Incorporating Waveform Constraints in the Optimal Design of Sampling and Hold Functions

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Abstract

This paper presents solutions to the sampled-data $H^2$ and $H^\infty$ control problems where the sampling function, the discrete-time part of the controller and the hold function are all design parameters. The generalized sampling and hold functions are constrained to have piecewise impulse and piecewise constant waveforms, respectively. The resulting sampled data controller has improved control capabilities in comparison to those based on zero-order hold and ideal sampler devices, yet it is readily implementable on digital hardware, in contrast to those based on unconstrained sampling and hold functions.

1 Introduction

Digital controllers connected to continuous-time plants via A/D (sampler) and D/A (hold) converters are widely used in today industry owing to their lower price, enhanced reliability and flexibility in comparison with their analog counterparts. Therefore, it is not surprising that over the last three decades the sampled-data control theory has been the subject of wide debate and activity (see (Åström and Wittenmark, 1989; Chen and Francis, 1995; Feuer and Goodwin, 1996) for an introduction to the subject and pointers to relevant literature). In the great majority of these works, however, only the discrete part of the sampled-data controller is designed, while the sampler and the hold are usually selected to be the “ideal” sampler and the zero-order hold respectively, without taking into consideration the plant dynamics or the control objectives. Having fixed sampler and hold has several advantages: the behavior of these devices is fairly well understood and hence can be incorporated (at least heuristically) in the controller design process, the design of hardware is considerably simplified, adequate simulation tools exist, etc. At the same time, fixing the sampler and hold devices irrespectively of control considerations may limit the closed-loop performance, especially when the sampling rate is not “fast” enough with respect to the plant dynamics.

Early attempts to incorporate the sampler and, particularly, the hold in the design process (Chammas and Leondes, 1978; Kabamba, 1987) took into consideration only the discrete-time closed-loop performance. As a result, significant improvements of the discrete-time performance were usually achieved at the expense of a poor inter-sampling behavior and undesirable robustness properties (Feuer and Goodwin, 1994). The design of sampling and/or hold functions on the basis of continuous-time $H^2$ and $H^\infty$ (sub)optimal performance was treated
in (Juan and Kabamba, 1991; Tadmor, 1992; Sun et al., 1993; Mirkin et al., 1999a). On one hand, analysis and simulations of those results show that the “sufficiently fast sampling” assumption can be relaxed by using the generalized A/D and D/A converters instead of the zero-order hold and the ideal sampler. On the other hand, these generalized sampling and hold functions are not easily implementable, mainly because they all have continuous waveforms.

A possible way to implement the generalized sampling and hold functions designed in (Tadmor, 1992; Mirkin et al., 1999a) is to use custom built analog hardware. Such an approach, however, may diminish the advantages of using a sampled-data controller instead of an analog one. Hence, programmable digital hardware is likely to be more adequate for the realization of those functions. The simplest way to implement them, using existing A/D and D/A converters that can digitally modulate the controller and the plant outputs, is to discretize the continuous waveform of the generalized hold and sampling functions. That is, to approximate them by piecewise constant and piecewise impulse functions, respectively. This method however may inherit the pitfalls of the approximation based sampled-data control design.

The alternative approach is to incorporate the waveform constraints into the design of the generalized sampling and hold functions. This yields a more realistic and easy to implement class of A/D and D/A converters. This paper presents the solutions to the $H^2$ and to the $H^\infty$ optimization problems where the sampling and hold functions are free design parameters yet constrained to have piecewise constant/impulse waveforms, respectively. The solutions are obtained by transforming the control problems into pure discrete, LTI optimizations, by using the continuous- and the discrete-time lifting techniques (Chen and Francis, 1995). The computations are carried out using advanced continuous-time lifting techniques (Mirkin and Palmor, 1998) and some recently developed mathematical tools for the discrete-time lifting (Kahane et al., 1999). Consequently, the resulting formulae are explicit (i.e., in terms of the original plant parameters) and provide meaningful interpretations to the solutions.

This paper is organized as follows. Section 2 assembles the mathematical background needed in this paper. The $H^2$ and the $H^\infty$ optimization problems of the sampled-data setup based on piecewise impulse/constant sampling and hold functions are formulated in Section 3. In Section 4 these problems are reduced to standard $H^2$ and $H^\infty$ optimization problems in the lifted domain and their lifted solutions are presented and discussed. The lifted solutions are “peeled-off” back to the time domain in Section 5, which presents the complete solutions and various properties and interpretations of the $H^2$ and $H^\infty$ (sub)optimal piecewise constant hold and piecewise impulse sampler. The potential benefits of these devices are demonstrated through an illustrative example in Section 6. Some concluding remarks are presented in Section 7.

1.1 Notation

The notation throughout the paper is fairly standard. $M'$ means the transpose of a matrix $M$ and $O^*$ — the adjoint of a Hilbert space operator $O$. $\rho(M)$ is the spectral radius of a square matrix $M$. As usual, $\mathbb{D}$ denotes the open unit disc. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, $L_2^2[0,\nu]$ denotes the Hilbert space of square integrable $\mathbb{R}^n$-valued functions on the interval $[0,\nu]$ and $l_2^2[0,\nu-1]$ denotes the Hilbert space of $\mathbb{R}^n$ valued sequences defined over a finite time interval $[0,\nu-1]$. When the dimensions are irrelevant or clear from the context we will write $\mathbb{R}$, $L_2^2[0,\nu]$ and $l_2^2[0,\nu-1]$. The notation $L_2^2$ is the shortcut for $L_2^2[0,\nu]$.

A “bar” above a variable (\bar{\zeta}) denotes discrete-time signals in $\mathbb{R}^n$, while “vector” and “breve” (\vec{\zeta} and \breve{\zeta}) denote discrete-time signals in the discrete- and continuous-lifted domains, respectively. When operators with both finite and infinite dimensional input/output spaces are in-
volved we use the following notation: a bar (or, in the discrete-lifted domain, a vector) indicates an operator $\bar{O}$ ($\bar{\bar{O}}$) with both input and output spaces finite dimensional; a grave accent — $\grave{O}$, when the input space is finite dimensional and the output infinite dimensional one; an acute accent — $\acute{O}$, when the input space is infinite dimensional and the output finite dimensional one; and, finally, a breve — $\breve{O}$, when both input and output spaces are infinite dimensional.

The compact block notation $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ denotes (matrix- or operator-valued) transfer functions either in $s$ or in $z$ domain in terms of their state-space realization. To distinguish LTI systems in the time domain from their corresponding transfer functions, the former are denoted by script capital letters, so $G(s)$ and $\bar{G}(z)$ imply the transfer functions of the LTI systems $\mathcal{G}$ and $\bar{\mathcal{G}}$, respectively. Finally, $\text{Ric}_D$ denotes the discrete-time Riccati function (for definition and properties see (Mirkin et al., 1999b)) defined over a subset called $\text{dom}(\text{Ric}_D)$ and which has a one-to-one correspondence with the stabilizing solution of the discrete-time algebraic Riccati equations (DARE).

## 2 Preliminaries

This section contains some preliminary results which are required in the sequel. Subsection 2.1 briefly reviews the discrete-time lifting technique, while subsections 2.2 and 2.3 contain some of the results derived in (Kahane et al., 1999): Subsection 2.2 is concerned with the representation of the parameters of discrete-lifted plants as discrete-time dynamical systems operating over a finite time interval and Subsection 2.3 presents some important connections between the discrete-time lifted domain and the discrete time domain.

### 2.1 The discrete-time lifting technique

The notion of discrete-time lifting consists on establishing a one-to-one correspondence between a discrete-time periodically shift-varying system and a shift-invariant one (but with higher input and output dimensions). The application of lifting enables one to use the well-established LTI tools for the analysis and design of periodically shift-varying systems, like those arising in many multi-rate sampled-data control problems.

Define the discrete-time lifting operator $\bar{W}_\nu$ (Chen and Francis, 1995), which transforms the $\mathbb{R}^n$ valued sequences to the $\mathbb{R}^{n\nu}$ valued sequences, as follows:

$$
\bar{\xi} = \bar{W}_\nu \xi \iff \bar{\xi}[k] = \begin{bmatrix} \xi[\nu k] \\ \xi[\nu k + 1] \\ \vdots \\ \xi[\nu k + \nu - 1] \end{bmatrix}.
$$

The usefulness of this operator follows from the fact that for a $\nu$-periodic system $\mathcal{G}$ its lifting $\bar{\mathcal{G}} = \bar{W}_\nu \mathcal{G} \bar{W}_\nu^{-1}$ is shift-invariant. Also, since $\bar{W}_\nu$ is an isomorphism, the stability properties are preserved under lifting, and since the restriction of $\bar{W}_\nu$ to $l^p$ is an isometry, induced norms of the original system are equivalent to norms of the lifted one.

