

Control of MIMO Systems (00860289)

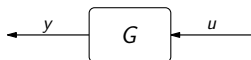
chapter no. 2

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Static systems



is static (memoryless) if

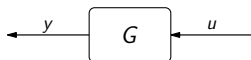
- $y(t_1)$ depends only on $u(t)$ only at $t = t_1$ for all t_1

Hence,

- frozen-time analysis

(no need in time dependence)

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Outline

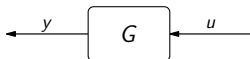
Static MIMO systems: basic notions

Size matters

Direction matters too

Systems as modeling tools

Signals & systems



Signals are vectors:

$$u \in \mathbb{F}^m \quad \text{and} \quad y \in \mathbb{F}^p.$$

Systems are mappings:

$$G : \mathbb{F}^m \rightarrow \mathbb{F}^p$$

with $\mathbb{F}^m$ as its **domain** and $\mathbb{F}^p$ as its **co-domain**, meaning

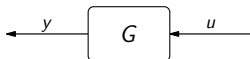
- every $u \in \mathbb{F}^m$ is mapped to $y = G(u) \in \mathbb{F}^p$ (i.e. $G : u \mapsto y$),

Sometimes we simplify $y = G(u)$ to $y = Gu$ when the input vector is clear.

G is linear if (superposition)

$$G(a_1 u_1 + a_2 u_2) = a_1 G(u_1) + a_2 G(u_2), \quad \forall u_1, u_2 \in \mathbb{F}^m, a_1, a_2 \in \mathbb{F}$$

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i.e. additivity $G(u_1 + u_2) = G(u_1) + G(u_2)$ + homogeneity $G(\alpha u) = \alpha G(u)$

Visualizing signals and systems in the standard basis $\{e_i\}$

Given $x \in \mathbb{F}^n$,

$$x = \sum_{i=1}^n x_i e_i = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} x_1 + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} x_n = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Given $G : \mathbb{F}^m \rightarrow \mathbb{F}^p$,

$$y = G\left(\sum_{j=1}^m u_j e_j\right) = \sum_{j=1}^m u_j G(e_j) = \sum_{j=1}^m u_j \sum_{i=1}^p g_{ij} \tilde{e}_i = \sum_{i=1}^p \left(\sum_{j=1}^m g_{ij} u_j\right) \tilde{e}_i$$

whence

$$\begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & & \vdots \\ g_{p1} & \cdots & g_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}.$$

system matrix (also coordinate matrix)

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Matrix notation and terminology

$G \in \mathbb{F}^{p \times m}$ is

- **square** if $p = m$, **tall** if $p > m$, and **fat** if $p < m$
- **upper (lower) triangular** if its elements $g_{ij} = 0$ whenever $i > j$ ($i < j$)
- **diagonal** if $g_{ij} = 0$ whenever $i \neq j$, denoted $\text{diag}\{g_{ii}\}$

For square G ,

- **trace** $\text{tr}(G) := \sum_{i=1}^m g_{ii} \in \mathbb{F}$
- **spectrum** $\text{spec}(G)$ is the set of all m eigenvalues, $\{\lambda_i(G)\}$
- **spectral radius** $\rho(G) := \max_i |\lambda_i(G)|$

Similarity transformations

Given a (non-standard) basis $\{v_1, \dots, v_m\}$ on $\mathbb{F}^m$,

$$x = \tilde{x}_1 v_1 + \dots + \tilde{x}_m v_m = \begin{bmatrix} v_1 & \dots & v_m \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_m \end{bmatrix} =: T \tilde{x}$$

and $T \in \mathbb{F}^{m \times m}$ is invertible. The vector

- $\tilde{x} = T^{-1}x$ represents the same signal, from a different viewpoint.

If $\tilde{u} := T_u^{-1}u$ and $\tilde{y} := T_y^{-1}y$, then

$$y = Gu \iff \tilde{y} = T_y^{-1}GT_u\tilde{u},$$

where $T_y^{-1}GT_u$ is the matrix representation of G in these new coordinates.

If $p = m$, we may take $T_u = T_y = T$ and $T^{-1}GT$ is called similar to G .

