

Control of MIMO Systems (00860289)

chapter no. 1

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Technion—IIT



Outline

Course info

Introduction

Review of signals and systems

Review of control principles

Naïve MIMO

Course info

- Credit points: 3
- Formal prerequisite: Control Theory (00840738)
- Background to know:
 - calculus
 - linear algebra !
 - signals and systems in time, Fourier, and Laplace domains
 - basic control principles (modal and frequency-domain analyses)
- Grading policy: homework 100% (5 best out of 6 assignments)
Homework solutions must be submitted electronically via Moodle
- Course site: <http://leo.technion.ac.il/Courses/CMIMOS/>
- Literature (main):
 1. My lecture notes (available at the course site)

Syllabus

1. Stand-alone systems

- static MIMO systems
- dynamic MIMO systems
 - basic notions (stability, causality, domain) in time and transformed domains
 - coprime factorization in H_∞
 - poles and zeros of rational transfer functions
 - state-space realizations and their structural and computational properties
 - model order reduction via balanced truncation

2. Interconnected systems

- basic interconnections and their effects on dynamics, LFTs
- stability and stabilization
 - internal stability
 - general stability results (Small Gain and Passivity)
 - all stabilizing controllers (Youla–Kučera)
- optimization-based performance
 - weighted / mixed sensitivity problems, the standard problem
 - balanced sensitivity (H_∞ loop shaping)

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Why and for whom

Course goals

1. MIMO literacy badly lacking, especially in industry
2. Power and limitations of optimization-based design methods

Background needed

- linear algebra (see Appendix A of Lecture Notes)
- SISO systems
- classical SISO control methods

Hazard (or opportunity, depends on preferences)

- analytic material no choice; hand-waving hurts in MIMO more

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Why and for whom

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- classical SISO control methods

Hazard (or opportunity, depends on preferences)

- analytic material no course, homework, extra in lab/TA office

Why and for whom

Course goals

1. MIMO literacy badly lacking, especially in industry
2. Power and limitations of optimization-based design methods

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Hazard (or opportunity, depends on preferences)

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A quiz

What is the order and what are poles and zeros of the transfer functions:

1. $G(s) = \begin{bmatrix} 1/s & 0 \\ 0 & 1/s \end{bmatrix}$

2. $G(s) = \begin{bmatrix} 1/s & 1/s \\ 1/s & 1/s \end{bmatrix}$

3. $G(s) = \begin{bmatrix} 1 & 1/s \\ 0 & 1 \end{bmatrix}$

Nomenclature

$\mathbb{R}$	the set of real numbers, $\mathbb{R} = (-\infty, \infty)$
$\mathbb{R}_+$	the set of nonnegative real numbers, $\mathbb{R}_+ = [0, \infty)$
$\mathbb{R}_-$	the set of nonpositive real numbers, $\mathbb{R}_- = (-\infty, 0]$
$j\mathbb{R}$	the set of pure imaginary numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}_\alpha$	the half plane to the right of $\alpha \in \mathbb{R}$, i.e. $\mathbb{C}_\alpha: \{z \in \mathbb{C} \mid \operatorname{Re} z > \alpha\}$
$\bar{\mathbb{C}}_\alpha$	the closure of $\mathbb{C}_\alpha$, i.e. $\bar{\mathbb{C}}_\alpha: \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \alpha\}$
$\mathbb{T}$	the unit circle, $\mathbb{T} := \{z \in \mathbb{C} \mid z = 1\}$
$\mathbb{D}_\alpha$	the open α -disk, $\mathbb{D}_\alpha := \{z \in \mathbb{C} \mid z < \alpha\}$
$\bar{\mathbb{D}}_\alpha$	the closed α -disk, $\bar{\mathbb{D}}_\alpha := \{z \in \mathbb{C} \mid z \leq \alpha\} = \mathbb{D}_\alpha \cup (\alpha\mathbb{T})$
$\mathbb{F}$	alias of either $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of positive integers (natural numbers)
$\mathbb{Z}_+$	the set of nonnegative integers
$\mathbb{Z}_-$	the set of nonpositive integers, $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}$
$\mathbb{Z}_{i_1..i_2}$	the interval $\{i_1, i_1 + 1, \dots, i_2\}$