Lifting, however, increases the input and output dimensions. For example, let $\mathcal{G}$ be a discrete-time LTI system with the following transfer matrix:

$$
\bar{G}(z) \doteq \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (1)
$$
where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$. Its lifting, $\vec{G}$, is also shift-invariant (Chen and Francis, 1995), and:

$$
\vec{G}(z) = \begin{bmatrix}
A^v & A^{v-1}B & A^{v-2}B & \ldots & B \\
C & D & 0 & \ldots & 0 \\
CA & CB & D & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
CA^{v-1} & CA^{v-2}B & CA^{v-3}B & \ldots & D
\end{bmatrix}.
$$

(2)

In principle, (2) describes a standard discrete-time system. Hence, dealing with $\vec{G}$ is conceptually not more complicated than with $\overline{G}$. Moreover, the state dimensions in (1) and (2) are equal. Yet the input and output dimensions of $\vec{G}$ increase by the factor $v$ with respect to those of $\overline{G}$. That, in turn, “blows up” the matrices $\vec{B}$, $\vec{C}$, $\vec{D}$. Consequently, numerical difficulties associated with lifted solutions increase rapidly as $v$ grows. This fact reduces the effectiveness of lifting, especially for large lifting frames $v$. Moreover, when $\vec{G}$ is not finite dimensional, but rather the result of a continuous-time lifting, the direct treatment of $\vec{B}$, $\vec{C}$, and $\vec{D}$ as block-matrices does not appear to be helpful.

To overcome this difficulty, (Kahane et al., 1999) proposed an alternative representation for the discrete-lifted parameters. The idea is to treat the parameters of the lifted systems as dynamical systems operating over a finite time interval rather than unstructured matrices. Consequently, the manipulations over these parameters can be reduced to manipulations over dynamical systems which, in turn, can efficiently be performed using state-space machinery. In the sequel we present a simplified version of this representation, which is sufficient for the purpose of this paper.

2.2 A representation of the lifted parameters using dynamical systems

The simplified version of the representation in (Kahane et al., 1999) is based upon the following three components:

i) The causal discrete-time systems operating over a finite-time interval (DSOFTI) are linear operators $\vec{O} : l^2[0, v - 1] \mapsto l^2[0, v - 1]$, described by the state equations

$$
\vec{O} : \begin{cases}
x[k + 1] = Ax[k] + Bu[k], & x[0] = 0, \quad k = 0, \ldots, v - 1, \\
y[k] = Cx[k] + Du[k].
\end{cases}
$$

In the sequel, these systems will be denoted using the compact block notation

$$
\vec{O} = \begin{pmatrix} A & B \end{pmatrix}^{v-1}.
$$

and the time interval $v$ will be omitted from the notation when it is clear from the context.

ii) The discrete impulse operator $\overline{I}_{\theta} : \mathbb{R}^n \mapsto l^2[0, v - 1], \theta = 0, \ldots, v - 1$ is defined in the following manner:

$$
\zeta = \overline{I}_{\theta} \eta \iff \zeta[k] = \begin{cases} 
\eta, & \text{if } k = \theta \\
0, & \text{else}
\end{cases}.
$$
iii) The adjoint $\mathcal{J}_0^*: \mathbb{R}^2[0, \nu - 1] \rightarrow \mathbb{R}^n$ of the Hilbert space operator $\mathcal{J}_0$, is given by:

$$\eta = \mathcal{J}_0^* \zeta[k] \iff \eta = \zeta[0],$$

and, in fact, it is the discrete-time sampling operator.

These three components allows one to introduce the following representation for the parameters of the discrete-lifted plant $\mathcal{J}$ in (2):

$$\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} = \begin{bmatrix} \mathcal{J}_{\nu-1}^* & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & A & B \\ A & A & B \\ C & C & D \end{bmatrix}^{-1} \begin{bmatrix} A & A \\ A & A \\ C & C \end{bmatrix} \mathcal{J}_0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3}$$

By using this representation, the involved manipulations over the high dimensional matrix parameters of the lifted plant can be replaced by operations over DSOFTI. Many of the latter manipulations can be performed in the state space, in terms of the low-dimensional parameters of the original plant. Consequently, the efficiency of the analytical and numerical computations in the lifted domain is improved, and the structures of the original problems are preserved.

To illustrate the advantages of the DSOFTI-based representation, we compute the matrix $\bar{A} - \bar{B}\bar{D}^{-1}\bar{C}$ (assuming that $\det(D) \neq 0$, otherwise $\bar{D}$ is singular). The computation of this matrix using the matrix-based representation (2) is rather cumbersome since it requires the inversion of the high dimensional matrix $\bar{D}$. One the other hand, by using the DSOFTI-based representation one gets the following analytical result

$$\begin{align*}
\bar{A} - \bar{B}\bar{D}^{-1}\bar{C} &= \mathcal{J}_{\nu-1}^* \left[ \begin{bmatrix} A & A \\ A & A \end{bmatrix} - \begin{bmatrix} A & B \\ A & B \end{bmatrix} \begin{bmatrix} A & B \\ C & C \end{bmatrix}^{-1} \begin{bmatrix} A & A \\ A & A \end{bmatrix} \right] \mathcal{J}_0 \\
&= \mathcal{J}_{\nu-1}^* \left[ \begin{bmatrix} A & A \\ A & A \end{bmatrix} - \begin{bmatrix} A & B \\ A & B \end{bmatrix} \begin{bmatrix} -BD^{-1}C & BD^{-1}C \\ -D^{-1}C & D^{-1} \end{bmatrix} \begin{bmatrix} A & A \\ C & C \end{bmatrix} \right] \mathcal{J}_0 \\
&= \mathcal{J}_{\nu-1}^* \left( \begin{bmatrix} A & -BD^{-1}C & BD^{-1}C \\ 0 & A - BD^{-1}C & BD^{-1}C \\ 0 & 0 & A \end{bmatrix} \right) \mathcal{J}_0 \\
&= \mathcal{J}_{\nu-1}^* \left( \begin{bmatrix} A & 0 & 0 \\ 0 & A - BD^{-1}C & 0 \\ 0 & 0 & A \end{bmatrix} \right) \mathcal{J}_0 \\
&= \mathcal{J}_{\nu-1}^* \left( \begin{bmatrix} A & BD^{-1}C & A - BD^{-1}C \\ A & BD^{-1}C & A - BD^{-1}C \\ A & BD^{-1}C & A - BD^{-1}C \end{bmatrix} \right) \mathcal{J}_0 = (A - BD^{-1}C)^\nu,
\end{align*}$$

which is conceptually simple and easy to calculate numerically.

It is worth stressing that not all the operations over the discrete-lifted parameters can be performed using this DSOFTI-based representation. The reason is that the class of the causal DSOFTI is not closed under the adjoint (when $\det(A) = 0$) and inverse (when $\det(D) = 0$) operations. At the same time, singular $A$ and especially $D$ matrices appear frequently in the sampled-data control problems (Chen and Francis, 1995). To overcome this difficulty, (Kahane et al., 1999) developed a representation for the discrete-lifted parameters based on a broader class of discrete-time dynamical systems: the class of the discrete-time implicit descriptor systems with two point boundary conditions (DIDS). This broader representation preserves the advantages.
of that based on DSOFTI and, in addition, covers all possible singularities in the description of $\tilde{G}$.

By using the DIDS-based representation for the discrete-lifted parameters, (Kahane et al., 1999) were able to establish some important connections between the discrete-time lifted domain and the discrete time domain. One of those, concerning the relations between the stabilizing solution to the DARE associated with the discrete-lifted plant $\tilde{G}$ and that to the DARE associated with the original plant $\bar{G}$, is presented below.

2.3 DARE’s in the lifted domain

Let $J$ be an appropriately dimensioned square matrix and associate with $\bar{G}$, (1), the equation

$$A'X A - X + C'J C + (A'XB + C'JD)F = 0,$$

where $F$ is the gain matrix associated with (4):

$$F = -(B'XB + D'JD)^{-1}(B'XA + D'JC).$$

Equation (4) is the well known DARE, and finding its stabilizing solution is a crucial step in solving various discrete-time control problems, such as $H^2$ ($J = I$) and $H^\infty$ ($J = [1 \ -1]$) optimizations (Chen and Francis, 1995; Zhou et al., 1995). Recall that $X$ is said to be the stabilizing solution of (4) if $X = X'$, $\det(B'XB + D'JD) \neq 0$ and $A + BF$ is Hurwitz.

Similarly, associate with $\tilde{G}$ the equation:

$$\tilde{A}'X\tilde{A} - X + \tilde{C}'J \tilde{C} + (\tilde{A}'XB + \tilde{C}'JD)\tilde{F} = 0,$$

where

$$\tilde{F} = -(\tilde{B}'X\tilde{B} + \tilde{D}'J\tilde{D})^{-1}(\tilde{D}'J\tilde{C} + \tilde{B}'X\tilde{A}).$$

Equation (5) is also a DARE, which arises in many multi-rate sampled-data optimization problems, see, e.g., (Chen and Qiu, 1994).

Since $\tilde{G}$ is equivalent to $\tilde{\bar{G}}$ from the input-output point of view, it is likely that the DARE’s (4) and (5) are closely connected. The internal structures of $\tilde{G}$ and $\tilde{\bar{G}}$, however, are different: the state vector of $\tilde{G}$ is the sampled version of that of $\tilde{\bar{G}}$. Hence, the relation between (4) and (5) might not be obvious. Lemma 1, presented below, clarifies this relation.

