Control Theory (00350188)
lecture no. 3

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Outline

Pole-zero cancellations

"Flexible" loops

Pendulum on cart

DC motor with flexible transmission

Strong stabilization

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Naïve shaping of T

Assume that the "ideal" closed-loop system is T_{dream}. We may want to ask

- whether there is C for which $T = T_{\text{dream}}$?

The answer is affirmative,

$$T_{\text{dream}}(s) = \frac{L(s)}{1 + L(s)} \iff L(s) = \frac{T_{\text{dream}}(s)}{1 - T_{\text{dream}}(s)}.$$

Thus, controller

$$C_{\text{dream}}(s) = \frac{1}{P(s)} \frac{T_{\text{dream}}(s)}{1 - T_{\text{dream}}(s)}$$

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What's behind C_{dream}

If s_0 is a pole of $P(s)$ and $T_{\text{dream}}(s_0) \neq 1$, then s_0 is a zero of $C_{\text{dream}}(s)$, i.e.

– $C_{\text{dream}}(s)$ tends to cancel poles of $P(s)$.

If s_1 is a zero of $P(s)$ and not that of $T_{\text{dream}}(s)$, then s_1 must be a pole of $C_{\text{dream}}(s)$, i.e.

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Pole-zero cancellations: what's legal

Obviously,

- only stable pole-zero cancellations are legal.

This implies that

- every RHP zero of $P(s)$ must be a zero of $T(s)$ (multiplicity counted)
- at every unstable pole s_p of $P(s)$ the equality

$$T(s_\mathrm{p}) = 1$$

  must hold (multiple poles impose additional conditions on $\frac{\mathrm{d}^i}{\mathrm{d}s^i} T(s)$).

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Canceled plant poles

Let s_p ($\operatorname{Re} s_\mathrm{p} < 0$) be a pole of $P(s)$ canceled by $C(s)$. In other words, let

$$P(s) = \frac{P_1(s)}{s - s_\mathrm{p}} \quad \text{and} \quad C(s) = (s - s_\mathrm{p})C_1(s)$$

for some $P_1(s)$ such that $P_1(s_\mathrm{p}) \neq 0$ and $C_1(s)$ such that $|C_1(s_\mathrm{p})| < \infty$. In this case

$$S(s) = \frac{1}{1 + P_1(s)C_1(s)} \quad \text{and} \quad T(s) = 1 - S(s)$$

do not depend on s_p, but

$$T_\mathrm{d}(s) = P(s)S(s) = \frac{1}{s - s_\mathrm{p}} \, \frac{P_1(s)}{1 + P_1(s)C_1(s)}$$

still has s_p as its pole.

Canceled plant zeros

Let s_z ($\operatorname{Re} s_z < 0$) be a zero of $P(s)$ canceled by $C(s)$. In other words, let

$$P(s) = (s - s_z)P_2(s) \quad \text{and} \quad C(s) = \frac{C_2(s)}{s - s_z}$$

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What does it mean?

Thus, for each canceled plant pole s_p and zero s_z we have that

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Typically,

- If $s_\bullet$ is "fast," its presence in $T_\mathrm{d}(s)$ or $T_\mathrm{c}(s)$ can in general be ignored
- If $s_\bullet$ is "slow," it slows down disturbance response / control input decay
- If $s_\bullet$ is "oscillatory," oscillations show up in $d \mapsto y$ or $r \mapsto u$

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- Fast well-damped poles/zeros can typially be safely canceled
  (though this might produce spikes in the control signal)
- Slow well-damped poles/zeros can be canceled with certain care
  (there are pros and cons in canceling slow plant poles vs. shifting them by feedback)
- Lightly-damped poles/zeros shall not be canceled
  (unless you reeeally know what you're doing)

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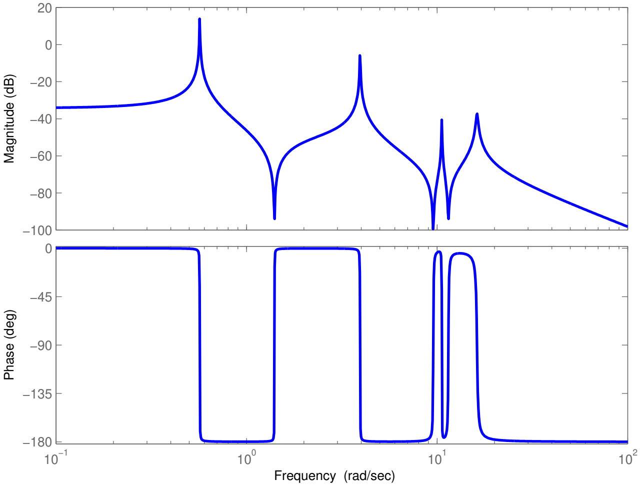
What we understand by "flexible" loops

– Loops with one or several resonant frequencies[1]:

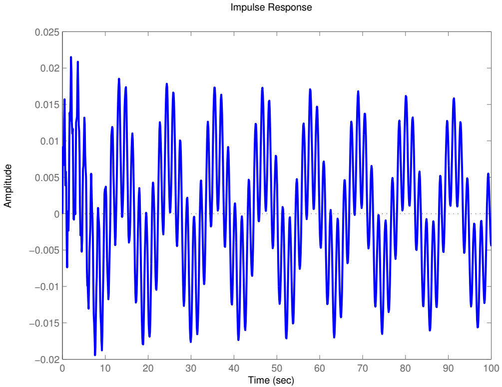


[1] Typical example is flexible mechanical structures.