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Introduction

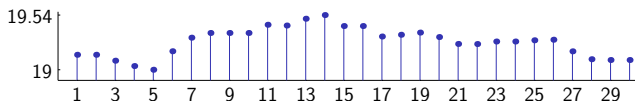
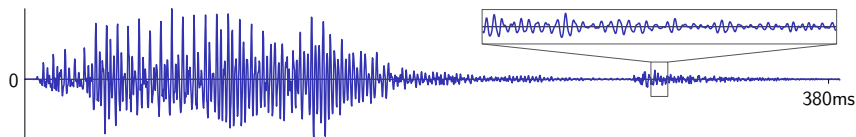
Review of signals and systems

Review of control principles

Naïve MIMO

Signals

Represent **evolving information**:

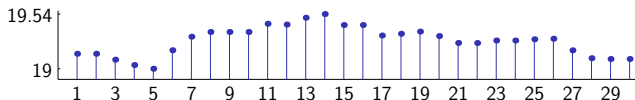
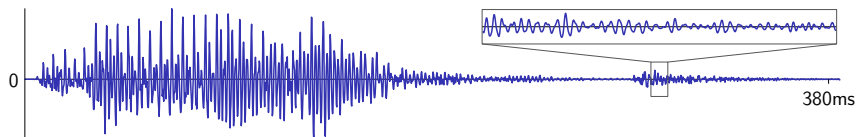


Mathematically,

— functions of independent variables, $f(t)$ or $f[n]$

Signals

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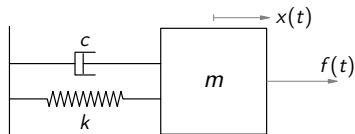


Mathematically,

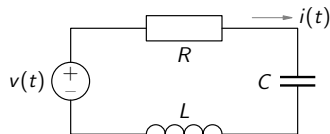
- functions of independent variables, $f(t)$ or $f[t]$

Systems

Constraints imposed on signals:



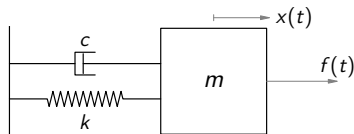
position x and force f



current i and voltage v

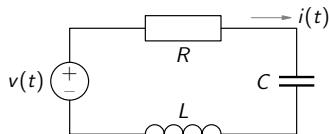
Systems

Constraints imposed on signals:



position x and force f

$$x \mapsto f \text{ or } f \mapsto x$$



current i and voltage v

$$v \mapsto i \text{ or } i \mapsto v$$

I/O view on systems:

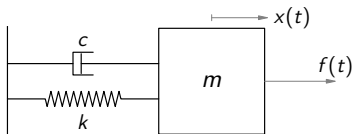
- some signals act
- some signals react

inputs

outputs

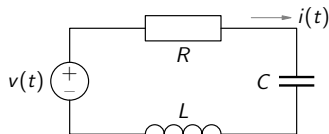
Mathematical models

(Approximate) description in a **mathematical language**:



position and force linked as

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$$



charge ($\dot{q} = i$) and voltage linked as

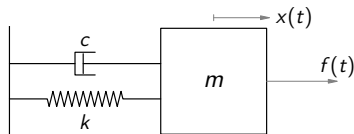
$$L\ddot{q}(t) + R\dot{q}(t) + \frac{1}{C}q(t) = v(t)$$

Abstract form:

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = k_{eq}\omega_n^2 u(t)$$

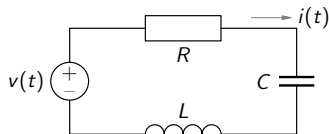
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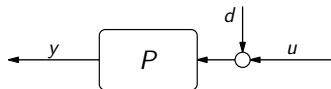
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Prototype control problem



y : regulated signal

u : control signal (means)

d : load disturbance

P : plant

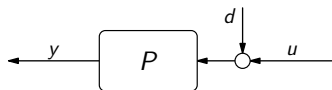
Goal:

$$u \longrightarrow y = y_r$$

where

y_r : reference signal (goal)

Ultimate methodology: plant inversion



$$y = P(d + u) \wedge y = y_r$$

$$\Downarrow$$

$$y_r = P(d + u)$$

$$\Downarrow$$

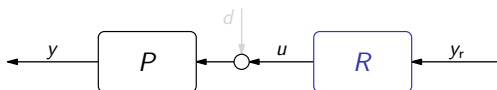
$$u = P^{-1}y_r - d$$

where

— P^{-1} is the inverse system

defined via $y = Pu \iff u = P^{-1}y$, with $P^{-1}(s) = \frac{1}{P(s)}$.