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Similarity transformations: example

DFT basis

$$\phi_i := \frac{1}{n} \begin{bmatrix} 1 \\ (e^{j2\pi/n})^{i-1} \\ \vdots \\ (e^{j2\pi(n-1)/n})^{i-1} \end{bmatrix}, \quad i \in \mathbb{Z}_{1..n}.$$

Let

$$G = \frac{1}{3} \begin{bmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{bmatrix}.$$

and consider

$$T = \frac{1}{6} \begin{bmatrix} 2 & 2 & 2 \\ 2 & -1+j\sqrt{3} & -1-j\sqrt{3} \\ 2 & -1-j\sqrt{3} & -1+j\sqrt{3} \end{bmatrix}, \quad T^{-1} = \frac{1}{6} \begin{bmatrix} 2 & 2 & 2 \\ 2 & -1-j\sqrt{3} & -1+j\sqrt{3} \\ 2 & -1+j\sqrt{3} & -1-j\sqrt{3} \end{bmatrix}.$$

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$$T^{-1}GT = \text{diag}\{1, 2, 2\}.$$

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Signal norms

Hölder norms $\|x\|_q$, $q \geq 1$, of $x \in \mathbb{F}^n$:

$$\|x\|_1 := \sum_{i=1}^n |x_i|, \quad \|x\| = \|x\|_2 := \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}, \quad \|x\|_\infty := \max_{1 \leq i \leq n} |x_i|.$$

x is **unit vector** in q -metric if $\|x\|_q = 1$.

All vector norms are equivalent, that is

$$\exists \gamma_2 > \gamma_1 > 0 \text{ such that } \gamma_1 \|x\|_b \leq \|x\|_a \leq \gamma_2 \|x\|_b, \quad \forall x.$$

e.g. if $x \in \mathbb{F}^n$, then

$$\|x\|_{q_2} \leq \|x\|_{q_1} \leq n^{1/q_1 - 1/q_2} \|x\|_{q_2} \text{ whenever } q_1 < q_2$$

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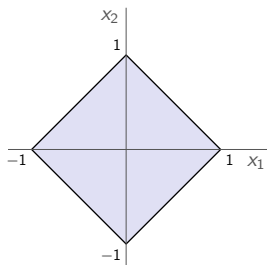
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Unit ball

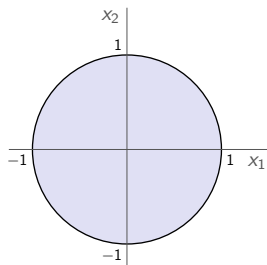
$$\mathcal{B}_q := \{x \mid \|x\|_q \leq 1\}.$$

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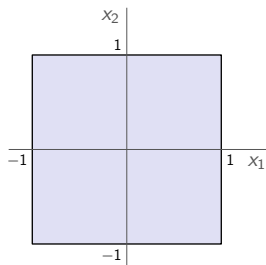
$\mathcal{B}_1$:



$\mathcal{B}_2$:



$\mathcal{B}_\infty$:



System norms

Induced norms (largest gain):

$$\|G\|_q := \max_{u \in \mathbb{F}^m, u \neq 0} \frac{\|Gu\|_q}{\|u\|_q} = \sup_{\|u\|_q=1} \|Gu\|_q = \sup_{u \in \mathcal{B}_q} \|Gu\|_q.$$

e.g.

$$\|G\|_1 = \max_{1 \leq j \leq m} \sum_{i=1}^p |g_{ij}|, \quad \|G\| = \|G\|_2 = \sqrt{\rho(G'G)}, \quad \|G\|_\infty = \max_{1 \leq i \leq p} \sum_{j=1}^m |g_{ij}|$$

Sub-multiplicative property of induced norms: $\|G_2 G_1\|_q \leq \|G_2\|_q \|G_1\|_q$,

$$\|G_1 G_2\|_q \leq \|G_1\|_q \|G_2\|_q, \quad \|G_1 G_2\|_q \leq \|G_1\|_q \|G_2\|_q + \|G_1\|_q \|G_2\|_q$$

Frobenius norm (not induced)

$$\|G\|_F := \sqrt{\text{tr}(G'G)} = \left(\sum_{i=1}^m \|G e_i\|^2 \right)^{1/2} = \left(\sum_{i=1}^p \sum_{j=1}^m |g_{ij}|^2 \right)^{1/2}$$

All matrix norms are also equivalent, e.g. $\|G\| \leq \|G\|_F \leq \sqrt{\text{rank}(G)} \|G\|$.

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Induced system norms: example

If

$$G = \begin{bmatrix} 1 + \alpha & 1 - \alpha \\ 1 - \alpha & 1 + \alpha \end{bmatrix} \quad \text{for } \alpha \in [0, 1],$$

then

$$\text{spec}(G'G) = \{4, 4\alpha^2\} \implies \|G\| = 2.$$

If $u = \begin{bmatrix} u_0 \\ u_0 \end{bmatrix}$, then

$$y = \begin{bmatrix} 1 + \alpha & 1 - \alpha \\ 1 - \alpha & 1 + \alpha \end{bmatrix} \begin{bmatrix} u_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} u_0 \implies \|y\| = 2\|u\| = \|G\|\|u\|$$

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— the induced norm alone is not enough to understand amplification.

