# Control Theory (00840738)

HOMEWORK 1, PREREQUISITE FOR LABORATORY 1 (SUBMISSION DEADLINE: **19/11/2026**, STRICT)

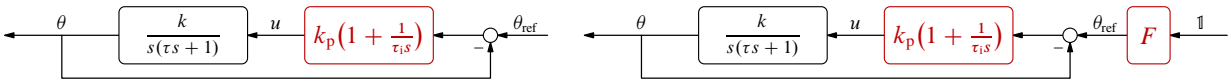
The purpose of Laboratory 1 is to identify parameters of a DC motor from experimental data. Assume that we have a DC motor connected to a rigid mechanical load with the armature voltage $u$ as the input and the shaft angle $\theta$ as the output. A general form of the transfer function of this system (see Lecture 2) is

$$P(s) = \frac{k}{s(\tau s + 1)}$$

for some $k > 0$ and $\tau > 0$, which are assumed to be unknown. Assume that input voltage is generated as the output of a linear system, call it controller, having the transfer function

$$R(s) = k_{\mathrm{p}}\left(1 + \frac{1}{\tau_{\mathrm{i}}\, s}\right) \quad \text{for } k_{\mathrm{p}} > 0 \text{ and } \tau_{\mathrm{i}} > 0,$$

whose input is the mismatch between an exogenous signal $\theta_{\mathrm{ref}}$ and the motor shaft angle, see Fig. 1(a). This is a (negative) unity feedback around the series interconnection of $P$ and $R$. The idea is to tune the




(a) (b)

Fig. 1: Identification setups

parameters $k_{\mathrm{p}}$ and $\tau_{\mathrm{i}}$ to end up with a marginally stable response of the system $T : \theta_{\mathrm{ref}} \mapsto \theta$ and then identify $k$ and $\tau$ from characteristics of that response and the knowledge of $k_{\mathrm{p}}$ and $\tau_{\mathrm{i}}$.

1. Derive the transfer function of the closed-loop system $T : \theta_{\mathrm{ref}} \mapsto \theta$. What is its static gain?
2. Under what choices of $k_{\mathrm{p}} > 0$ and $\tau_{\mathrm{i}} > 0$ the system $T$ is stable?
3. Under what choices of $k_{\mathrm{p}} > 0$ and $\tau_{\mathrm{i}} > 0$ the system $T$ has a pair of pure imaginary nonzero poles?
4. What is the response of the resulting $T$ to a unit step input $\theta_{\mathrm{ref}} = \mathbb{1}$?
5. Can the unknown parameters of $P(s)$ be identified from characteristics of the step response in the previous item (assuming the knowledge of $R(s)$, of course)? If yes, propose an algorithm.
6. Does the answer to the previous question change if the unit step input $\theta_{\mathrm{ref}} = \mathbb{1}$ is replaced by a filtered unit step of the form $\theta_{\mathrm{ref}} = F\mathbb{1}$, see Fig. 1(b), where $F$ is a stable system?

   Hint: take a look at the Appendix A to refresh your memory on the partial fraction expansion techniques / meaning.
7. The file `data1.txt` contains two vectors, the time $t$ and the response of $T$ satisfying the condition of Item 3 under $R(s) = 2 + 1/s$ to $\theta_{\mathrm{ref}} = F\mathbb{1}$ for an unknown stable $F$. Load the data to Matlab via

   ```
   data = readmatrix('data1.txt'); t = data(1,:); theta = data(2,:); clear data
   ```

   Determine $k$ and $\tau$ and plot the step response of the resulting closed-loop system $T$. What does this response have in common with $\theta$ from `data1.txt`?

   Submit a standalone Matlab file with your calculations and plot generation. Use a vector, not bitmap, PDF in your report, via `exportgraphics(gca,'filename.pdf','ContentType','vector')` from within the code (i.e. don't save figures manually).

# A Steady-state components of signals via partial fraction expansion

Let $y$ be a signal, whose Laplace transform $Y(s)$ has no poles in the open right-half plane $\{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$ and all its pure imaginary poles are simple, i.e. their multiplicity is one. We know that poles of $Y(s)$ in the open left-half plane (OLHP) $\{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$ correspond to decaying transient parts of $y$, while pure imaginary poles correspond to persistent steady-state components. A way to see that is to carry out partial fraction expansion of $Y(s)$ and use the linearity of the inverse Laplace transform. For example, let

$$Y(s) = \frac{s-2}{s(s+1)(s+2)}.$$

Because all poles are simple, we may use residues to carry out the partial fraction expansion. Specifically, the residue of $Y(s)$ at its simple pole $s = p$ is defined as

$$\operatorname{Res}(Y, p) := \lim_{s \to p} (s-p) Y(s).$$

In our case

$$Y(s) = \sum_{i=1}^{3} \frac{\operatorname{Res}(Y, p_i)}{s - p_i} = -\frac{1}{s} + \left( \frac{3}{s+1} - \frac{2}{s+2} \right) = -\frac{1}{s} + \frac{s+4}{(s+1)(s+2)}$$

and then

$$y(t) = -\mathbb{1}(t) + \left(3\mathrm{e}^{-t}\mathbb{1}(t) - 2\mathrm{e}^{-2t}\mathbb{1}(t)\right) \xrightarrow{t \to \infty} -1.$$