Open-loop plant inversion



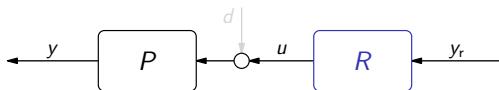
with

$$R = P^{-1}$$

with

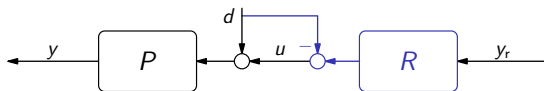
$$R = P^{-1}$$

Open-loop plant inversion



with

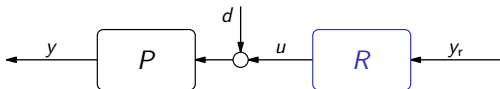
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with

$$R = P^{-1}$$

Limitations of open-loop plant inversion: stability



All signals,

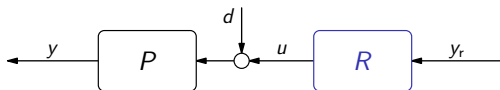
$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} PR & P \\ R & 0 \end{bmatrix} \begin{bmatrix} y_r \\ d \end{bmatrix},$$

bounded (**internal stability**).

Must have:

- P stable
- R stable, if $R = P^{-1} \Rightarrow P$ stably invertible

Limitations of open-loop plant inversion: stability



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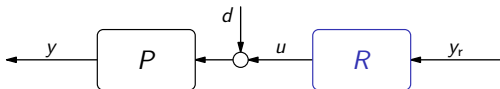
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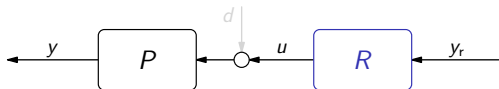
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Must have:

- P stable
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Approximate open-loop plant inversion



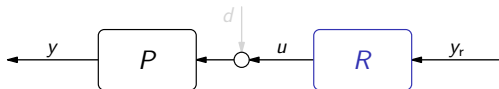
Pragmatic alternative:

$$y \approx y_r \quad \Rightarrow \quad R \approx P^{-1} \quad \Rightarrow \quad R = P^{-1} T_{\text{ref}}$$

Reference model:

- T_{ref} stable
- $P^{-1} T_{\text{ref}}$ stable (proper, poles in $\text{Re } s < 0$)
- $\lim_{s \rightarrow \infty} T_{\text{ref}} P^{-1} = 1$

Approximate open-loop plant inversion



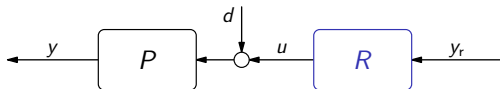
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Reference model:

- T_{ref} stable
- $P^{-1} T_{\text{ref}}$ stable (proper, poles in $\text{Re } s < 0$)
- $T_{\text{ref}} \approx 1$

Limitations of open-loop plant inversion: other



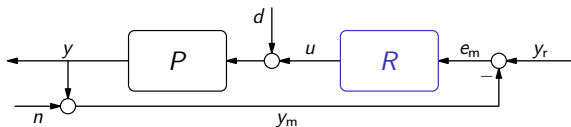
- unmeasured d
- uncertain P
- limited u

nothing to do

nothing to do

bandwidth limitations

Closed-loop control



Gang of four:

$$\begin{bmatrix} T_c(s) & T(s) \\ S(s) & T_d(s) \end{bmatrix} := \frac{1}{1 + P(s)R(s)} \begin{bmatrix} R(s) & P(s)R(s) \\ 1 & P(s) \end{bmatrix}$$

Signals:

$$\begin{bmatrix} y \\ u \\ e \end{bmatrix} = \begin{bmatrix} T & T_d & -T \\ T_c & -T & -T_c \\ S & -T_d & T \end{bmatrix} \begin{bmatrix} y_r \\ d \\ n \end{bmatrix},$$

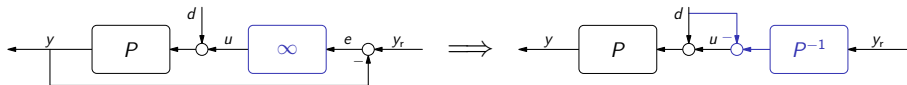
where $e := y_r - y = e_m + n$.