What is special about flexible systems

Resonances in frequency domain give rise to

– slowly decaying oscillations, dominating time response:



Ability to cope with oscillations (dampen them) is main leitmotiv in control of flexible systems, frequently more important than high / low gain tradeoff

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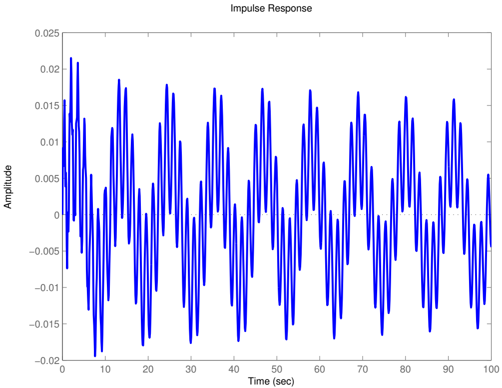

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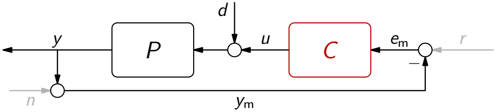
What is special about dampening flexible modes



At frequencies around *problematic* resonance peaks (where $|P(j\omega)| \gg 1$),

$$|T_d(j\omega)| = \left|\frac{P(j\omega)}{1 + P(j\omega)C(j\omega)}\right| = \frac{1}{|1/P(j\omega) + C(j\omega)|} \approx \frac{1}{|C(j\omega)|}.$$

This means that it's

– not sufficient to keep $|L(j\omega)| \gg 1$

(this may be achieved with a small $|C(j\omega)|$), but we should endeavor to

– keep $|C(j\omega)|$ high (or, at least, $|C(j\omega)| \not\ll 1$)

at ω's around meaningful resonances. This support the knowledge that

– canceling lightly damped poles of $P(s)$ by $C(s)$ is a bad idea

(as it normally leads to $|C(j\omega)| \ll 1$ at frequencies where $|P(j\omega)| \gg 1$).

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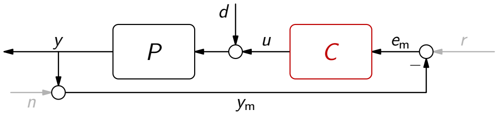

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What will we learn today

Some tricks about how to

– shape lightly damped loops with high-gain controllers

These tricks are rather non-obvious, sometimes contradicting conventional loop-shaping guidelines.

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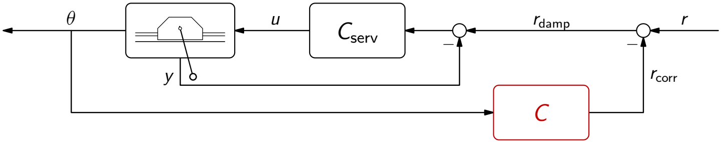
Strong stabilization

Experimental setup



Here pendulum is mounted on a cart, which is controlled by a DC motor. A local servo loop is already closed around the motor, yet it does not dampen oscillations of the pendulum. So, our goal here is to

- close the second loop, dampening pendulum oscillations.

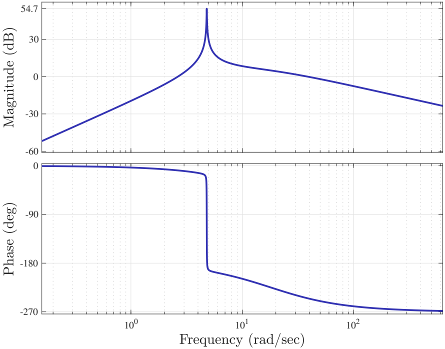
In this case

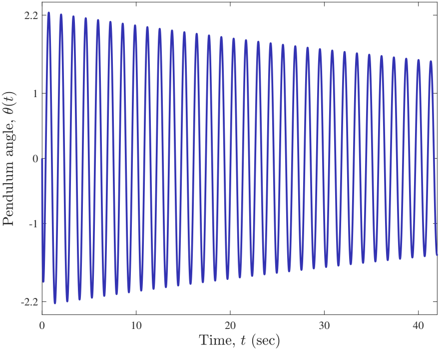
- the system from r_{damp} to θ plays the plant role
- the reference signal r plays the load disturbance role
- the correcting reference r_{corr} plays the control signal role

Plant

After closing the motor position loop, plant becomes of the form:

$$P(s) = \frac{-42s^2}{(s+18)(s^2+0.02s+23)}.$$





These oscillations should be dampened by feedback.

Loop shaping: before we start

Some observations are in order:

1. Static gain $P(s)|_{s=0}$ and velocity gain $P(s)/s|_{s=0}$ are both zero.
2. Gain at sub-resonance frequencies ($\omega < 2$) is pretty low.

Thus, plant filters out low-frequency disturbances well (no help required[2]).

Also,

3. gain at over-resonance frequencies ($\omega > 50$) is low.

Thus, plant filters out high-frequency noise also well (no help required).

Therefore,

– we only need to interfere around the resonance (i.e. in $2 < \omega < 50$).

---

[2] In fact, we cannot do much. For example, the static loop gain can only be increased by canceling the zeros at the origin, which is obviously illegal.