Lemma 1 ((Kahane et al., 1999)). A matrix $X = X'$ is the stabilizing solution to DARE (4) if and only if it is the stabilizing solution to DARE (5). Moreover, if $X$ is the stabilizing solution to those DARE’s, then $\tilde{A} + \tilde{B}F = (A + BF)^\nu$ and

$$\tilde{F} = \begin{bmatrix} F \\ F(A + BF) \\ \vdots \\ F(A + BF)^{\nu-1} \end{bmatrix}.$$

Lemma 1 states that, considering the stabilizing solution, the DARE (5) is equivalent to the considerably simpler and well understood DARE (4). Note that this equivalence does not hold true in general: the DARE (5) might have additional solutions to those of the DARE (4), however, they are not stabilizing (Kahane et al., 1999).
3 Problem formulation

The purpose of this section is to formulate the sampled-data $H^2$ and $H^\infty$ control problems where the sampling and hold functions are design parameters restricted to have piecewise impulse and piecewise constant waveforms, respectively. These problems will be defined in terms of the feedback setup illustrated in Fig. 1, where $P$ is a continuous-time generalized plant and $w$, $z$, $y$ and $u$ are the continuous-time exogenous input, the regulated output, the measured output and the control signal, respectively. The sampled-data controller consists of a digital controller $\bar{K}$, a sampler $S_h$ and a hold $H_h$ which are assumed to be synchronized and with a given sampling period $h$. The generalized plant $P$,

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} P_{11}^{\bullet} & P_{12}^{\bullet} \\ P_{21}^{\bullet} & P_{22}^{\bullet} \end{bmatrix} = \begin{bmatrix} P_{11}^* & P_{12}^* \\ P_{21}^* & P_{22}^* \end{bmatrix}$$

is assumed to be LTI, with the following state-space representation:

$$P(s) = \begin{bmatrix} A & \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\ C_1 & \begin{bmatrix} 0 & D_{12} \\ C_2 & 0 \end{bmatrix} \end{bmatrix}.$$  \hfill (6)

The matrix $D_{11}$ is taken to be zero in order to simplify the derivations and to obtain more transparent results. Moreover, for the $H^2$ problem $D_{11} = 0$ is also a necessary condition for the cost function to be finite. The conditions $D_{21} = 0$ and $D_{22} = 0$ ensure that the sampler $S_h$ operates over proper signals (i.e., the measured output is pre-filtered, if necessary, by an anti-aliasing filter before sampling).

In general, the generalized (zero-order) hold and the generalized sampler can be presented (see, e.g. (Kabamba, 1987; Araki, 1993)) as operators that act on the output of the digital controller $\bar{u}[k]$ and on the measured output $y(t)$ respectively, to generate:

$$(\mathcal{H}_h \bar{u})(kh + \tau) = \phi_H(\tau)\bar{u}[k], \quad \forall \tau \in [0, h)$$  \hfill (7a)

and

$$(S_h y)[k] = \int_0^{kh^-} \phi_S(\tau)y(kh^- - \tau) \, d\tau$$  \hfill (7b)

for some generalized hold and sampling functions $\phi_H$ and $\phi_S$ defined on the interval $[0, h)$. During the inter-sample, the hold function $\phi_H$ shapes the form of the control signal, while the sampling function $\phi_S$ is used to weight the measurements.
Motivated by the technological considerations, we constrain in this paper the hold and the sampling functions to have the following piecewise constant and piecewise impulse waveforms, respectively:

\[
\phi_H(\tau) = \phi_H[i], \quad \forall \tau \in [h \frac{i}{\nu}, h \frac{i+1}{\nu}), \quad i = 0, \ldots, \nu - 1
\]

\[
\phi_S(\tau) = \sum_{j=0}^{\mu-1} \phi_S[j] \delta(\tau - \frac{\mu - 1}{\nu} h^-),
\]

where \(\nu\) and \(\mu\) are any two natural numbers called, in the sequel, the constraint divisions. Hence, the sampler and the hold considered throughout this paper act in the following manner:

\[
u(\mu h + \tau) = \phi_H[i] \bar{u}[k], \quad \forall \tau \in [h \frac{i}{\nu}, h \frac{i+1}{\nu}), \quad i = 0, \ldots, (\nu - 1)
\]

and

\[
\bar{y}[k] = \sum_{j=0}^{\mu-1} \phi_S[j] y(\bar{k} - \frac{\mu - 1}{\nu} h^-).
\]

The operation of the A/D and the D/A converters (8) is as follows. The measured output \(y(t)\) is ideally sampled \(\mu\) times during one sampling period \(h\) by the piecewise impulse sampler \(\delta_h\). The samples are weighted by the gain function \(\phi_S[j]\) and then summed to generate \(\bar{y}[k]\). The output from the digital controller at the time instance \(kh\) is shaped by the gain function \(\phi_H[i]\) of the piecewise constant hold \(\mathcal{H}_h\) in order to generate the control signal \(u(t)\), which changes its value \(\nu\) times during one sampling period \(h\) in a piecewise constant manner. It is worth noticing that \(\bar{y}[k]\) in (8b) contains information about the measured output prior to \(kh\). This is due to the fact that the weighting and summation operations can not be processed instantaneously. Also note that the A/D and D/A converters (8) digitally modulate the outputs of the controller and the plant, hence they are readily implementable by means of digital hardware.

The purpose of this paper is to solve the following \(H^2\) and \(H^\infty\) optimization problems:

**OP\(_{H^\infty}(\mathcal{P})\):** Given the generalized plant \(\mathcal{P}\), (6), the sampling period \(h\) and the constraint divisions \(\nu\) and \(\mu\), find, if they exist, the discrete part of the controller \(\mathcal{K}\) and the hold and sampler gain functions \(\phi_H[i]\) and \(\phi_S[j]\), such that the sampled-data system in Fig. 1 is internally stable and the \(H^\infty\) \((H^2)\) norm of the closed-loop operator from \(w\) to \(z\) is less than a given \(\gamma > 0\) (is minimized).

It is worthwhile noting that under an arbitrary choice of the gain functions \(\phi_S[j]\), \(\phi_H[i]\) and any LTI \(\mathcal{K}\), the system in Fig. 1 is \(h\)-periodic in continuous-time. For such systems, the notion of the \(H^2\) norm can be introduced in a natural manner (Khargonekar and Sivashankar, 1991; Bamieh and Pearson, 1992). With a slight abuse of notation, we use the term \(H^\infty\) norm to denote the \(L^2\) induced norm of the periodic operator from \(w\) to \(z\) (see (Bamieh and Pearson, 1992) for a discussion on the extension of the \(H^\infty\) system norm notion to periodic systems).

In the sequel, the \(\delta_h\) and \(\mathcal{H}_h\) solving the \(\text{OP}_{H^2}\) are referred to as \(H^\infty\)-suboptimal sampler and hold, respectively. The \(H^\infty\) suboptimality of these devices is understood as the ability to design \(\delta_h\) and \(\mathcal{H}_h\) together with \(\mathcal{K}\) so that the overall sampled-data controller \(\mathcal{K}_{sd} = \mathcal{H}_h \mathcal{K} \delta_h\) is \(\gamma\)-suboptimal. Similarly, the sampler and hold solving the \(\text{OP}_{H^2}\) will be called the \(H^2\)-optimal ones. The \(H^2\)-optimality of \(\mathcal{H}_h\) and \(\delta_h\) here is understood as the ability to design \(\mathcal{K}_{sd}\), \(\delta_h\) and \(\mathcal{H}_h\) so that the \(H^2\) performance achieved by the sampled-data controller \(\mathcal{K}_{sd}\) supersedes the one achieved using any other hold and sampling devices.
The treatment of the $\mathcal{OP}_\mathcal{H}_\infty$ and the $\mathcal{OP}_\mathcal{H}_2$ is complicated by the inherent periodicity of the sampled-data system and by its hybrid continuous/discrete nature. To overcome these difficulties, the following approach will be used. In the next section, the feedback setup in Fig. 1 will be lifted by applying the continuous- and discrete-time lifting techniques. Then, the $\mathcal{OP}_\mathcal{H}_\infty$ and the $\mathcal{OP}_\mathcal{H}_2$ will be reformulated and solved in the lifted domain, where these problems reduce to rather standard, pure discrete and time-invariant $\mathcal{H}_\infty$ and $\mathcal{H}_2$ optimizations. These results will be “peeled-off” back to the time domain in Section 5.

### 4 Solution in lifted domain

The purpose of this section is to reformulate and solve the $\mathcal{OP}_\mathcal{H}_\infty$ and the $\mathcal{OP}_\mathcal{H}_2$ in the lifted domain. Toward this end, observe that

$$\mathcal{J}_h = \mathcal{J}_h^{\text{ZOH}} \mathcal{W}_y^{-1} \Phi_H, \quad \Phi_H = \begin{bmatrix} \phi_H[0] \\ \vdots \\ \phi_H[v - 1] \end{bmatrix}, \quad h_u = h / v,$$

where $\mathcal{J}_h^{\text{ZOH}}$ is the zero order hold ($\mathcal{J}_h^{\text{ZOH}} \bar{u}(ih_u + \tau) = \bar{u}[i], \forall \tau \in [ih_u, (i + 1)h_u]$), and

$$S_h = \bar{u}_h \Phi_S \mathcal{W}_y \mathcal{W}_v, \quad \Phi_S = \begin{bmatrix} \phi_S[0] & \ldots & \phi_S[m - 1] \end{bmatrix}, \quad h_y = h / \mu,$$

where $S_h^{\text{IS}}$ is the ideal predictive sampler ($S_h^{\text{IS}} y[j] = y((j + 1)h_y)$) and $\bar{u}_h$ is the backward unit shift operator.