Important is that if we are interested only in steady-state characteristics, then only partial fractions corresponding to pure imaginary poles are of interest. For example, let

$$Y(s) = \frac{F(s)}{s}$$

for some $F(s)$ having its poles in the OLHP. In this case $\operatorname{Res}(Y, 0) = F(0)$ and

$$Y(s) = \frac{F(0)}{s} + Y_{\mathrm{tr}}(s) = \frac{F(0)}{s} + \frac{F(s) - F(0)}{s}$$

where $Y_{\mathrm{tr}}(s)$ has its poles at the poles of $F(s)$, i.e. in the OLHP. Hence, $Y_{\mathrm{tr}}(s)$ is the Laplace transform of a decaying function $y_{\mathrm{tr}}$. The steady-state part of $y$ is thus $F(0)\mathbb{1}$, regardless of the specific form of $y_{\mathrm{tr}}$.

Likewise, let

$$Y(s) = \frac{F(s)}{s^2 + \omega^2}$$

for some $F(s)$ having real parameters and all poles in the OLHP. In this case $Y(s)$ has two pure imaginary poles, at $s = \pm \mathrm{j}\omega$, and $\operatorname{Res}(Y, \pm\mathrm{j}\omega) = \pm F(\pm\mathrm{j}\omega)/(\mathrm{j}2\omega)$. Thus,

$$Y(s) = \frac{F(\mathrm{j}\omega)/(\mathrm{j}2\omega)}{s - \mathrm{j}\omega} - \frac{F(-\mathrm{j}\omega)/(\mathrm{j}2\omega)}{s + \mathrm{j}\omega} + Y_{\mathrm{tr}}(s)$$

for some $Y_{\mathrm{tr}}(s)$ having its poles at the poles of $F(s)$ and thus corresponding to the decaying transient components of $y$. Note that because $F(-\mathrm{j}\omega) = \overline{F(\mathrm{j}\omega)}$, we have that

$$F(\pm\mathrm{j}\omega) = |F(\mathrm{j}\omega)| \mathrm{e}^{\pm \mathrm{j} \arg F(\mathrm{j}\omega)}.$$

Therefore,

$$\begin{aligned}
Y(s) &= \frac{|F(\mathrm{j}\omega)|}{\omega} \left( \frac{\mathrm{e}^{\mathrm{j}\arg F(\mathrm{j}\omega)}/(2\mathrm{j})}{s - \mathrm{j}\omega} - \frac{\mathrm{e}^{-\mathrm{j}\arg F(\mathrm{j}\omega)}/(2\mathrm{j})}{s + \mathrm{j}\omega} \right) + Y_{\mathrm{tr}}(s) \\
&= \frac{|F(\mathrm{j}\omega)|}{\omega} \frac{s(\mathrm{e}^{\mathrm{j}\arg F(\mathrm{j}\omega)} - \mathrm{e}^{-\mathrm{j}\arg F(\mathrm{j}\omega)})/(2\mathrm{j}) + \omega(\mathrm{e}^{\mathrm{j}\arg F(\mathrm{j}\omega)} + \mathrm{e}^{-\mathrm{j}\arg F(\mathrm{j}\omega)})/2}{s^2 + \omega^2} + Y_{\mathrm{tr}}(s) \\
&= \frac{|F(\mathrm{j}\omega)|}{\omega} \frac{\sin(\arg F(\mathrm{j}\omega))\, s + \cos(\arg F(\mathrm{j}\omega))\omega}{s^2 + \omega^2} + Y_{\mathrm{tr}}(s).
\end{aligned}$$

Hence,

$$y(t) = \frac{|F(\mathrm{j}\omega)|}{\omega} \sin\bigl(\omega t + \arg F(\mathrm{j}\omega)\bigr) + y_{\mathrm{tr}}(t) \xrightarrow{\text{steady state}} \frac{|F(\mathrm{j}\omega)|}{\omega} \sin\bigl(\omega t + \arg F(\mathrm{j}\omega)\bigr).$$