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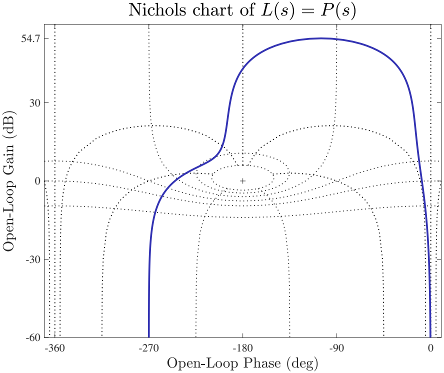
Loop shaping: proportional gain

Stabilizing P with negative feedback w/o lowering the resonance too much might not be trivial, even if a dynamics controller is used:



The task is way easier positive feedback is preferable here. Note that
— loop gain has two crossover frequencies in this case,
which means that we have two regions of interest for loop phase ($\arg L(j\omega)$)

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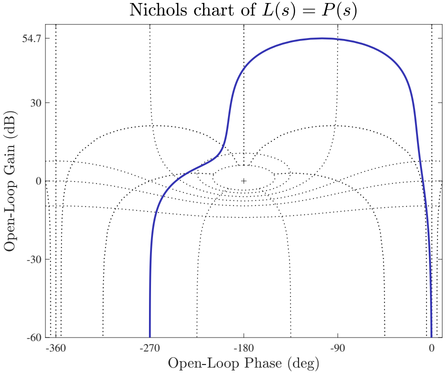

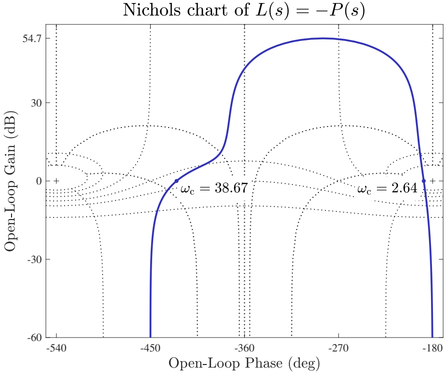

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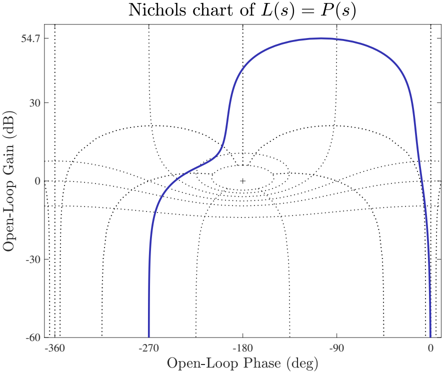

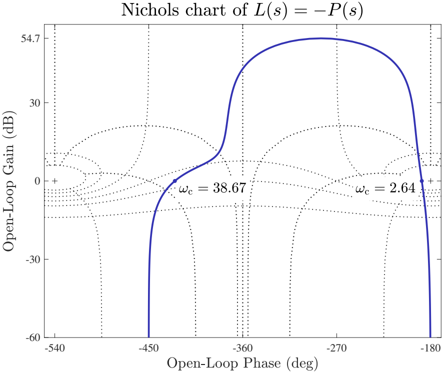

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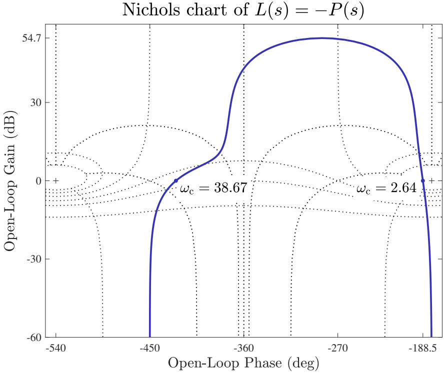
Loop shaping: keeping far from the critical point

If we keep plant's first crossover (≈ 2.64) and want phase margin of 50°,



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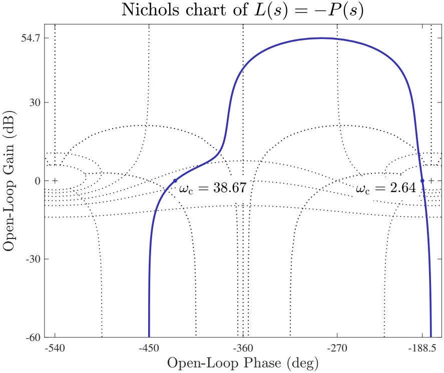

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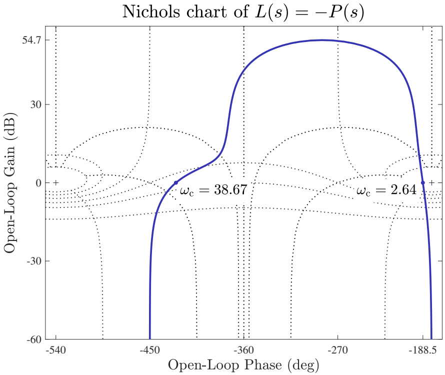
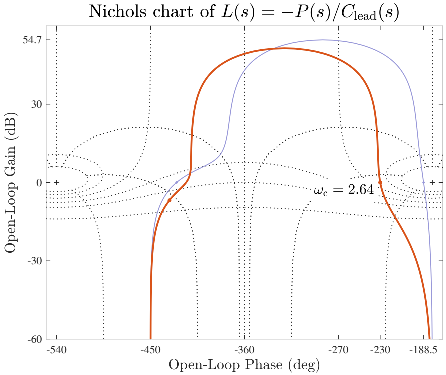




we need to add phase lag around ω_c! This promts the choice

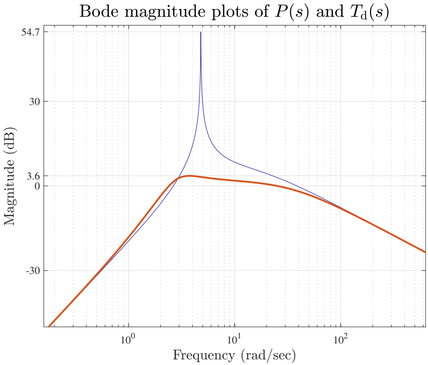
$$C(s) = -\frac{1}{C_{\text{lead}}(s)}, \qquad \text{for } \omega_m = \omega_c \text{ and } \phi_m = 41.5^\circ.$$