Using these relations and the continuous- and discrete-time lifting operations, the sampled-data control setup in Fig. 1 is converted to the equivalent one shown in Fig. 2. The usefulness of this conversion follows from the fact that, after lifting, all the subsystems in Fig. 2 are discrete-time LTI systems. Moreover, all the given information is contained in the lifted plant $\bar{\mathcal{P}}$,

$$\bar{\mathcal{P}} = \begin{bmatrix} \mathcal{W}_h \\ \mathcal{W}_y \mathcal{W}_v \end{bmatrix} \begin{bmatrix} \mathcal{P}_h^{\text{ZOH}} \mathcal{W}_v^{-1} \\ \Phi_h \end{bmatrix} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{P}}_{1*} \\ \bar{\mathcal{P}}_{2*} \end{bmatrix},$$

while all the design parameters are "absorbed" into the controller $\bar{\mathcal{K}}_{\text{eq}}$,

$$\bar{\mathcal{K}}_{\text{eq}}(z) = z^{-1} \Phi_H \bar{K}(z) \Phi_S = \begin{bmatrix} A_{K,\text{eq}} & B_{K,\text{eq}} \\ C_{K,\text{eq}} & 0 \end{bmatrix}.$$
The operator $z^{-1}$ was absorbed into the controller in order to preserve the state dimension of $\tilde{P}$ and to simplify the final formula of the controller.

In the sequel, the LTI plant $\tilde{P}$ is assumed to have the following state-space realization:

$$
\tilde{P}(z) = \begin{bmatrix}
\tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\
\tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\
\tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22}
\end{bmatrix} = \begin{bmatrix}
A & B \\
C_1 & D_{11} \\
C_2 & D_{21}
\end{bmatrix} = \begin{bmatrix}
\bar{A} & \bar{B}_1 & \bar{B}_2 \\
\bar{C}_1 & \bar{D}_{11} \\
\bar{C}_2 & \bar{D}_{21}
\end{bmatrix}.
$$

The $O_{P_{H^\infty}}$ and the $O_{P_{H^2}}$ reduce then to the following equivalent problems:

$O_{P_{H^\infty}}^{eq}$: Given the LTI discrete-time generalized plant $\tilde{P}$, (10), find, if such exists, an LTI strictly proper controller $\tilde{K}_{eq}$ that internally stabilizes the plant $\tilde{P}$ and for which the $H^\infty$ ($H^2$) norm of the closed-loop operator from $\hat{w}$ to $\hat{z}$ is less than a given $\gamma > 0$ (is minimized).

Note that, disregarding the fact that $\tilde{P}$ has infinite input and output dimensions, the $O_{P_{H^\infty}}^{eq}$ and the $O_{P_{H^2}}^{eq}$ are standard discrete-time optimization problems with a strictly proper controller. This kind of problems was discussed in detail for the case of finite dimensional pure discrete plants in (Mirkin, 1997) and for the case of continuous-lifted plants in (Mirkin et al., 1999a). Consequently, we present below the solutions to the $O_{P_{H^\infty}}^{eq}$ and to the $O_{P_{H^2}}^{eq}$ without proofs.

For the solution of the $O_{P_{H^\infty}}^{eq}$ and $O_{P_{H^2}}^{eq}$ the following assumptions are required:

(A1): The pair $(\bar{A}, \bar{B}_2)$ is stabilizable;

(A2): The pair $(\bar{A}, \bar{C}_2)$ is detectable;

(A3): The operator $\begin{bmatrix} \bar{A} - e^{j\theta}I & \bar{B}_2 \\ \bar{C}_1 & \bar{D}_{12} \end{bmatrix}$ is left invertible $\forall \theta \in [0, 2\pi]$;

(A4): The operator $\begin{bmatrix} \bar{A} - e^{j\theta}I & \bar{B}_1 \\ \bar{C}_2 & \bar{D}_{21} \end{bmatrix}$ is right invertible $\forall \theta \in [0, 2\pi]$.

These assumptions are the counterparts of the standard assumptions imposed on a discrete-time generalized plant in order to guarantee input-output stabilizability and non-singularity of the $H^2$ and $H^\infty$ optimization problems.

The solution to the $O_{P_{H^\infty}}^{eq}$ requires the following two $H^\infty$ DARE’s:

$$
\begin{align}
\bar{X} &= \bar{A}'\bar{X}\bar{A} + \bar{C}\bar{C}_1 - (\bar{B}'\bar{X}\bar{A} + \bar{D}\bar{C}_1)'(\bar{D}\bar{C}_1 - \gamma^2 E_{11} + \bar{B}'\bar{X}\bar{B})^{-1}(\bar{B}'\bar{X}\bar{A} + \bar{D}\bar{C}_1), \\
\bar{Y} &= \bar{A}\bar{Y}\bar{A}' + \bar{B}_1\bar{B}_1' - (\bar{A}\bar{Y}\bar{C} + \bar{B}_1\bar{D}_1)'(\bar{D}_1\bar{D}_1 - \gamma^2 E_{11} + \bar{C}\bar{Y}\bar{C})^{-1}(\bar{A}\bar{Y}\bar{C} + \bar{B}_1\bar{D}_1)' \tag{11a}
\end{align}
$$

where $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Using (11), necessary and sufficient conditions for the existence of a solution to the $O_{P_{H^\infty}}^{eq}$, as well as a particular solution can be established:

**Theorem 1.** Given plant (10) such that the assumptions (A1)–(A4) are satisfied, the following statements are equivalent:

i) There exists a controller $\tilde{K}_{eq}^{H^\infty}$ which solves $O_{P_{H^\infty}}^{eq}$.

ii) The DARE’s (11) have stabilizing solutions $\bar{X} \succeq 0$ and $\bar{Y} \succeq 0$ such that

$$
\left\| \begin{bmatrix} \bar{X}^{1/2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B}_1 \\ \bar{C}_1 & \bar{D}_{11} \end{bmatrix} \begin{bmatrix} \bar{Y}^{1/2} & 0 \\ 0 & 1 \end{bmatrix} \right\|_2 \leq \gamma.
$$

(12)
If the conditions of part ii) hold, then the matrix $\hat{Z} = (1 - \gamma^{-2} \hat{Y} \hat{X})^{-1}$ is well defined and one controller that solves the $\text{OP}_{\infty}$ is

$$\hat{R}^{\infty}_{eq}(z) = \left[ \frac{\hat{A} + \hat{B} \hat{F} + \hat{Z} \hat{L}_2 (\hat{C}_2 + \hat{D}_2 \hat{F})}{\hat{F}_2} - \hat{Z} \hat{L}_2 \right],$$

where:

$$\hat{F} = -(\hat{D}_1 \hat{D}_1 - \gamma^2 \hat{E}_{11} + \hat{B}^* \hat{X} \hat{B})^{-1} (\hat{B}^* \hat{X} \hat{A} + \hat{D}_1 \hat{C}_1) = \left[ \begin{array}{c} \hat{F}_1 \\ \hat{F}_2 \end{array} \right],$$

$$\hat{L} = - (\hat{ \Lambda } \hat{Y} \hat{C}^* + \hat{B}_1 \hat{D}_1^*) (\hat{D}_1 \hat{D}_1^* - \gamma^2 \hat{E}_{11} + \hat{C}_1 \hat{Y} \hat{C}^*)^{-1} = \left[ \begin{array}{c} \hat{L}_1 \\ \hat{L}_2 \end{array} \right].$$

**Remark 1.** The solution to the $\text{OP}_{\infty}$ presented in Theorem 1 contains the solution to the $\text{OP}^{eq}_{\infty}$ as a particular case. Indeed, it is easy to verify that as $\gamma \to \infty$ the $H^\infty$ DARE’s (11) reduce to the corresponding $H^2$ ones associated with $\hat{\Phi}_{12}$ and $\hat{\Phi}_{21}$. Furthermore, in this case, the conditions of statement ii) in Theorem 1 are automatically satisfied, $\hat{Z} = 1$, the $H^\infty$ suboptimal controller $\hat{R}^{\infty}_{eq}(z)$ approaches the $H^2$ optimal one, $\hat{R}^{\infty}_{eq}(z)$, and the optimal performance index $J^\infty_{H^2}$ is:

$$J^{opt}_{H^2} = \frac{1}{h} ||\hat{D}_1||^2_{H^2} + \frac{1}{h} \text{tr}\{\hat{X} \hat{B}_1 \hat{B}_1^* + (\hat{C}_1 \hat{C}_1 + \hat{A}^* \hat{X} \hat{A} - \hat{X}) \hat{Y}\}$$

Consequently, in order to reach the solutions to both the $\text{OP}_{H^2}$ and the $\text{OP}_{H^\infty}$ it is sufficient to “peel-off” back to the time domain only the solution presented in Theorem 1; the $H^2$ results can be obtained from the $H^\infty$ ones by the simple substitution $\gamma^{-1} = 0$.