Closed-loop plant inversion

Because

$$T_c = \frac{1}{1/R + P} \xrightarrow{R \rightarrow \infty} \frac{1}{P} \quad \text{and} \quad -T = -\frac{P}{1/R + P} \xrightarrow{R \rightarrow \infty} -1,$$

we have

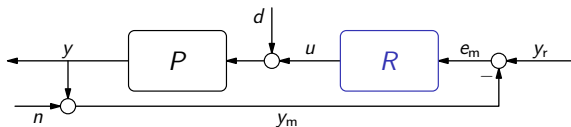


Thus,

$$T_d = \frac{P}{1 + PR} \xrightarrow{R \rightarrow \infty} 0 \quad \text{and} \quad S = \frac{1}{1 + PR} \xrightarrow{R \rightarrow \infty} 0,$$

independently of the plant and w/o explicit measurements of d .

Limitations of closed-loop plant inversion



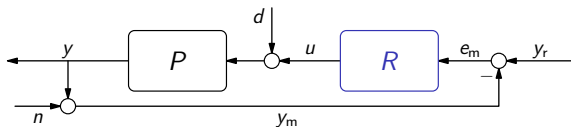
- closed-loop stability
- closed-loop stability
- closed-loop stability
- measurement noise sensitivity
- limited u
- ...

Hence,

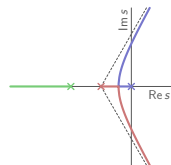
- (nontrivial) tradeoffs over different frequencies

cf. Bode's sensitivity integral $\int_0^{\infty} \ln |S(j\omega)| d\omega = \pi \sum_i \operatorname{Re} p_i \geq 0$ (waterbed effect)

Limitations of closed-loop plant inversion



- closed-loop stability
- closed-loop stability
- closed-loop stability
- measurement noise sensitivity
- limited u
- ...



Hence,

- (nontrivial) tradeoffs over different frequencies

cf. Bode's sensitivity integral $\int_0^\infty \ln |S(j\omega)| d\omega = \pi \sum_i \operatorname{Re} p_i \geq 0$ (waterbed effect)

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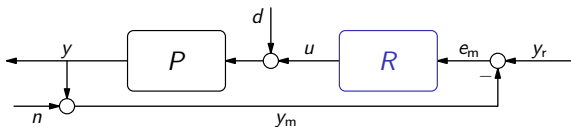
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Setup



Closed-loop relations:

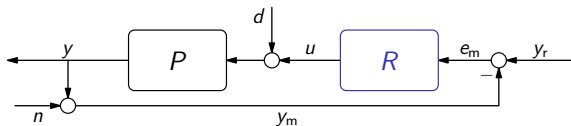
$$\begin{bmatrix} u \\ e \end{bmatrix} = \begin{bmatrix} T_c & -T_i \\ S_o & -T_d \end{bmatrix} \begin{bmatrix} y_r \\ d \end{bmatrix},$$

where

$$\begin{bmatrix} T_c(s) & T_i(s) \\ S_o(s) & T_d(s) \end{bmatrix} := \begin{bmatrix} R(s) \\ I \end{bmatrix} (I + P(s)R(s))^{-1} \begin{bmatrix} I & P(s) \end{bmatrix}.$$

($S_o \neq I - T_i$ in general, because $PR \neq RP$).

Example 1



If

$$P(s) = \begin{bmatrix} 1 + \alpha & 1 - \alpha \\ 1 - \alpha & 1 + \alpha \end{bmatrix} \quad \text{and} \quad R(s) = kl$$

for $\alpha \in [0, 1]$, then

$$\begin{bmatrix} T_c & T_i \end{bmatrix} = \frac{k}{2k+1} \begin{bmatrix} \frac{(1+\alpha)k+1}{2\alpha k+1} & -\frac{(1-\alpha)k}{2\alpha k+1} & \frac{4\alpha k+1+\alpha}{2\alpha k+1} & \frac{1-\alpha}{2\alpha k+1} \\ -\frac{(1-\alpha)k}{2\alpha k+1} & \frac{(1+\alpha)k+1}{2\alpha k+1} & \frac{1-\alpha}{2\alpha k+1} & \frac{4\alpha k+1+\alpha}{2\alpha k+1} \end{bmatrix}$$