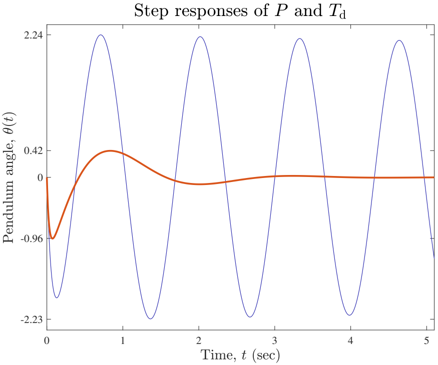
Loop shaping: keeping far from the critical point (contd)

The resulting controller,

$$C(s) = -0.451\,\frac{s+5.85}{s+1.188},$$

does its job:





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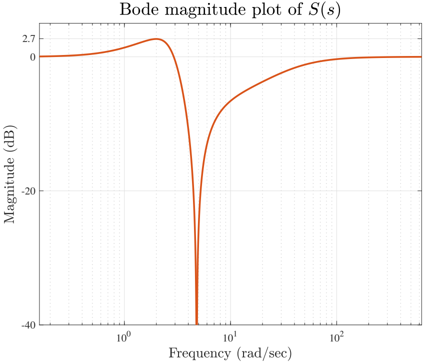
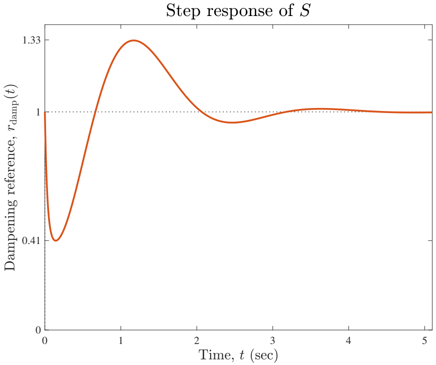
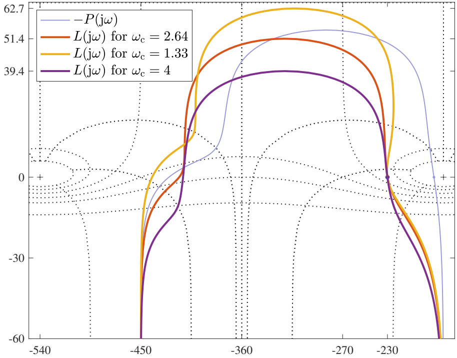




Loop shaping: let's play with ω_c

We may want to

- increase damping/accelerate response—by decrease of the first ω_c, or
- decrease control efforts—by increase of the first ω_c.



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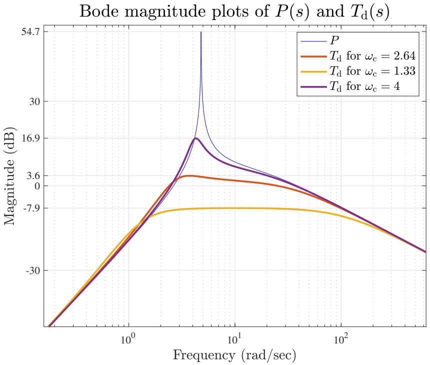

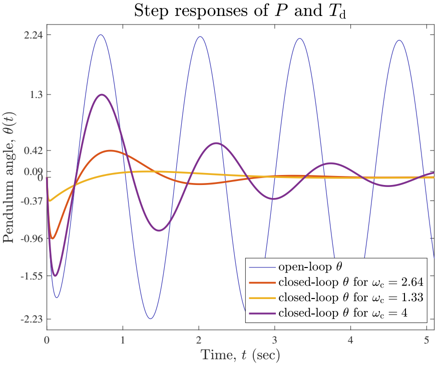

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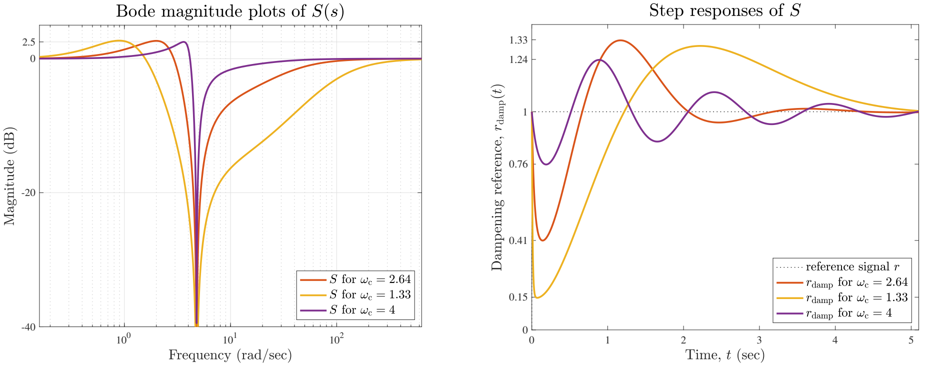

What can we learn from this example?

Shaping flexible loops is characterized by

– over-emphasizing resonance frequencies.

This means that design should not hinge upon increasing gain in the whole region, but rather can

– target narrow frequency bands, around resonances.

Flexible loops typically involve

– multiple crossover regions

with

– alternating regions of high- and low-gains.

This property is of great importance as it enables us to

– exploit phase lag (even nonminimum-phase or delay) elements

in feedback loops, thus circumventing Bode's gain-phase relation bounds.