The solution presented in Theorem 1 is not readily treatable, since it is not clear how to verify assumptions (A1)–(A4) and how to compute the parameters of $\hat{K}^{\infty}_{eq}$ in (13). Nevertheless, it reveals some interesting properties of the solutions to the $\text{OP}_{H^2}$ and the $\text{OP}_{H^\infty}$. In particular:

**Remark 2.** Theorem 1 presents a sampled-data controller $\hat{K}^{\infty}_{eq}$ which solves, in the lifted domain, the $\text{OP}_{H^\infty}$. It is clear, however, that the separation of this controller into $\hat{R}^{\infty}_{eq}$, $\Phi^{\infty}_{H}$, and $\Phi^{\infty}_{S}$ is not unique. By comparing (13) with (9) one possible separation is found:

$$\Phi^{\infty}_{H} = \hat{F}_2,$$  

$$\Phi^{\infty}_{S} = -\hat{L}_2,$$

and then

$$\hat{R}^{\infty}_{eq}(z) = z \left[ \frac{\hat{A} + \hat{B} \hat{F} + \hat{Z} \hat{L}_2 (\hat{C}_2 + \hat{D}_2 \hat{F})}{\hat{F}_2} - \hat{Z} \hat{L}_2 \right].$$

From implementation point of view, however, other separations might be advantageous.

**Remark 3.** It is clear from (9) that the hold function $\Phi_H$ is completely absorbed into the ‘C’-part of the sampled-data controller $\hat{K}_{eq}$, while the sampling function $\Phi_S$ is contained in its ‘B’-part. Hence, by comparing this equation with (13) in Theorem 1, the $\gamma$-suboptimal hold and sampler are characterized by the operators $\hat{F}_2$ and $\hat{L}_2$, respectively. On the other hand, by inspecting (14) and (11), it is seen that $\hat{F}_2$ (hence, also the $\gamma$-suboptimal piecewise constant hold) depends

\footnote{Although the ‘B’-part of $\hat{R}^{\infty}_{eq}$ contains the coupling term $Z$, the latter can be absorbed into the discrete-time part of the controller.}
only on the parameters of $\mathcal{P}_{1*}$ — the subsystem from $w$ and $u$ to $z$. Similarly, $\bar{L}_2$ (hence, also the $\gamma$-suboptimal piecewise impulse sampler) depends only on the parameters of $\mathcal{P}_{*1}$ — the subsystem from $w$ to $z$. Hence, there is a separation between the designs of the $H^\infty$ suboptimal $\mathcal{H}_1$ and $\mathcal{S}_1$ in the sense that the hold design does not depend on the measurement $y(t)$, and the sampler design does not depend on the control action $u(t)$. However, both designs are affected by the subsystem from $w$ to $z$, hence the separation is not complete. This is similar to the separation between the designs of the $H^\infty$ suboptimal $H^\infty$ and $S^\infty$ in the sense that the hold design does not depend on the measurement $y(t)$, and the sampler design does not depend on the control action $u(t)$. However, both designs are affected by the subsystem from $w$ to $z$, hence the separation is not complete. This is similar to the separation between the designs of the $H^\infty$ suboptimal generalized sampler and hold in the unconstrained case (Mirkin et al., 1999a) and in contrast to other works in the literature (Tadmor, 1992; Mirkin and Rotstein, 1997), where the $H^\infty$ suboptimal unconstrained sampler depends on the hold. Note that as $\gamma \to \infty$, $\bar{F}_2$ and $\bar{L}_2$ become the stabilizing matrix gains of the $H^2$ DARE’s associated with $\hat{P}_{12}$ and $\hat{P}_{21}$. Since these DARE’s are independent of the parameters of the subsystem from $w$ to $z$, the separation between the designs of the $H^2$ optimal generalized sampler and hold in the unconstrained case (Mirkin et al., 1999a) and it is a reminiscence of the separation between the state feedback and state estimation in the standard $H^2$ (LQG) design.

Note that the formal solution to $\text{OP}^\infty_{H^\infty(H^2)}$ is exactly the same as the solution to the $H^\infty$ ($H^2$) optimization problem defined in the lifted domain by (Mirkin et al., 1999a) for the case of the unconstrained generalized sampling and hold functions. This is due to the fact that in both the unconstrained and the constrained cases the same idea was used: to reduce a periodically varying control problem to an equivalent one which is time-invariant in the lifted domain. Yet the lifted domain in which the optimization problems were defined in this paper is different from that used by (Mirkin et al., 1999a): for the optimization problems of the unconstrained generalized sampling and hold functions only the continuous-time lifting was required, while for the $\text{OP}^\infty_{H^\infty(H^2)}$ the discrete-time lifting had to be used, in addition, due to its multi-rate nature. Consequently, even though the lifted problems in both papers have the same formal solutions, the “peeling-off” process of the solutions to the $\text{OP}^\infty_{H^\infty(H^2)}$ will be different. The peeling-off is discussed below.

5 Main results

This section is devoted to peeling-off the lifted solution given in Theorem 1. This will result in the readily implementable solution to $\text{OP}^\infty_{H^\infty}$ in terms of the original plant parameters. The solution to the $\text{OP}^\infty_{H^2}$ can easily be obtained from that of the $\text{OP}^\infty_{H^\infty}$ by the simple substitution $\gamma^{-1} = 0$ (see Remark 1). The only part of the $H^2$ solution which has to be treated independently is the computation of the optimal performance index $J^\text{opt}_{H^2}$. This issue is the subject matter of Lemma 3. A short discussion on the properties and the interpretations of the $H^2$ and the $H^\infty$ (sub)optimal piecewise impulse sampling and piecewise constant hold functions will follow.

Let

$$\Sigma_H \doteq \exp\left(\begin{bmatrix} 0 & -D_{12}C_1 & -B_1^T & -D_{12}D_{12} \\ 0 & A & \gamma^{-2}B_1B_1^T & B_2 \\ 0 & -C_1^T & -A^T & -C_1^TD_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix} h_u \right) = \begin{bmatrix} 1 & \Sigma_{H12} & \Sigma_{H13} & \Sigma_{H14} \\ 0 & \Sigma_{H22} & \Sigma_{H23} & \Sigma_{H24} \\ 0 & \Sigma_{H32} & \Sigma_{H33} & \Sigma_{H34} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{17a}$$
\[ \Sigma_S \triangleq \exp \left( \begin{bmatrix} A & -B_1B_1' \\ \gamma^{-2}C_1C_1 & -A' \end{bmatrix} h_y \right) = \begin{bmatrix} \Sigma_{S11} & \Sigma_{S12} \\ \Sigma_{S21} & \Sigma_{S22} \end{bmatrix}, \]  

(17b)

where: \( h_u \triangleq h/\nu \) and \( h_y \triangleq h/\mu \). The computation of the \( L^2[0,h] \) induced norm of the subsystem from \( w \) to \( z \) is also required:

\[ \gamma_0 \triangleq \|P_{11}\|_{L^2[0,h]}. \]

This quantity can be computed as described in (Chen and Francis, 1995), and it is the lower bound for the \( H^\infty \) performance in sampled-data systems under an arbitrary choice of \( \delta_h \) and \( \gamma_h \). Hence it is natural to consider only the cases where \( \gamma > \gamma_0 \).

In the sequel we assume that:

(A1'): The pair \( (\Sigma_{H22}, \Sigma_{H24}) \) is stabilizable for any \( \gamma > \gamma_0 \);

(A2'): The pair \( (C_2, \Sigma_{S11}) \) is detectable for any \( \gamma > \gamma_0 \);

(A3'): The matrix

\[
\begin{bmatrix}
\Sigma_{H12} & \Sigma_{H14} \\
\Sigma_{H22} - e^{i\theta}1 & \Sigma_{H24} \\
\Sigma_{H32} & \Sigma_{H34}
\end{bmatrix}
\]

is left invertible \( \forall \theta \in [0,2\pi) \) and any \( \gamma > \gamma_0 \);

(A4'): The matrix

\[
\begin{bmatrix}
\Sigma_{S11} - e^{i\theta}1 & \Sigma_{S12} \\
C_2 & 0
\end{bmatrix}
\]

is right invertible \( \forall \theta \in [0,2\pi) \) and any \( \gamma > \gamma_0 \);

Assumptions (A1') and (A2') are, in a sense (see (Mirkin et al., 1999b, Subsection 4.2)), the counterparts of the standard assumptions on the stabilizability and detectability of \( P_{22} \). They are necessary for the existence of solutions to the \( \text{OP}_{1}^{H^\infty} \). Moreover, as \( \gamma \to \infty \) these assumptions become necessary and sufficient for the existence of sampled-data stabilizing controllers for the setup in Fig. 1. Assumptions (A3') and (A4') are the counterparts of the standard assumptions on the absence of unit circle (or the imaginary axis, in the continuous-time case) zeros of the subsystems from the control signal to the regulated output and from the exogenous input to the measured output, respectively.

The next Lemma establishes an important relation between assumptions (A1'), (A3') and (A1), (A3), which were required for the solution to the \( \text{OP}_{1}^{H^\infty} \) in the lifted domain.

**Lemma 2.** Whenever \( \gamma > \gamma_0 \) and the \( \text{OP}_{1}^{H^\infty} \) is solvable, plant (6) satisfies assumptions (A1) and (A3) if and only if it satisfies assumptions (A1') and (A3').