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Signal direction

The **direction** of $x \in \mathbb{F}^n$ is the

- 1-dimensional **subspace** $\text{span}(x) \subset \mathbb{F}^n$.

Relative direction may be quantified by the **inner product** notion,

$$\langle x, y \rangle := \sum_{i=1}^n \bar{y}_i x_i = y'x,$$

with

$$\cos \theta = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

x and y are

orthogonal if $\langle x, y \rangle = 0$

co-directed if $|\langle x, y \rangle| = \|x\| \|y\|$

i.e. $\exists \alpha \in \mathbb{F}$ such that $x = \alpha y$

Note,

- if u_1 and u_2 co-directed, then $\|u_1\| = |\alpha| \|u_2\| \implies \|Gu_1\| = |\alpha| \|Gu_2\|$

Special I/O directions

Given $G : \mathbb{F}^m \rightarrow \mathbb{F}^p$,

- kernel (null space) $\ker G := \{u \in \mathbb{F}^m \mid Gu = 0\}$
- image (range) $\operatorname{Im} G := \{y \in \mathbb{F}^p \mid \exists u \in \mathbb{F}^m \text{ such that } y = Gu\}$

and $\text{rank } G := \dim(\operatorname{Im} G)$ quantifies the richness of G .

Kernel defines freedom of choice for the input (or its ambiguity),

$$y = Gu \iff u = u_0 + u_n \text{ for some } u_n \in \ker G$$

for any u_0 such that $y_0 = Gu_0$ (unique iff $\ker G = \{0\}$, i.e. trivial).

Image defines consistency of measurements,

$$y_m = Gu \iff y_m \in \operatorname{Im} G$$

Proposition 2.1

$(\ker G)^\perp = \operatorname{Im} G'$ and $(\operatorname{Im} G)^\perp = \ker G'$ for all $G \in \mathbb{F}^{p \times m}$.

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Special I/O directions

Given $G : \mathbb{F}^m \rightarrow \mathbb{F}^p$,

- kernel (null space) $\ker G := \{u \in \mathbb{F}^m \mid Gu = 0\}$
- image (range) $\operatorname{Im} G := \{y \in \mathbb{F}^p \mid \exists u \in \mathbb{F}^m \text{ such that } y = Gu\}$

and $\text{rank } G := \dim(\operatorname{Im} G)$ quantifies the richness of G .

Kernel defines **freedom of choice** for the input (or its ambiguity),

$$y = Gu \iff u = u_0 + u_n \text{ for some } u_n \in \ker G$$

for any u_0 such that $y_0 = Gu_0$ (unique iff $\ker G = \{0\}$, i.e. **trivial**).

Image defines **consistency** of measurements,

$$y_m = Gu \iff y_m \in \operatorname{Im} G$$

Proposition 2.1

$(\ker G)^\perp = \operatorname{Im} G'$ and $(\operatorname{Im} G)^\perp = \ker G'$ for all $G \in \mathbb{F}^{p \times m}$.

Adjoint system

Given $G \in \mathbb{F}^{m \times p}$, there is a unique $G' \in \mathbb{F}^{p \times m}$ such that

$$\langle Gu, y \rangle = \langle u, G'y \rangle, \quad \text{for all } u \in \mathbb{F}^m \text{ and } y \in \mathbb{F}^p.$$

Because

$$\begin{aligned} \langle Gu, y \rangle &= \overline{y_1}(g_{11}u_1 + \cdots + g_{1m}u_m) + \cdots + \overline{y_p}(g_{p1}u_1 + \cdots + g_{pm}u_m) \\ &= \overline{(g_{11}y_1 + \cdots + g_{p1}y_p)}u_1 + \cdots + \overline{(g_{1m}y_1 + \cdots + g_{pm}y_p)}u_m \\ &= \left\langle u, \begin{bmatrix} \overline{g_{11}} & \cdots & \overline{g_{p1}} \\ \vdots & & \vdots \\ \overline{g_{1m}} & \cdots & \overline{g_{pm}} \end{bmatrix} y \right\rangle, \end{aligned}$$

we have

$$\begin{bmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & & \vdots \\ g_{p1} & \cdots & g_{pm} \end{bmatrix}' = \begin{bmatrix} \overline{g_{11}} & \cdots & \overline{g_{p1}} \\ \vdots & & \vdots \\ \overline{g_{1m}} & \cdots & \overline{g_{pm}} \end{bmatrix}$$

(complex conjugate transpose).