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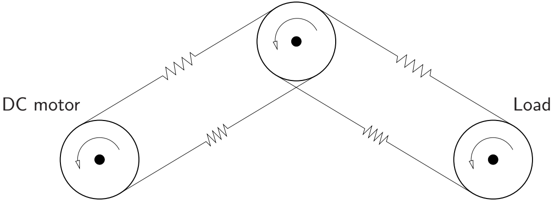
Strong stabilization

Experimental setup



DC motor is connected with load via flexible transmission. We want:

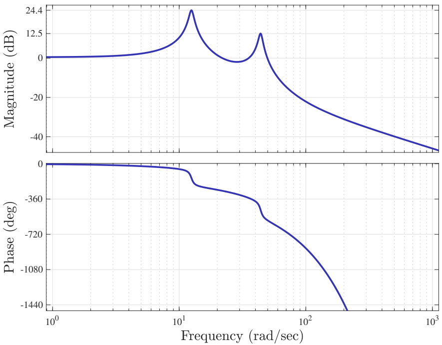
- complete steady-state rejection of step disturbances,
- dampened output response.

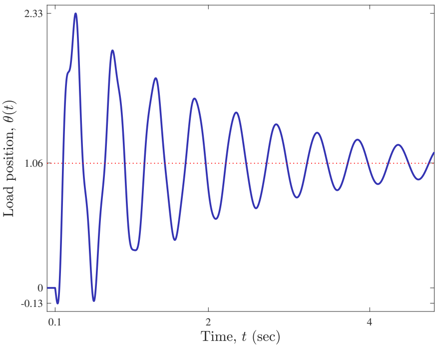
The plant

Plant transfer function (obtained experimentally) is

$$P(s) = \frac{5(-s+40)(s+40)^2}{(s^2+s+156.25)(s^2+3.172s+1936)}\mathrm{e}^{-0.1s},$$

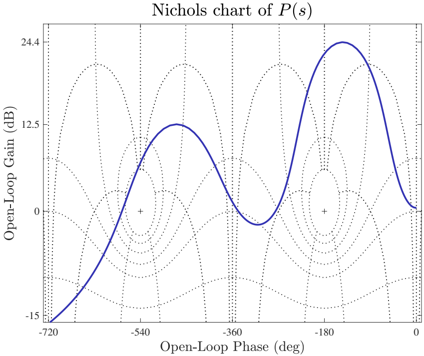
with the following Bode plot and step response:





Loop shaping logic

Delay and NMP zero add considerable phase lag:



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Adding an integral action makes this even clearer, leading to the need to

- add phase lag, again.

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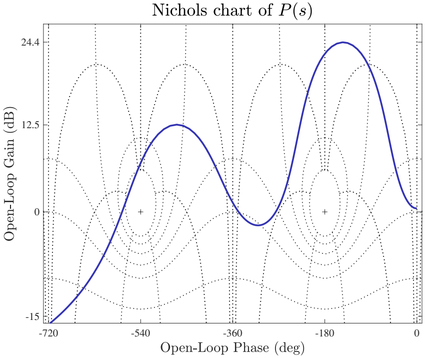

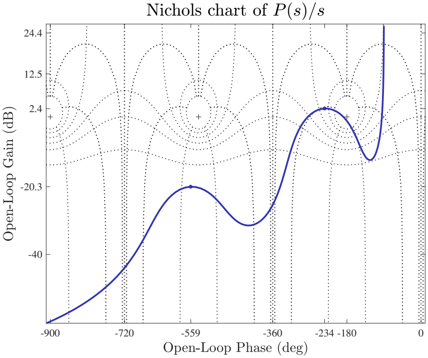

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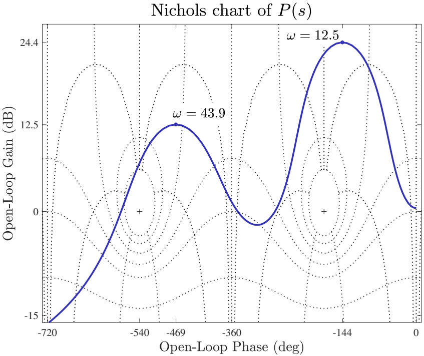
What lag do we need?



Endeavoring to locate resonances far from the critical points, we may try to

- move the resonance at $\omega = 12.5$ to $\arg L(\mathrm{j}12.5) \approx -360^\circ$
- move the resonance at $\omega = 43.9$ to $\arg L(\mathrm{j}43.9) \approx -720^\circ$

This requires

- a phase lag of 216° at $\omega = 12.5$ and a phase lag of 251° at $\omega = 43.9$.

And we also need to have high enough controller gains at those frequencies.

Tool: nonminimum-phase PID

This can be achieved by a PID controller with RHP zeros. Consider

$$C_{\mathrm{PID}}(s) = k\Big(-1 + \frac{1}{\tau_{\mathrm{i}} s} + \tau_{\mathrm{d}} s\Big) \implies C_{\mathrm{PID}}(\mathrm{j}\omega) = -k + \mathrm{j}k\Big(\tau_{\mathrm{d}}\omega - \frac{1}{\tau_{\mathrm{i}}\omega}\Big).$$

If we need phase to be in $(-270^\circ, -90^\circ)$, then $k > 0$ and

$$\arg C_{\mathrm{PID}}(\mathrm{j}\omega) = -180^\circ - \arctan\frac{\tau_{\mathrm{d}}\omega^2 - 1/\tau_{\mathrm{i}}}{\omega} \quad \text{[in degrees]}.$$