and

$$\begin{bmatrix} S_o & T_d \end{bmatrix} = \frac{1}{2k+1} \begin{bmatrix} \frac{(1+\alpha)k+1}{2\alpha k+1} & -\frac{(1-\alpha)k}{2\alpha k+1} & \frac{4\alpha k+1+\alpha}{2\alpha k+1} & \frac{1-\alpha}{2\alpha k+1} \\ -\frac{(1-\alpha)k}{2\alpha k+1} & \frac{(1+\alpha)k+1}{2\alpha k+1} & \frac{1-\alpha}{2\alpha k+1} & \frac{4\alpha k+1+\alpha}{2\alpha k+1} \end{bmatrix}$$

Example 1: $\alpha \neq 0$

Here

$$\lim_{k \rightarrow \infty} T_c = \frac{1}{4\alpha} \begin{bmatrix} 1 + \alpha & 1 - \alpha \\ -1 + \alpha & -1 - \alpha \end{bmatrix} = P^{-1} \quad \text{and} \quad \lim_{k \rightarrow \infty} T_i = I$$

and

$$S_o \rightarrow 0 \leftarrow T_d$$

Hence,

$$u \rightarrow P^{-1}y_r - d \quad \text{and} \quad e \rightarrow 0,$$

exactly as in the SISO case.

Example 1: $\alpha = 0$

Here

$$\begin{bmatrix} T_c & T_i \end{bmatrix} = \frac{k}{2k+1} \begin{bmatrix} k+1 & -k & 1 & 1 \\ -k & k+1 & 1 & 1 \end{bmatrix}$$

and

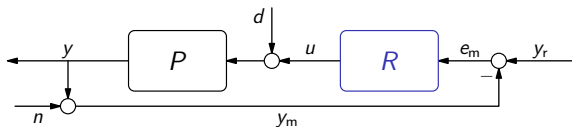
$$\begin{bmatrix} S_o & T_d \end{bmatrix} = \frac{1}{2k+1} \begin{bmatrix} k+1 & -k & 1 & 1 \\ -k & k+1 & 1 & 1 \end{bmatrix} \xrightarrow{k \rightarrow \infty} \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$$

Hence

$$u \rightarrow \frac{k}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} y_r - \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} d \quad \text{and} \quad e \rightarrow \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} y_r$$

- e decoupled from d like in SISO
- $e \rightarrow 0$ only if $y_{r,1} = y_{r,2}$ different from SISO
- u might be unbounded different from SISO

Example 2



If

$$P(s) = \begin{bmatrix} 1 + \alpha & 1 - \alpha \\ -1 + \alpha & -1 - \alpha \end{bmatrix} \quad \text{and} \quad R(s) = kI$$

for $\alpha \in [0, 1]$, then

$$\begin{bmatrix} T_c & T_i \end{bmatrix} = \frac{k}{4\alpha k^2 - 1} \begin{bmatrix} (1 + \alpha)k - 1 & (1 - \alpha)k & 4\alpha k - 1 - \alpha & -1 + \alpha \\ -(1 - \alpha)k & -(1 + \alpha)k - 1 & 1 - \alpha & 4\alpha k + 1 + \alpha \end{bmatrix}$$

and

$$\begin{bmatrix} S_o & T_d \end{bmatrix} = \frac{1}{4\alpha k^2 - 1} \begin{bmatrix} (1 + \alpha)k - 1 & (1 - \alpha)k & 4\alpha k - 1 - \alpha & -1 + \alpha \\ -(1 - \alpha)k & -(1 + \alpha)k - 1 & 1 - \alpha & 4\alpha k + 1 + \alpha \end{bmatrix}$$

Example 2: $\alpha \neq 0$

Here

$$\lim_{k \rightarrow \infty} T_c = \frac{1}{4\alpha} \begin{bmatrix} 1 + \alpha & 1 - \alpha \\ -1 + \alpha & -1 - \alpha \end{bmatrix} = P^{-1} \quad \text{and} \quad \lim_{k \rightarrow \infty} -T_i = -I$$

and

$$S_o \rightarrow 0 \leftarrow T_d$$

Hence

$$u \rightarrow P^{-1}y_r - d \quad \text{and} \quad e \rightarrow 0,$$

exactly as in the SISO case.