Matrix notation and terminology (contd)

We say that $G \in \mathbb{F}^{m \times m}$ is

- Hermitian (symmetric, if $\mathbb{F} = \mathbb{R}$) if $G' = G$;
- skew-Hermitian (skew-symmetric, if $\mathbb{F} = \mathbb{R}$) if $G' = -G$;
- normal if $GG' = G'G$;

Diagonal matrices

G is diagonal if

$$Ge_i = g_i e_i, \quad \forall i \in \mathbb{Z}_{1..m}$$

so

$$y = G \left(\sum_{i=1}^m u_i e_i \right) = \sum_{i=1}^m (u_i g_i) e_i$$

Because $\{e_i\}$ is orthonormal,

$$\|y\|^2 = \sum_{i=1}^m |g_i|^2 |u_i|^2.$$

so

— $|g_i|$ is the gain of the component of u in the direction $\text{span}(e_i)$

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Eigen-diagonalization

Find $0 \neq v \in \mathbb{F}^m$ such that

$$Gv = \lambda v$$

for some $\lambda \in \mathbb{C}$. Equivalent to finding λ for which $\ker(\lambda I - A)$ is nontrivial.

- there are m such λ_i (eigenvalues)
- spectrum $\text{spec}(A) = \{\lambda_i\}$
- corresponding v_i , i.e. such that $Gv_i = \lambda_i v_i$, are eigenvectors

Hence,

- $|\lambda_i|$ is the gain of G in the direction $\text{span}(v_i)$

However,

1. not every G has eigenvectors (e.g. rotation)
 2. not every G has m linearly independent eigenvectors (e.g. non-diagonalizable)
 3. not every G has orthonormal eigenbasis (e.g. non-normal, $GG^H \neq G^H G$)
- $(x = \sum_{i=1}^m u_i \lambda_i v_i \Rightarrow \|x\|^2 = \sum_{i=1}^m |\lambda_i|^2 \|u_i\|^2)$

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Hence,

- $|\lambda_i|$ is the gain of G in the direction $\text{span}(v_i)$

However,

1. not every G has eigenvectors square
 2. not every G has m linearly independent eigenvectors diagonalizable
 3. not every G has orthonormal eigenbasis normal, $GG' = G'G$
- $(y = \sum_{i=1}^m u_i \lambda_i v_i \not\Rightarrow \|y\|^2 = \sum_{i=1}^m |\lambda_i|^2 |u_i|^2)$

Unitary matrices

$G \in \mathbb{F}^{m \times m}$ is said to be

- **unitary** if $G'G = I$ (equivalently, $G^{-1} = G'$).

If G is unitary, then

$$\|Gu\|^2 = \langle Gu, Gu \rangle = \langle G'Gu, u \rangle = \langle u, u \rangle = \|u\|^2, \quad \forall u \in \mathbb{F}^m$$

i.e. has unit norm in all directions.

$\mathbb{R}^{2 \times 2}$ examples:

plain rotation

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

plain reflection

$$Q_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

In real case, either rotations (if $\det G = 1$) or reflections (if $\det G = -1$).

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$\mathbb{R}^{2 \times 2}$ examples:

	plain rotation	plain reflection
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$	$Q_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

In real case, either rotations (if $\det G = 1$) or reflections (if $\det G = -1$).

Unitary matrices (contd)

If

$$G = \begin{bmatrix} g_1 & g_2 & \cdots & g_m \end{bmatrix}, \quad g_i \in \mathbb{F}^m$$

is unitary, then

$$G'G = \begin{bmatrix} g'_1 \\ \vdots \\ g'_m \end{bmatrix} \begin{bmatrix} g_1 & \cdots & g_m \end{bmatrix} = \begin{bmatrix} g'_1 g_1 & \cdots & g'_1 g_m \\ \vdots & \ddots & \vdots \\ g'_m g_1 & \cdots & g'_m g_m \end{bmatrix} = I,$$

i.e.

- columns of unitary G constitute an **orthonormal basis** in $\mathbb{F}^m$.

E.g.

- identity $I_2 \implies$ the standard basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
- rotation $P_\theta \implies \left\{ \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}, \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \right\}$

Singular value decomposition / Matlab: `svd(G)`

Every $G \in \mathbb{F}^{p \times m}$ can be decomposed as

$$G = U \Sigma V'$$

for *unitary*

$$U = \begin{bmatrix} u_1 & u_2 & \dots & u_p \end{bmatrix} \in \mathbb{F}^{p \times p} \quad \text{and} \quad V = \begin{bmatrix} v_1 & v_2 & \dots & v_m \end{bmatrix} \in \mathbb{F}^{m \times m}$$

and $\Sigma \in \mathbb{R}^{p \times m}$ is of the form

$$\Sigma = \begin{cases} \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix} & \text{if } p \leq m \\ \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} & \text{if } p \geq m \end{cases}$$

and $\Sigma_1 = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_{\min\{p,m\}}\}$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\min\{p,m\}} \geq 0$.