Given $\omega_2 > \omega_1 > 0$, the equations

$$\arg C_{\mathrm{PID}}(\mathrm{j}\omega_i) = \phi_i \in (-270^\circ, -90^\circ), \quad i = 1, 2$$

are solved by

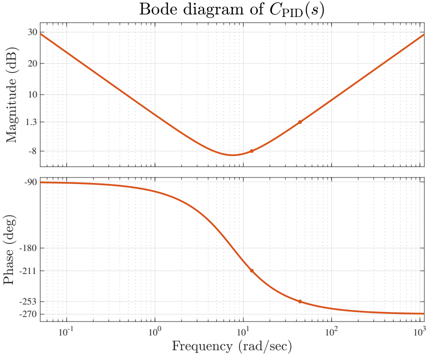
$$\tau_{\mathrm{i}} = \frac{\omega_2^2 - \omega_1^2}{\omega_1\omega_2(\omega_2 \tan\phi_1 - \omega_1 \tan\phi_2)} \quad \text{and} \quad \tau_{\mathrm{d}} = \frac{\omega_1 \tan\phi_1 - \omega_2 \tan\phi_2}{\omega_2^2 - \omega_1^2}.$$

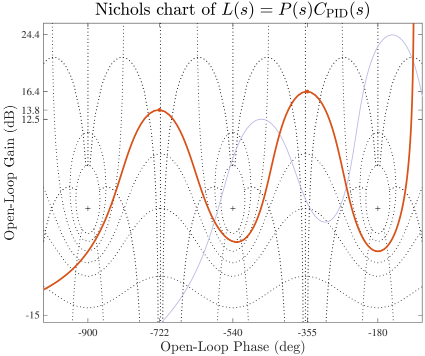
The controller

For our data ($\phi_1 = -211^\circ$, $\phi_2 = -253^\circ$, tuned manually) we end up with

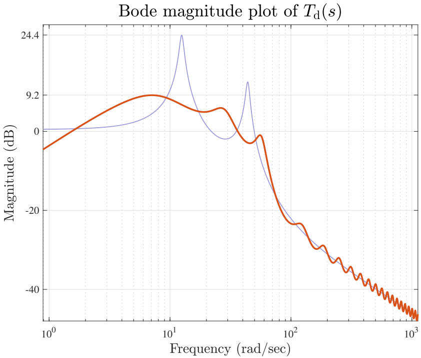
$$C_{\mathrm{PID}}(s) = 0.343\left(-1 + \frac{1}{0.229s} + 0.0757s\right) = \frac{0.026(s^2 - 13.21s + 57.68)}{s}$$

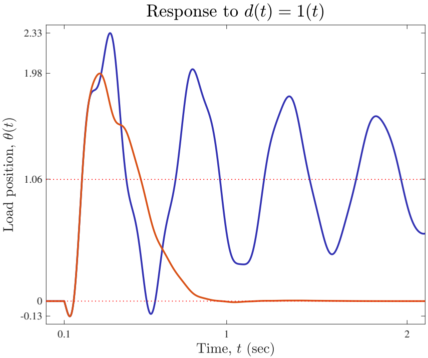
(k was also tuned manually), for which





Closed-loop disturbance response





show substantially better dampening. But

– the PID controller transfer function is non-proper.

Still, there are simple methods to have a proper controller.

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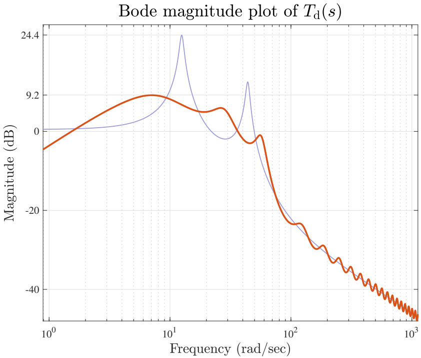

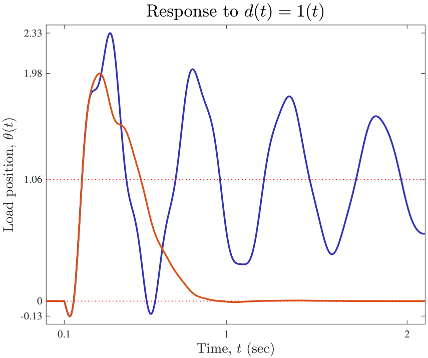

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Proper D-part

The non-proper D-part, $\tau_\mathrm{d}s$, is normally implemented as

$$\frac{\tau_\mathrm{d}s}{\alpha\tau_\mathrm{d}s+1} \quad \text{for } 0.05 \le \alpha \le 0.3.$$

Let's choose α to render $|C(\infty)| = 10\,[\mathrm{dB}]$. To this end,

$$C(s) = k\Bigl(-1+\frac{1}{\tau_\mathrm{i}s}+\frac{\tau_\mathrm{d}s}{\alpha\tau_\mathrm{d}s+1}\Bigr) \implies C(\infty) = k\,\frac{1-\alpha}{\alpha} > 0,$$

so that

$$\alpha = \frac{k}{k+C(\infty)}.$$

Thus, we need

$$\alpha = \frac{k}{k+3.1623} = 0.0979 \approx 0.1,$$

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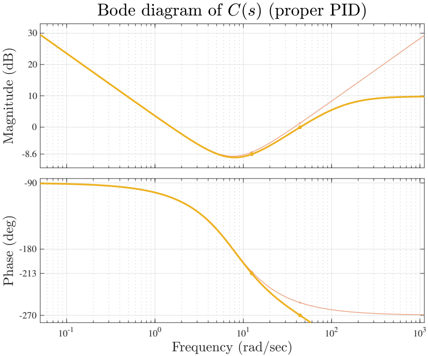
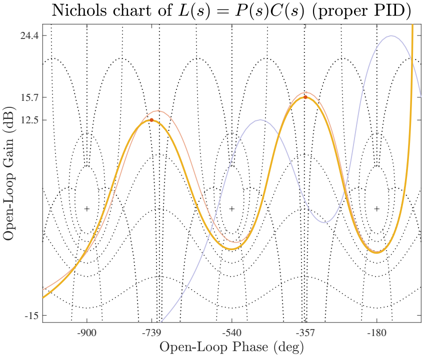
The proper controller