**Proof.** First, we will reformulate assumptions (A1) and (A3) in terms of the parameters of

\[ \mathcal{P}_{1*H} \triangleq W_{h_u} P_{1*} \begin{bmatrix} W_{h_u}^{-1} \\ \gamma(t_{h_u}) \end{bmatrix}, \quad \mathcal{P}_{1*H}(z) = \begin{bmatrix} \tilde{A}_H & \tilde{B}_{1H} & \tilde{B}_{2H} \\ \tilde{C}_{1H} & \tilde{D}_{11H} & \tilde{D}_{12H} \end{bmatrix}. \]

To this end, note that \( \mathcal{P}_{1} = \mathcal{W}_v \mathcal{P}_{1*H} \begin{bmatrix} W_v^{-1} \\ \gamma_v^{-1} \end{bmatrix} \). Also note that assumptions (A1) and (A3) are actually (Lancaster and Rodman, 1995) the necessary and sufficient conditions for the existence of the stabilizing solution to the \( H^2 \) DARE associated with \( \mathcal{P}_{12} \). According to Lemma 1, this solution exists if and only if the \( H^2 \) DARE associated with \( \mathcal{P}_{12H} \) possesses a stabilizing solution. This solution, however, exists if and only if the following conditions are satisfied:
(A1''') The pair \((\bar{A}_H, \bar{B}_{2H})\) is stabilizable;

(A3'') The operator \(\begin{bmatrix} \bar{A}_H - e^{i\theta}I & \bar{B}_{2H} \\ \bar{C}_{1H} & \bar{D}_{12H} \end{bmatrix}\) is left invertible \(\forall \theta \in [0, 2\pi)\).

Hence, assumptions (A1) and (A3) both\(^2\) hold true if and only if so do (A1''') and (A3'').

Assume now that plant (6) satisfies assumptions (A1'), (A3') and that the \(\text{OP}_{\text{H}\infty}\) is solvable. Consequently:

i) (A3'') is satisfied, since it is equivalent to (A3') (Mirkin et al., 1999b, Lemma 6), and

ii) (A1) holds true, since it is a necessary condition for the existence of stabilizing controllers for plant (6).

Thus, in order to complete the proof to the first part of this Lemma, we only have to show that if assumption (A1) is satisfied, so is (A1'''). We prove that by contradiction. Assume that the pair \((\bar{A}_H, \bar{B}_{2H})\) is not stabilizable. Hence, there exist \(|\lambda| \geq 1\) and \(\eta \in \mathbb{R}^n\), \(\eta \neq 0\) such that \(\eta' [\bar{A}_H - \lambda I \ \bar{B}_{2H}] = 0\). It means that \(\eta'\) is the left eigenvector of \(\bar{A}_H\) associated with the eigenvalue \(\lambda\). Thus \(\eta' \bar{A}_H^k \eta' = \lambda^k \eta'\). Consequently, \(\eta' \bar{A}_H^k \bar{B}_{2H} = 0\), \(\forall k = 0, \ldots, \nu - 1\) and \(\eta' (\bar{A}_H^\nu - \lambda^\nu I) = 0\). This, in turn, leads to

\[
\eta' [\bar{A} - \lambda^\nu I \ \bar{B}_2] = \eta' [\bar{A}_H^\nu - \lambda^\nu I \ \bar{A}_H^{\nu-1} \bar{B}_{2H} \ \ldots \ \bar{A}_H \bar{B}_{2H} \ \bar{B}_{2H}] = 0,
\]

which means that \((\bar{A}, \bar{B}_2)\) is also not stabilizable, since \(|\lambda| \geq 1 \Rightarrow |\lambda^\nu| \geq 1\).

To prove the second part of this Lemma, we consider the state-feedback single-rate sampled-data \(\text{H}\infty\) optimization problem for the plant

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_1w(t) + B_2u(t), \\
\ell(t) &= C_1x(t) + D_{12}u(t),
\end{align*}
\]

where both the zero-order hold and the ideal sampler operate with the sampling period \(h_u\). We claim that, if assumptions (A1'''), (A3'') are satisfied and the \(\text{OP}_{\text{H}\infty}\) for plant (18) possesses a \(\gamma\) suboptimal solution, so does this single-rate optimization problem. Then, according to (Mirkin et al., 1999b, Lemma 5), the solution to this problem exists only if assumption (A1') holds true.

The fact that (A3') and (A3'') are equivalent, completes the proof.

To prove this claim, let assumptions (A1''') and (A3'') be satisfied and suppose that the \(\text{OP}_{\text{H}\infty}\) for plant (18) possesses a solution. According to Theorem 1, the stabilizing solution \(\bar{X}\) to the \(\text{H}\infty\) DARE (11a), satisfies

\[
\begin{bmatrix} \bar{X}^{1/2} \bar{B}_1 \\ \bar{D}_{11} \end{bmatrix} \leq \gamma \iff \rho(\bar{B}_1^* \bar{X} \bar{B}_1 + \bar{D}_{11}^* \bar{D}_{11}) \leq \gamma.
\]

Note that the hermitian matrix \(\bar{B}_1^* \bar{X} \bar{B}_1 + \bar{D}_{11}^* \bar{D}_{11}\) has the form

\[
\begin{bmatrix} \bar{B}_1^* \bar{X} \bar{B}_1 + \bar{D}_{11}^* \bar{D}_{11} \\
\end{bmatrix} = \begin{bmatrix} \bar{B}_1^* \bar{X} \bar{B}_1 + \bar{D}_{11}^* \bar{D}_{11} \\ \bar{B}_1^* \bar{X} \bar{B}_1 + \bar{D}_{11}^* \bar{D}_{11} \end{bmatrix},
\]

where ? denotes irrelevant block terms. According to the Cauchy Theorem of separation (Gantmacher, 1974):

\[
\rho(\bar{B}_1^* \bar{X} \bar{B}_1 + \bar{D}_{11}^* \bar{D}_{11}) \leq \rho(\bar{B}_1^* \bar{X} \bar{B}_1 + \bar{D}_{11}^* \bar{D}_{11}) \leq \gamma.
\]

\(^2\)In fact, it is possible to prove that (A1) is equivalent to (A1''') and (A3) is equivalent to (A3''). Yet this proof is more involved and not essential for the reasoning to follow.
Since, as a direct application of Lemma 1, $\bar{X}$ is also the stabilizing solution to the $H^\infty$ DARE associated with the plant $\mathcal{P}_{\kappa\mu}$, we conclude, according to (Mirkin et al., 1999b, Theorem 2), that the single-rate optimization problem defined for plant (18) possesses a $\gamma$ suboptimal solution.

We are now in the position to present the solution to the $\text{OP}_{H^\infty}$. For this purpose, we also define the following two extended symplectic pencils (Mirkin et al., 1999b):

$$
(\Lambda_\nu, \Delta_\nu) \doteq \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \Sigma_{H12} & \Sigma_{H13} & \Sigma_{H14} \\ \Sigma_{H22} & \Sigma_{H23} & \Sigma_{H24} \\ \Sigma_{H32} & \Sigma_{H33} & \Sigma_{H34} \end{pmatrix}, \quad (19a)
$$

$$
(\Lambda_\mu, \Delta_\mu) \doteq \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & C_2 & 0 \\ \Sigma'_{S11} & \Sigma'_{S21} & \Sigma'_{S11}C_2' \\ \Sigma'_{S12} & \Sigma'_{S22} & \Sigma'_{S12}C_2' \end{pmatrix}, \quad (19b)
$$

Theorem 2. Suppose plant (6) satisfies assumptions (A1')–(A4') and let $\nu = \kappa\mu$ for some $\kappa \in \mathbb{Z}^+/(0)$. Then, for any $\gamma > \gamma_0$, the following statements are equivalent:

i) There exist $\bar{X}^H$, $\phi_H[i]$, and $\phi_S[i]$ which solve the $\text{OP}_{H^\infty}$.

ii) $(\Lambda_\nu, \Delta_\nu) \in \text{dom}(\text{Ric}_D)$, $(\Lambda_\mu, \Delta_\mu) \in \text{dom}(\text{Ric}_D)$ and the following conditions are satisfied:

(a) $\bar{X}_\nu \geq 0$ and $\rho(\bar{X}_\nu \Pi_{12} \Pi_{22}^{-1}) < 1$;

(b) $\bar{Y}_\mu \geq 0$ and $\rho(\Pi_{22}^{-1} \Pi_{21} \bar{Y}_\mu) < \gamma^2$;

(c) $\rho(\bar{Y}_\mu(\Pi_{22} + \gamma^{-2} \Pi_{21} \bar{Y}_\mu)^{-1} \bar{X}_\nu(\Pi_{22} - \Pi_{12} \bar{X}_\nu)^{-1}) < \gamma^2$;

where $(\bar{X}_\nu, \bar{F}_2) = \text{Ric}_D(\Lambda_\nu, \Delta_\nu)$, $(\bar{Y}_\mu, \bar{L}_2) = \text{Ric}_D(\Lambda_\mu, \Delta_\mu)$ and

$$
\Pi = \begin{bmatrix} \Sigma_{H12} & \Sigma_{H13} \\ \Sigma_{H32} & \Sigma_{H33} \end{bmatrix}^\gamma = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}.
$$