Example 2: $\alpha = 0$

Here

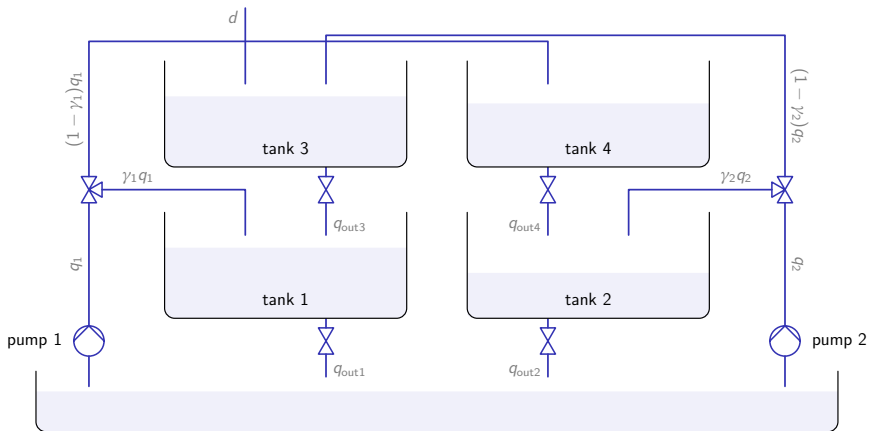
$$T_d = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad S_o = \begin{bmatrix} -k+1 & -k \\ k & k+1 \end{bmatrix}, \quad T_i = kT_d, \quad T_c = kS_o$$

and

$$u \rightarrow \left(\begin{bmatrix} -k(k-1) & -k^2 \\ k^2 & k(k+1) \end{bmatrix} y_r - k \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} d \right) \Big|_{k \rightarrow \infty} \quad \text{and} \quad e \rightarrow \frac{1}{k} u$$

different from the SISO case.

Quadruple-tank system

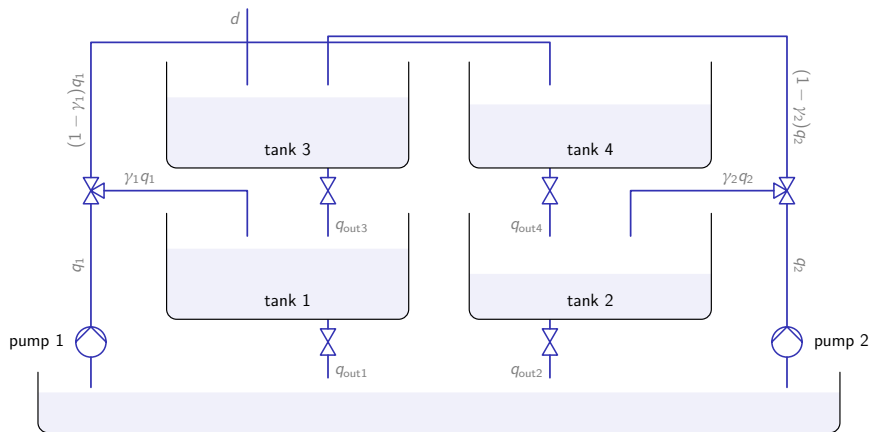


with flow division proportionality $\gamma_i \in (0, 1)$ for $i = 1, 2$.

Goal:

control water levels h_1 and h_2 in Tanks 1 and 2 via pump flow rates q_1 and q_2 despite disturbance d .

Quadruple-tank system



with flow division proportionality $\gamma_i \in (0, 1)$ for $i = 1, 2$. Goal:

- control water levels h_1 and h_2 in Tanks 1 and 2 via pump flow rates q_1 and q_2 despite disturbance d .

Quadruple-tank system: nonlinear model

by Torricelli's law,

$$q_{out i} = r_i \sqrt{h_i}, \quad i \in \mathbb{Z}_{1..4}$$

volumes balance:

$$\begin{cases} A_1 \dot{h}_1(t) = -r_1 \sqrt{h_1(t)} + r_3 \sqrt{h_3(t)} + \gamma_1 q_1(t) \\ A_2 \dot{h}_2(t) = -r_2 \sqrt{h_2(t)} + r_4 \sqrt{h_4(t)} + \gamma_2 q_2(t) \\ A_3 \dot{h}_3(t) = -r_3 \sqrt{h_3(t)} + (1 - \gamma_2) q_2(t) + d(t) \\ A_4 \dot{h}_4(t) = -r_4 \sqrt{h_4(t)} + (1 - \gamma_1) q_1(t) \end{cases}$$