Singular value decomposition / Matlab: `svd(G)`

Every $G \in \mathbb{F}^{p \times m}$ can be decomposed as

$$G = U \Sigma V' = \sum_{i=1}^{\min\{p,m\}} \sigma_i u_i v_i',$$

for *unitary*

$$U = \begin{bmatrix} u_1 & u_2 & \dots & u_p \end{bmatrix} \in \mathbb{F}^{p \times p} \quad \text{and} \quad V = \begin{bmatrix} v_1 & v_2 & \dots & v_m \end{bmatrix} \in \mathbb{F}^{m \times m}$$

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and $\Sigma_1 = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_{\min\{p,m\}}\}$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\min\{p,m\}} \geq 0$.

Singular value decomposition (contd)

- $\sigma_i \geq 0$ are called the **singular values** of G
- u_i are called **left singular vectors** of G
- v_i are called **right singular vectors** of G

Because (assume for simplicity that $p = m$)

$$G'G = V\Sigma^2V' \implies G'Gv_i = Ve_i\sigma_i^2 = \sigma_i^2v_i$$

and

$$GG' = U\Sigma^2U' \implies GG'u_i = Ue_i\sigma_i^2 = \sigma_i^2u_i$$

σ_i^2 are the eigenvalues of both $G'G$ and GG' .

Special cases:

- if G is normal ($G'G = GG'$), then $\sigma_i = |\lambda_i|$
- if G is hermitian and $G \geq 0$, then $\sigma_i = \lambda_i$

Interpretation

If

$$U = \begin{bmatrix} \cos \theta_{\text{out}} & -\sin \theta_{\text{out}} \\ \sin \theta_{\text{out}} & \cos \theta_{\text{out}} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \cos \theta_{\text{inp}} & -\sin \theta_{\text{inp}} \\ \sin \theta_{\text{inp}} & \cos \theta_{\text{inp}} \end{bmatrix}$$

then

$$G = \begin{bmatrix} \cos \theta_{\text{out}} & -\sin \theta_{\text{out}} \\ \sin \theta_{\text{out}} & \cos \theta_{\text{out}} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos \theta_{\text{inp}} & \sin \theta_{\text{inp}} \\ -\sin \theta_{\text{inp}} & \cos \theta_{\text{inp}} \end{bmatrix}$$

and every input w

- rotated by V' by $-\theta_{\text{inp}}$
- scaled by Σ along standard directions
- rotated by U by θ_{out}

i.e.

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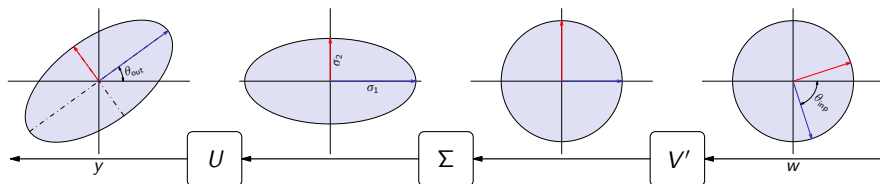
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i.e.



System action via SVD

Given $G : w \mapsto y$ and

$$w = \sum_{i=1}^m w_i v_i, \quad \text{where } w_i = v_i' w = \langle w, v_i \rangle \text{ are coordinates of } w \text{ in } \{v_i\}$$

By orthonormality of $\{v_i\}$, i.e. $v_i' v_j = \delta_{ij}$ (Kronecker delta),

$$y = \left(\sum_{i=1}^{\min\{p,m\}} \sigma_i u_i v_i' \right) \left(\sum_{i=1}^m w_i v_i \right) = \sum_{i=1}^{\min\{p,m\}} (\sigma_i w_i) u_i$$

($y_i = \sigma_i w_i$ are coordinates of y in $\{u_i\}$) and

$$\|y\|^2 = \sum_{i=1}^{\min\{p,m\}} \sigma_i^2 |w_i|^2.$$

like in the diagonal case

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like in the diagonal case.

System gains via SVD

Proposition 2.4

Given $G \in \mathbb{F}^{p \times m}$,

$$\sigma_1 =: \bar{\sigma}(G) = \max_{\|w\|=1} \|Gw\|$$

(i.e. $\bar{\sigma}(G) = \|G\|$) with a maximizing $w = v_1$. If, in addition, $p \geq m$, then

$$\sigma_m =: \underline{\sigma}(G) = \min_{\|w\|=1} \|Gw\|$$

with a minimizing $w = v_m$. Moreover, if $p = m$, then $\underline{\sigma}(G) = 1/\|G^{-1}\|$.

