(s-2)^2(s+1)^3}$. It has 5 RHP zeros, 3 of them real at $\{1, 1, \infty\}$. Between 1 and 1 lies 0 poles, while between 1 and ∞ lie 2 poles (at 2). Hence this plant is strongly stabilizable.

Inverted pendulum without angle measurement



M :	cart mass
m :	pendulum mass
ℓ :	pendulum length
y :	cart position
θ :	pendulum angle
u :	force applied to the cart
g :	standard gravity

Linearized transfer function of the inverted pendulum from u to y is

$$P(s) = \frac{\ell s^2 - g}{M\big(\ell s^2 - g(1 + \frac{m}{M})\big)s^2}.$$

It has 3 real RHP zeros in $\{\sqrt{g/\ell}, \infty, \infty\}$. Between the first two of them $P(s)$ has one pole at $s = \sqrt{(1 + m/M)g/\ell}$. Thus,

- pendulum is not strongly stabilizable.