Thus, we have

$$C(s) = 0.343\left(-1 + \frac{1}{0.229s} + \frac{0.0757s}{0.0074s+1}\right) \approx \frac{3.1(s^2 - 14s + 64)}{s(s+132.1)},$$

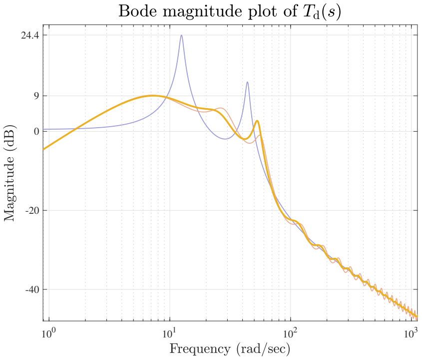
for which

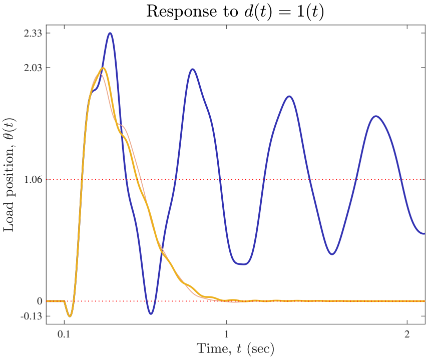




with a slight deterioration of the last phase margin and controller gain.

Closed-loop disturbance response (contd)





have comparable dampening.

Remark 1 There are other approaches to render $C(s)$ proper. For example, we may add a low-pass filter $F_{lp}(s)$ to the plant, design a PID $C_{PID}(s)$ for $P(s)F_{lp}(s)$, and then implement the proper $C(s) = F_{lp}(s)C_{PID}(s)$. Try it with $F_{lp}(s) = 1/(s/175 + 1)$ and $\phi_1 = 197^\circ$ and $\phi_2 = 242^\circ$ at the resonances.

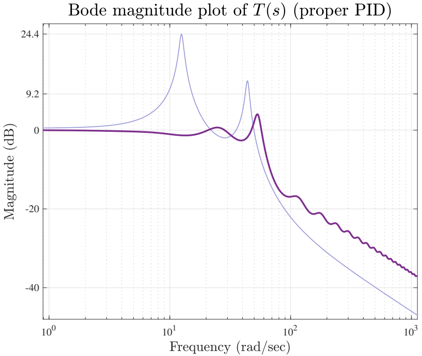
Remark 2 Small fast oscillations are the result of getting closer to the last critical point. To get rid of them, we may use a (complex) lead around the last crossover. Try it.

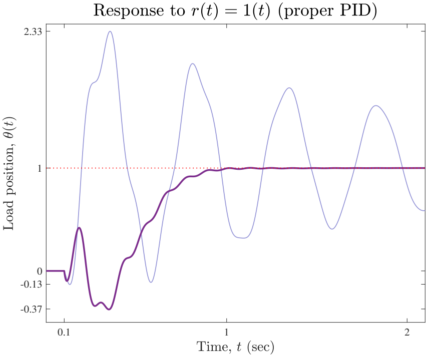
Command response

The complementary sensitivity transfer function in this case is

$$T(s) = \frac{15.5\,(-s+40)(s+40)^2(s^2-14s+64)}{\chi_{\rm cl}(s)}\,e^{-0.1s}$$

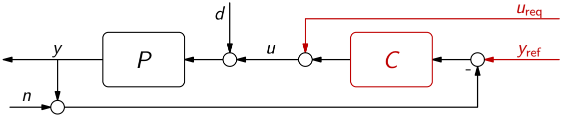
for some Hurwitz $\chi_{\rm cl}(s)$ and its step response is not quite satisfactory:





Let's improve that without altering C ...

2DOF architecture



Signals of interest (as long as $y_{\text{ref}} = P u_{\text{req}}$):

$$y = y_{\text{ref}} + T_{\text{d}} d - T n \quad \text{and} \quad u = u_{\text{req}} - T d - T_{\text{c}} n.$$

Let's use

$$y_{\text{ref}} = T_{\text{ref}} \mathbb{1} \quad \text{and} \quad u_{\text{req}} = C_{\text{ol}} \mathbb{1} = \frac{T_{\text{ref}}}{P} \mathbb{1}$$

for some reference model such that

- all nonminimum-phase zeros, and the delay, of $P(s)$ are those of $T_{\text{ref}}(s)$
- pole excess of $T_{\text{ref}}(s) \geq$ poles excess of $P(s)$
- $T_{\text{ref}}(0) = 1$ (zero steady-state error)
- T_{ref} has smooth and sufficiently fast transients

Example 2: 2DOF design

With

$$P(s) = \frac{5(-s+40)(s+40)^2}{(s^2+s+12.5^2)(s^2+3.172s+44^2)}\,\mathrm{e}^{-0.1s},$$

the reference model T_{ref} has three stability-related constraints

1. it must have a zero at $s = 40$,
2. it must have a delay of 0.1 sec,
3. its pole excess must be at least 1.