Proof: With $w = \sum_{i=1}^m w_i v_i$,

$$\begin{aligned} \|y\|^2 &= \sum_{i=1}^{\min\{p,m\}} \sigma_i^2 |w_i|^2 \leq \sum_{i=1}^{\min\{p,m\}} \sigma_1^2 |w_i|^2 = \sigma_1^2 \sum_{i=1}^{\min\{p,m\}} |w_i|^2 \leq \sigma_1^2 \|w\|^2 \\ &\geq \sum_{i=1}^m \sigma_m^2 |w_i|^2 = \sigma_m^2 \sum_{i=1}^m |w_i|^2 = \sigma_m^2 \|w\|^2 \end{aligned}$$

...



Singular vectors

If

$$w = v_j, \quad j \in \mathbb{Z}_{1..m}$$

(i.e. $w_i = \delta_{ij}$), then

$$y = \left(\sum_{i=1}^{\min\{p,m\}} (\sigma_i w_i) u_i \right) = \begin{cases} \sigma_j u_j & \text{if } j \leq \min\{p, m\} \\ 0 & \text{otherwise} \end{cases} \implies \|y\| = \sigma_j$$

Hence,

- $\text{span}(v_j)$, $j \leq p$, is the input direction corresponding to gain σ_j
- $\text{span}(v_j)$, $j > p$, belongs to $\ker(G)$
- $\text{span}(u_j)$ is the output direction corresponding to gain σ_j

Structural properties via SVD

Proposition 2.5

If singular values of $G \in \mathbb{F}^{p \times m}$ satisfy $\sigma_r > \sigma_{r+1} = \dots = 0$ for some r , then

- $\text{Im } G = \text{span}(u_1, \dots, u_r)$;
- $\text{rank}(G) = r$;
- $\ker G = \text{span}(v_{r+1}, \dots, v_m)$.

Proof: For every $w = \sum_{i=1}^m w_i v_i$,

$$y = \sum_{i=1}^r \sigma_i w_i u_i = \sum_{i=1}^r (\sigma_i w_i) u_i \quad \text{and} \quad \|y\|^2 = \sum_{i=1}^r \sigma_i^2 |w_i|^2$$

Hence

(remember: $\sigma_i \neq 0$ for all $i \in \mathbb{Z}_{1,r}$).



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Hence

- $y \in \text{span}(u_1, \dots, u_r)$ and $\text{Im } G = \text{span}(u_1, \dots, u_r)$, for w_i are arbitrary

$\dim(\text{span}(u_1, \dots, u_r)) = r$, for u_i are linearly independent

$y = 0 \iff w_i = 0, \quad \forall i \in \mathbb{Z}_{1..r}$

(remember, $\sigma_i \neq 0$ for all $i \in \mathbb{Z}_{1..r}$).



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- $\dim(\text{span}(u_1, \dots, u_r)) = r$, for u_i are linearly independent
- $y = 0 \iff w_i = 0, \quad \forall i \in \mathbb{Z}_{1..r}$

(remember, $\sigma_i \neq 0$ for all $i \in \mathbb{Z}_{1..r}$).



Rank decomposition

Given a $G \in \mathbb{F}^{p \times m}$ such that $\text{rank}(G) = r \leq \min\{m, p\}$, there are **full row rank** $G_{\text{inp}} \in \mathbb{F}^{r \times m}$ and **full column rank** $G_{\text{out}} \in \mathbb{F}^{p \times r}$ such that

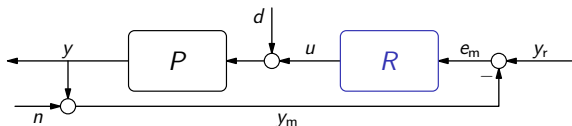
$$G = G_{\text{out}} G_{\text{inp}}.$$

In this case $\ker G = \ker G_{\text{inp}}$ and $\text{Im } G = \text{Im } G_{\text{out}}$. This follows directly from the SVD of G (as $\sigma_i \neq 0 \iff i \leq r$):

$$\begin{aligned} G &= \sum_{i=1}^r \sigma_i u_i v_i' = \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \begin{bmatrix} v_1' \\ \vdots \\ v_r' \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} u_1 \sigma_1^k & \cdots & u_r \sigma_r^k \end{bmatrix}}_{G_{\text{out}}} \underbrace{\begin{bmatrix} \sigma_1^{1-k} v_1' \\ \vdots \\ \sigma_r^{1-k} v_r' \end{bmatrix}}_{G_{\text{inp}}} \end{aligned}$$

for any $k \in \mathbb{R}$.