Furthermore, if the conditions of part ii) hold, then the matrix $\bar{Z}_{\nu\mu} \doteq (I - \gamma^{-2} \bar{Y}_\mu \bar{X}_\nu)^{-1}$ is well defined and one possible choice for the sampled-data controller which solves the $\text{OP}_{H^\infty}$ consists of the discrete-time part:

$$
\bar{R}^H(z) = z \left[ \frac{\bar{Z}_{\nu\mu} \Theta_{12} + \Theta_{22}}{1} \frac{\bar{Z}_{\nu\mu}}{0} \right], \quad (20a)
$$

the generalized hold of the form (8a) with

$$
\phi_H[i] = \bar{F}_2(\Sigma_{H22} + \Sigma_{H24} \bar{F}_{2v} + \Sigma_{H23} \bar{X}_v)^i, \quad i = 0, \ldots, \nu - 1, \quad (20b)
$$

and of the generalized sampler of the form (8b) with

$$
\phi_S[j] = (\Sigma_{S11} + \bar{L}_2 \mu C_2 \Sigma_{S11} + \bar{Y}_\mu \Sigma_{S21})^j \bar{L}_2, \quad j = 0, \ldots, \mu - 1, \quad (20c)
$$

where

$$
\Theta = \begin{bmatrix} \Sigma_{S11} + \bar{L}_2 \mu C_2 \Sigma_{S11} + \bar{Y}_\mu \Sigma_{S21} & \bar{L}_2 \mu C_2 (\Sigma_{H22} + \Sigma_{H24} \bar{F}_{2v} + \Sigma_{H23} \bar{X}_v)^\kappa \\ 0 & (\Sigma_{H22} + \Sigma_{H24} \bar{F}_{2v} + \Sigma_{H23} \bar{X}_v)^\kappa \end{bmatrix}^{\mu} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ 0 & \Theta_{22} \end{bmatrix}.
$$
Proof. Although assumptions (A1')–(A4') are, in general, not equivalent to (A1)–(A4), according to Lemma 2 and its dual, the replacement of (A1)–(A4) with the more readily checkable conditions (A1')–(A4') does not affect the solution to the OPH∞ presented in Theorem 1. Hence, it is actually sufficient to prove that the solution presented in this Theorem is equivalent to the one in Theorem 1. To this end, note that $\tilde{P}_{\ast 1} = \begin{bmatrix} \tilde{W}_\mu \\ \tilde{W}_\mu \end{bmatrix} \tilde{P}_{\ast 1S} \tilde{W}_\mu^{-1}$, where

$$\tilde{P}_{\ast 1S} = \begin{bmatrix} \tilde{W}_\mu \end{bmatrix} \tilde{P}_{\ast 1S} \tilde{W}_\mu^{-1}, \quad \tilde{P}_{\ast 1S}(z) = \begin{bmatrix} \bar{A}_S \\ \bar{C}_1S \\ \bar{D}_{11S} \\ \bar{C}_2S \\ \bar{D}_{21S} \end{bmatrix}.$$ 

Thus, as a direct application of Lemma 1, $\tilde{X}$ and $\tilde{Y}$ are the stabilizing solutions to the H∞ DARE's (11) if and only if they are the stabilizing solutions to the H∞ DARE's associated with the plants $\tilde{P}_{\ast 1H}$ and $\tilde{P}_{\ast 1S}$, respectively. Since it was assumed that $\gamma > \gamma_0$ and by a direct application of (Mirkin et al., 1999b, Lemma 7), the latter DARE's have stabilizing solutions if and only if $(\Lambda_v, \Delta_v) \in \text{dom}(\text{Ric}_D)$ and $(\Lambda_\mu, \Delta_\mu) \in \text{dom}(\text{Ric}_D)$. Moreover, in this case, $\tilde{X} = \tilde{X}_\nu, \tilde{Y} = \tilde{Y}_\mu$ and $\bar{Z} = \bar{Z}_\nu$. Then, according to (Mirkin et al., 1999b, Lemma 8), the items (a)–(c) are equivalent to the coupling condition (12).

Now, consider the separation of $\bar{X}_{eq}^{\infty}$ into $\bar{X}_{eq}^{\infty}, \Phi_H^{\infty}$ and $\Phi_S^{\infty}$ suggested in (16). Denote by $\tilde{F}_v = \begin{bmatrix} \tilde{F}_{1v} \\ \tilde{F}_{2v} \end{bmatrix}$ and $\tilde{L}_\mu = \begin{bmatrix} \tilde{L}_{1\mu} \\ \tilde{L}_{2\mu} \end{bmatrix}$ the stabilizing gain matrices of the H∞ DARE's associated with these plants, respectively. By using (Mirkin et al., 1999b, Lemma 7) again, the following relations are established:

$$\bar{A}_H = \bar{A}_H + \bar{B}_{1H} \bar{F}_{1v} + \bar{B}_{2H} \bar{F}_{2v} = \Sigma_{H22} + \Sigma_{H23} \bar{F}_{2v} + \Sigma_{H12} \bar{F}_{1v},$$
$$\bar{A}_S = \bar{A}_S + \bar{L}_{1\mu} \bar{C}_{1S} + \bar{L}_{2\mu} \bar{C}_{2S} = \Sigma_{S11} + \Sigma_{S12} \bar{C}_{1S} + \Sigma_{S13} \bar{C}_{2S} + \Sigma_{S21}.$$ 

On the other hand, it follows from Lemma 1 that

$$\tilde{F}_2 = \begin{bmatrix} \tilde{F}_{2v} \\ \tilde{F}_{2v} \bar{A}_H \\ \vdots \\ \tilde{F}_{2v} \bar{Y}_v^{-1} \end{bmatrix}, \quad \tilde{L}_2 = \begin{bmatrix} \tilde{L}_{2\mu} \\ \tilde{L}_{2\mu} \bar{A}_S \tilde{L}_{2\mu} \vdots \tilde{L}_{2\mu} \bar{Y}_\mu^{-1} \tilde{L}_{2\mu} \end{bmatrix},$$

which proves (20b) and (20c).

To complete the proof, the computational formula for the ‘A’ part of the digital controller, given in (20a), will be derived. To this end, assume that $\mu$ is a divisor of $\nu$ or, in other words, that $\nu/\mu = \kappa$, where $\kappa$ is a natural number. Also note that, in this case, $\tilde{P}_{22} = \bar{W}_\mu \bar{W}_\mu^\dagger \bar{P}_{22} \bar{H}_\kappa(t_{nu}) \bar{W}_\kappa^{-1}$ can be written as $\tilde{P}_{22} = \bar{W}_\mu \bar{P}_{22S} \bar{W}_\mu^{-1}$, where $\tilde{P}_{22S} = \bar{W}_{h_\mu} \bar{P}_{22S} \bar{H}_\kappa(t_{nu}) \bar{W}_\kappa^{-1}$,

$$\tilde{P}_{22S}(z) = \begin{bmatrix} \bar{A}_S \\ \bar{B}_{2S} \\ \bar{C}_{2S} \\ \bar{D}_{22S} \end{bmatrix}.$$ 

Denote $\bar{B}_S = \begin{bmatrix} \bar{B}_{1S} \\ \bar{B}_{2S} \end{bmatrix}$ and $\bar{D}_{2S} = \begin{bmatrix} \bar{D}_{21S} \\ \bar{D}_{22S} \end{bmatrix}$. Using the DSOFTI representation (3) of the discrete-lifted parameters $3$, $\bar{C}_2, \bar{D}_2$, and $\bar{L}_2$ can be represented as follows:

$$\bar{C}_2 = \left( \begin{array}{c} \bar{A}_S \\ \bar{C}_S \\ \bar{C}_2 \\ \bar{C}_{2S} \end{array} \right)^{\mu-1} 0, \quad \bar{D}_2 = \left( \begin{array}{c} \bar{A}_S \\ \bar{C}_S \\ \bar{D}_{2S} \end{array} \right)^{\mu-1} 0, \quad \bar{L}_2 = \tilde{f}_\mu^{-1} \left( \begin{array}{c} \bar{A}_S \\ \bar{C}_S \\ \bar{L}_2 \end{array} \right)^{\mu-1} 0.$$ 

3This representation suffices since the derivation of the ‘A’ part of the digital controller requires only the multiplication and the addition operations.
Hence, the problem, which is a particular case of the general multi-rate problem treated in (Chen and Qiu, 1994; Voulgaris and Bamieh, 1993). It is worthwhile noting that Theorem 2 provides a solution to some known optimization problems already solved in the literature as particular cases. When \( \nu = \mu = 1 \), the sampled-data controller \( \mathcal{H}_H \mathcal{S}_H \) is the solution to the \( H^\infty \) suboptimal single-rate sampled-data control problem (Bamieh and Pearson, 1992) based on the zero-order hold and on the ideal sampler converters (\( \Phi_H \) and \( \Phi_S \) are absorbed into the digital controller, thus becoming its ‘C’ and ‘B’ coefficients, respectively). In the case where \( \mu = 1 \), Theorem 2 actually solves the input multi-rate (Araki, 1993) \( H^\infty \) problem, which is a particular case of the general \( H^\infty \) multi-rate problem treated in (Chen and Qiu, 1994; Voulgaris and Bamieh, 1993). It is worthwhile noting that Theorem 2 provides a simpler solution in this case both from the computational and conceptual point of view.

As follows from Remark 1, the solution to the \( OP_{H^\infty} \) approaches that to the \( OP_{H^2} \) as \( \gamma \to \infty \). Hence, the \( H^2 \) results can be obtained by the simple substitution \( \gamma^{-1} = 0 \) in Theorem 2.