where

- A_i are cross-sectional areas
- r_i are outflow valve resistances

Quadruple-tank system: linearized model

deviation variables:

$$y = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} - \begin{bmatrix} h_{eq1} \\ h_{eq2} \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - \begin{bmatrix} q_{eq1} \\ q_{eq2} \end{bmatrix}$$

linearized dynamics:

$$y = Pu + P_d d,$$

where

$$P(s) = \begin{bmatrix} \gamma_1 \frac{k_1}{T_1 s + 1} & (1 - \gamma_2) \frac{k_1}{(T_1 s + 1)(T_3 s + 1)} \\ (1 - \gamma_1) \frac{k_2}{(T_2 s + 1)(T_4 s + 1)} & \gamma_2 \frac{k_2}{T_2 s + 1} \end{bmatrix}$$

and

$$P_d(s) = \begin{bmatrix} \frac{k_d}{(T_1 s + 1)(T_3 s + 1)} \\ 0 \end{bmatrix}.$$

(stable, simple entries).

Quadruple-tank system: control strategy

ideal control law:

$$u = -P^{-1}P_d d$$

closed-loop relations:

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} -T_c \\ S_o \end{bmatrix} P_d d,$$

for

$$\begin{bmatrix} T_c \\ S_o \end{bmatrix} = \begin{bmatrix} R \\ I \end{bmatrix} (I + PR)^{-1}$$

high-gain feedback:

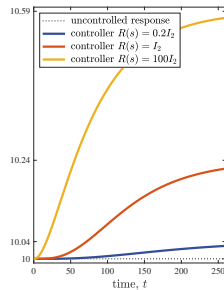
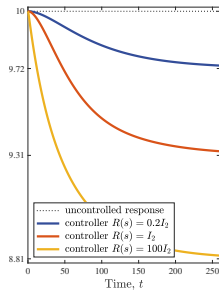
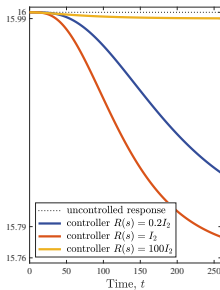
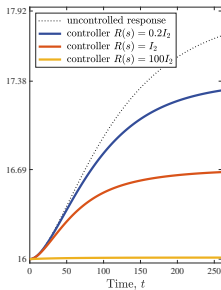
$$R \rightarrow \infty \implies \begin{cases} T_c \rightarrow P^{-1}P_d \\ S_o \rightarrow 0 \end{cases}$$

Example 3

If $\gamma_1 = \gamma_2 = 2/3$ and $h_{eq1} = h_{eq2} = 16$, then

$$T_1 = T_2 = 96, \quad T_3 = T_4 = 32, \quad k_1 = k_2 = 3.2, \quad k_d = 96.$$

Applying $d = 0.021$,



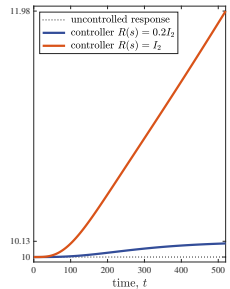
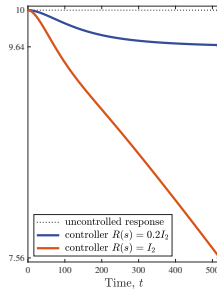
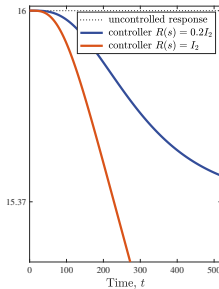
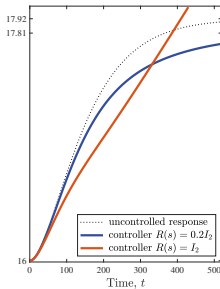
— high-gain feedback works

Example 4

If $\gamma_1 = \gamma_2 = 1/3$ and $h_{eq1} = h_{eq2} = 16$, then

$$T_1 = T_2 = 96, \quad T_3 = T_4 = 64, \quad k_1 = k_2 = 3.2, \quad k_d = 96.$$

Applying $d = 0.021$,



- high-gain feedback does not work