Example 1 from Ch. 1: $\alpha = 0$



If

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad R = kI$$

then (remember, $u = T_c y_{\text{ref}} - T_i d$ and $e = S_o y_{\text{ref}} - T_d d$)

$$\begin{bmatrix} T_c & T_i \end{bmatrix} = \frac{k}{2k+1} \left[\begin{array}{cc|cc} k+1 & -k & 1 & 1 \\ -k & k+1 & 1 & 1 \end{array} \right] \xrightarrow{k \rightarrow \infty} \frac{1}{2} \left[\begin{array}{cc|cc} \infty & -\infty & 1 & 1 \\ -\infty & \infty & 1 & 1 \end{array} \right]$$

and

$$\begin{bmatrix} S_o & T_d \end{bmatrix} = \frac{1}{2k+1} \left[\begin{array}{cc|cc} k+1 & -k & 1 & 1 \\ -k & k+1 & 1 & 1 \end{array} \right] \xrightarrow{k \rightarrow \infty} \frac{1}{2} \left[\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{array} \right]$$

and tracking only if $y_{\text{ref},1} = y_{\text{ref},2}$, all disturbances rejected.

Example 1 from Ch. 1: plant geometry

SVD of P :

$$P = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right) \begin{bmatrix} 2 & 0 \\ 0 & 2\alpha \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right)' \rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} 2 \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

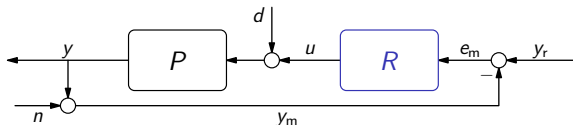
If $\alpha = 0$,

- only inputs in the direction $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ pass
- everything in $\text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is filtered out
- only outputs in the direction $\text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ exist

Because $R = kI$ does not alter inputs directions,

- everything entering in $\text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ processed as in SISO
- everything entering in $\text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ processed in open loop

Example 2 from Ch. 1: $\alpha = 0$



If

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad R = kI$$

then (remember, $u = T_c y_{\text{ref}} - T_i d$ and $e = S_o y_{\text{ref}} - T_d d$)

$$\begin{bmatrix} T_c & T_i \end{bmatrix} = -k \left[\begin{array}{cc|cc} k-1 & k & -1 & -1 \\ -k & -k-1 & 1 & 1 \end{array} \right] \xrightarrow{k \rightarrow \infty} \left[\begin{array}{cc|cc} \infty^2 & -\infty^2 & -\infty & -\infty \\ -\infty^2 & \infty^2 & \infty & \infty \end{array} \right]$$

and

$$\begin{bmatrix} S_o & T_d \end{bmatrix} = \left[\begin{array}{cc|cc} k-1 & k & -1 & -1 \\ -k & -k-1 & 1 & 1 \end{array} \right] \xrightarrow{k \rightarrow \infty} \left[\begin{array}{cc|cc} \infty & -\infty & -1 & -1 \\ -\infty & \infty & 1 & 1 \end{array} \right]$$

and tracking only if $y_{\text{ref},1} = -y_{\text{ref},2}$, only disturbances with $d_1 = -d_2$ rejected.

Example 2 from Ch. 1: plant geometry

SVD of P :

$$P = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right) \begin{bmatrix} 2 & 0 \\ 0 & 2\alpha \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)' \rightarrow \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} 2 \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

If $\alpha = 0$,

- only inputs in the direction $\text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ pass
- everything in $\text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is filtered out
- only outputs in the direction $\text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ exist

Because $R = kI$ does not alter inputs directions and $\text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \perp \text{span} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

- feedback does not exist!
- everything processed in open loop

Low-rank approximation

Given a $G \in \mathbb{F}^{p \times m}$ with $\text{rank } G = r$, then for every $l < r \leq \min\{p, m\}$

$$\min_{\text{rank}(H) \leq l} \|G - H\| = \sigma_{l+1} \quad \text{and} \quad \min_{\text{rank}(H) \leq l} \|G - H\|_F = \left(\sum_{i=l+1}^r \sigma_i^2 \right)^{1/2}$$

and the minimizing

$$H = G_l := \sum_{i=1}^l \sigma_i u_i v_i'$$

in both cases.

Outline

Static MIMO systems: basic notions

Size matters

Direction matters too

Systems as modeling tools

Generating subspaces

Proposition 2.7

A set $\mathcal{S} \subset \mathbb{F}^n$ is a subspace iff either of the following conditions holds:

- $\exists d \leq n$ and a full-rank matrix $S_i \in \mathbb{F}^{n \times d}$ such that $\mathcal{S} = \text{Im } S_i$
- $\exists d \leq n$ and a full-rank matrix $S_k \in \mathbb{F}^{(n-d) \times n}$ such that $\mathcal{S} = \ker S_k$

Moreover, this d is the dimension of $\mathcal{S}$.