With the requirement $T_{\mathrm{ref}}(0) = 1$ we may pick

$$T_{\mathrm{ref}}(s) = \frac{-s+40}{40(\tau s+1)^2}\,\mathrm{e}^{-0.1s}$$

and tune $\tau > 0$ to have a desired settling time of its step response. Then

$$C_{\mathrm{ol}}(s) = \frac{T_{\mathrm{ref}}(s)}{P(s)} = \frac{(s^2+s+12.5^2)(s^2+3.172s+44^2)}{200(\tau s+1)^2(s+40)^2}$$

is proper and has all its poles in the OLHP, hence stable as well.

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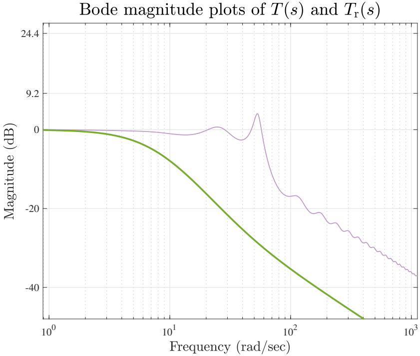
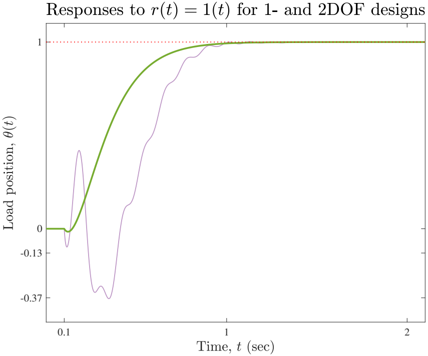
Example 2: 2DOF design (contd)

For $\tau = 1/8$ (picked to remain with about the same settling time) we have:



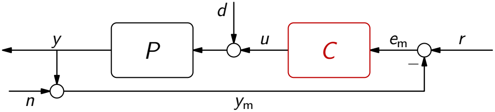


which is a much better[2] transient response.

[2] Virtually no effect of the zero at $s = 40$, the poles of $T_{\text{ref}}(s)$ at $s = -8$ are dominant.

Outline

Stabilization with stable controllers



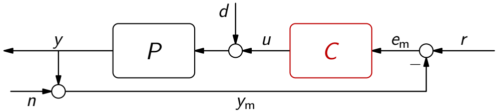
Stable controllers, especially for stable plants, are preferable since we want to maintain stability during

- sensor / actuator failures
- sensor / actuator saturation

We say that

- P is strongly stabilizable if it can be stabilized by a stable controller.

Stabilization with stable controllers



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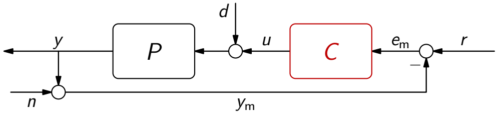
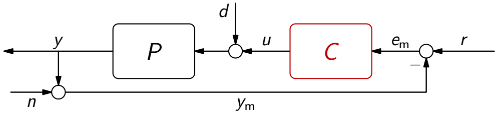
Parity interlacing property



P is strongly stabilizable iff its transfer function has

– even number of real poles between every pair of real zeros in RHP (including $+\infty$). This property called the parity interlacing property.

Parity interlacing property



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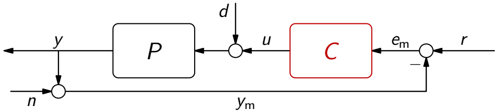
Example 1: Let $P(s) = \frac{s-1}{s(s-2)}$. It has 2 RHP zeros at $\{1, \infty\}$ and between them one pole at 2.

Parity interlacing property



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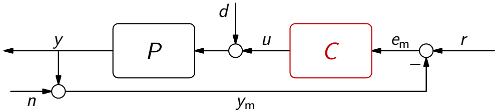
Example 1: Let $P(s) = \frac{s-1}{s(s-2)}$. It has 2 RHP zeros at $\{1, \infty\}$ and between them one pole at 2. Hence it is not strongly stabilizable.

Parity interlacing property



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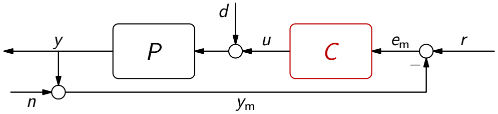
Example 2: Let $P(s) = \frac{(s-1)^2(s^2-s+1)}{(s-2)^2(s+1)^3}$. It has 5 RHP zeros, 3 of them real at $\{1, 1, \infty\}$. Between 1 and 1 lies 0 poles, while between 1 and ∞ lie 2 poles (at 2).

Parity interlacing property



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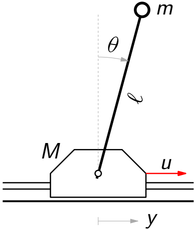
Inverted pendulum without angle measurement



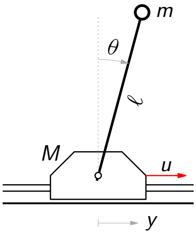
M	:	cart mass
m	:	pendulum mass
ℓ	:	pendulum length
y	:	cart position
θ	:	pendulum angle
u	:	force applied to the cart
g	:	standard gravity

Linearized transfer function of the inverted pendulum from u to y is

$$P(s) = \frac{\ell s^2 - g}{M\bigl(\ell s^2 - g(1 + \frac{m}{M})\bigr)s^2}.$$

It has 3 real RHP zeros in $\{\sqrt{g/\ell}, \infty, \infty\}$. Between the first two of them $P(s)$ has one pole at $s = \sqrt{(1 + m/M)g/\ell}$. Thus, pendulum is not strongly stabilizable.

Inverted pendulum without angle measurement



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– pendulum is not strongly stabilizable.