Proof.

Because both image and kernel are subspaces, the “if” part in both cases is immediate as well as the fact that $\dim(\mathcal{S}) = d$. To show the “only if” part, let $\mathcal{S}$ be a d -dimensional subspace and $\{s_1, \dots, s_d\}$ be its basis. By the very definition, $\mathcal{S} = \text{Im } S_i$ for $S_i = \begin{bmatrix} s_1 & \cdots & s_d \end{bmatrix}$. Likewise, if $\{s_{d+1}, \dots, s_n\}$ is a basis of the $(n - d)$ -dimensional space $\mathcal{S}^\perp$, then $S_k = \begin{bmatrix} s_{d+1} & \cdots & s_n \end{bmatrix}'$ is what we need. □

Example: freedom of choice

Let $y = Gu$ for $G = \begin{bmatrix} 3 & 0 & 4 \end{bmatrix}$. Find all u such that $y = 1$. Remember that $y = Gu \iff u = u_0 + u_n$ for $u_n \in \ker G$. A particular solution is

$$u_0 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

To characterize the kernel, bring in SVD

$$G = 1 \cdot \begin{bmatrix} 5 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0 & 0 & 1 \\ 0.8 & 0.6 & 0 \end{bmatrix}',$$

so

$$\ker G = \operatorname{span} \left(\begin{bmatrix} -0.8 \\ 0 \\ 0.6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \operatorname{Im} \begin{bmatrix} -0.8 & 0 \\ 0 & 1 \\ 0.6 & 0 \end{bmatrix}$$

Hence,

$$u = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.8 & 0 \\ 0 & 1 \\ 0.6 & 0 \end{bmatrix} v$$

for an arbitrary $v \in \mathbb{R}^2$.

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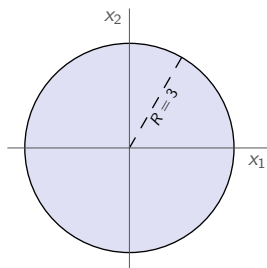
for an arbitrary $v \in \mathbb{R}^2$.

Exotic metrics

There may be reasons to define unorthodox metrics, like a metric in which

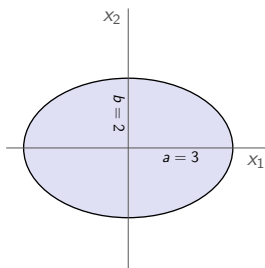
- x is “small” if $\|x\| \leq \gamma$ for some $\gamma > 0$ (uniform scaling)
- x_i is more important than x_j (non-uniform scaling)
- important are various linear combinations of x_i

$\mathcal{B}_a$:



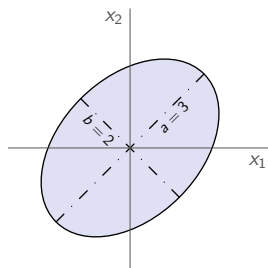
$$\sqrt{\left(\frac{x_1}{3}\right)^2 + \left(\frac{x_2}{3}\right)^2}$$

$\mathcal{B}_b$:



$$\sqrt{\left(\frac{x_1}{3}\right)^2 + \left(\frac{x_2}{2}\right)^2}$$

$\mathcal{B}_c$:



$$\sqrt{\left(\frac{x_1+x_2}{3\sqrt{2}}\right)^2 + \left(\frac{x_1-x_2}{2\sqrt{2}}\right)^2}$$

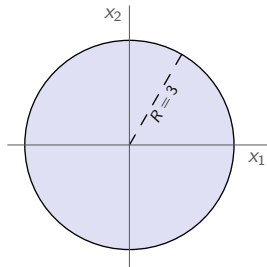
Exotic metrics modeled via standard metrics

Rather than dreaming up new metrics (unhandy), we can **recycle** existing. For example, we can define scaled unit balls as

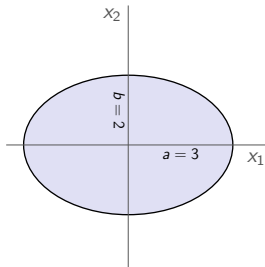
$$G\mathcal{B}_q := \{x \mid x = Gu, \|u\|_q \leq 1\},$$

which would result in

$$\mathcal{B}_a = 3\mathcal{B}_2:$$



$$\mathcal{B}_b = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \mathcal{B}_2:$$



$$\mathcal{B}_c = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 & -2 \\ 3 & 2 \end{bmatrix} \mathcal{B}_2:$$