Using again the fact that \( \nu / \mu = \kappa \), the operator \( \hat{\mathcal{F}} \) can also be represented as a DSOFTI operating over the finite time interval \([0, \mu - 1]\):

\[
\hat{\mathcal{F}} = \left( \begin{array}{c|c}
\overline{A}_H^\kappa & \overline{A}_H^\kappa \\
-\frac{\bar{F}_\nu}{\bar{F}_\kappa} & -\frac{\bar{F}_\nu}{\bar{F}_\kappa}
\end{array} \right)^{-1} \mathcal{J}_0,
\]

\[
\hat{\mathcal{F}}_\kappa = \left[ \begin{array}{c}
\hat{\mathcal{F}}_\nu \\
\vdots \\
\hat{\mathcal{F}}_\nu \overline{A}_H^{-1}
\end{array} \right].
\]

Based on the definition of \( \hat{\mathcal{F}}_{2\nu} \), the relations \( \hat{C}_{2\nu} = C_2 \overline{A}_S, \hat{D}_{2\nu} = C_2 \overline{B}_S \) and \( \overline{B}_S \hat{F}_\kappa = \overline{A}_H^\kappa - \overline{A}_S \) can be derived. Using these relations and the standard formulas for the addition and the multiplication of state-space systems, it is found that

\[
\mathcal{L}_2(\hat{C} + \hat{D} \hat{F}) = \mathcal{J}_\mu^{-1} \left( \begin{array}{c|c}
\overline{A}_S & \mathcal{L}_{2\mu} C_2 \overline{A}_H^\kappa \\
0 & \overline{A}_H^\kappa
\end{array} \right)^{-1} \mathcal{J}_0,
\]

which proves (20a), since \( \bar{A} + \bar{B} \bar{F} = \bar{A}_H + \Theta_{22} \) (Lemma 1).

Remark 4. Note that each of the matrices given by (17) plays a different role in the solution process. The matrix exponential \( \Sigma_H \) in (17a) is required only for the computation of the hold gain function \( \phi_H^{H^\infty} [i] \), while \( \Sigma_S \) in (17b) is required only for the computation of the sampler gain function \( \phi_S^{H^\infty} [j] \).

Remark 5. It is worthwhile noting that the assumption \( \nu = \kappa \mu \) is required only to simplify the derivation and the final formula of the main coefficient (the ‘A’ matrix) of the discrete-time part of the controller \( \mathcal{F}_H^{H^\infty} \). Indeed, this assumption affects neither formulae (20b) and (20c) nor conditions (a)-(c), which still apply in the general case.

Remark 6. It is worth mentioning that the “peeling-off” process of the ‘A’ matrix of the discrete-time part of the controller can be performed also in the general case. Thus, the assumption \( \nu = \kappa \mu \) is not a limitation due to the lifting-based approach used to solve the \( OP_{H^\infty} \). The general formula for this matrix is not presented in this paper since it turns out to be quite complicated. This is due to the fact that, in the general case, the operators \( \mathcal{L}_2, \mathcal{H}_2, \) and \( \mathcal{F} \) can only be represented by DSOFTI’s operating over different time intervals (\([0, \nu - 1]\) and \([0, \mu - 1]\)), even though, as block-operators, they have compatible dimensions. Since the multiplication of DSOFTI’s defined over different time intervals is meaningless, only the matrix-based approach can be used to calculate \( \mathcal{L}_2 \mathcal{H}_2 \mathcal{F} \). It is believed, however, that a simpler expression also exists in the general case, and its derivation is currently under investigation.

Remark 7. Theorem 2 contains the solutions to some known optimization problems already solved in the literature as particular cases. When \( \nu = \mu = 1 \), the sampled-data controller \( \mathcal{H}_H \mathcal{S}_H \) is the solution to the \( H^\infty \) suboptimal single-rate sampled-data control problem (Bamieh and Pearson, 1992) based on the zero-order hold and on the ideal sampler converters (\( \Phi_H \) and \( \Phi_S \) are absorbed into the digital controller, thus becoming its ‘C’ and ‘B’ coefficients, respectively). In the case where \( \mu = 1 \), Theorem 2 actually solves the input multi-rate (Araki, 1993) \( H^\infty \) problem, which is a particular case of the general \( H^\infty \) multi-rate problem treated in (Chen and Qiu, 1994; Voulgaris and Bamieh, 1993).
already explained at the beginning of this section, the only part of the $H^2$ solution which has to be treated independently of the $H^\infty$ one is the computation of the optimal performance index $J_{H^2}^{opt}$. The next lemma presents a simple formula for $J_{H^2}^{opt}$, in terms of the original plant parameters.

**Lemma 3.** Let (A1′)-(A4′) be satisfied and $(\bar{X}_v, \bar{F}_v) = \text{Ric}_D(\Lambda_v, \Delta_v)$. $(\bar{Y}_\mu, L_\mu) = \text{Ric}_D(\Lambda_\mu, \Delta_\mu)$ for $\gamma^{-1} = 0$ and for some general constraint divisions $\nu$ and $\mu$. The optimal value of the performance index $J_{H^2} = \|F_1(P, J_{h_1}hS_h)\|_{H^2}^2$ is:

$$J_{H^2}^{opt} = \frac{1}{h} \text{tr}\{\Sigma_{22}\Sigma_{31}\} + \frac{1}{h} \text{tr}\{\bar{X}_v\Sigma_{22}\Sigma_{21} + (\Sigma_{32}\Sigma_{22} + \Sigma_{22}^\prime\bar{X}_v\Sigma_{22} - \bar{X}_v)Y_\mu\},$$

where

$$\Sigma = \exp\left(\begin{bmatrix} -A' & 0 & 0 \\ B_1B_1' & A & 0 \\ 0 & C_1'C_1 & -A' \end{bmatrix}h\right) = \begin{bmatrix} \Sigma_{11} & 0 & 0 \\ \Sigma_{21} & \Sigma_{22} & 0 \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{bmatrix}.$$ 

**Proof.** This formula is a direct application of (Mirkin and Palmor, 1997, Th. 2) (see also (Bamieh and Pearson, 1992; Mirkin et al., 1999a)), since $\bar{P}_{11} = W_h\bar{P}_{11}W_h^{-1}$. \hfill $\square$

The solutions to the piecewise constant hold function in (20b) and the piecewise impulse sampling function in (20c) have interesting properties and interpretations. The remainder of this section is devoted to the discussion of those properties.

The first property is the separation between the designs of the $H^\infty$ suboptimal piecewise impulse sampler and piecewise constant hold in the sense that the hold design does not depend on the measurement $y(t)$, and the sampler design does not depend on the control action $u(t)$. The separation property, which has already been discussed in Remark 3, can be explained by the fact that both the sampler and the hold are, in a sense, open-loop devices.

A nice interpretation for the $H^\infty$ suboptimal piecewise constant hold (20b) can be obtained from the solution to the state-feedback single-rate sampled-data $H^\infty$ optimization problem for the plant (18). Assume that, in this problem, both the zero-order hold and the ideal sampler operate with the sampling period $h_u$. Hence, the state vector satisfies the following equation:

$$\bar{x}[i + 1] = \bar{A}_H\bar{x}[i] + \bar{B}_1h\bar{w}[i] + \bar{B}_2h\bar{u}[i].$$

It was shown by (Mirkin et al., 1999a) that the solution to this problem is based on the DARE $(\bar{X}_v, \bar{F}_v)$ (the same as the one used in Theorem 2) and that the resulting state-feedback control law is $\bar{\tilde{u}}[i] = \bar{F}_v\bar{x}[i]$. Assume now that the disturbance $w$ is given by $\bar{w}[i] = \bar{F}_v\bar{x}[i]$. The closed-loop state vector satisfies

$$\bar{x}[i + 1] = (\bar{A}_H + \bar{B}_1h\bar{F}_1v + \bar{B}_2h\bar{F}_2v)\bar{x}[i] = (\Sigma_{H22} + \Sigma_{H24}\bar{F}_2v + \Sigma_{H23}\bar{X}_v)\bar{x}[i],$$

and, consequently, the control signal $u(t)$ satisfies

$$u(kh + ih_u + \tau) = \bar{F}_v(\Sigma_{H22} + \Sigma_{H24}\bar{F}_2v + \Sigma_{H23}\bar{X}_v)^{\dagger}\bar{x}[k], \quad \forall \tau \in [0, h_u).$$

On the other hand, it follows from (8a) and (20b) that the $H^\infty$ suboptimal piecewise constant hold $J_{h_1}$ produces the control signal

$$u(kh + ih_u + \tau) = \bar{F}_v(\Sigma_{H22} + \Sigma_{H24}\bar{F}_2v + \Sigma_{H23}\bar{X}_v)^{\dagger}\bar{u}[k], \quad \forall \tau \in [0, h_u).$$
The comparison between the latter two expressions yields that the $H_\infty$ suboptimal piecewise constant hold with a sampling period $h$ attempts to "reconstruct" the $H_\infty$ state-feedback control law of the single-rate sampled-data control system with a $\nu$ times faster sampling period, assuming that i) the digital controller produces at the $k$-th sampling instance an estimate of the state vector of the plant at $t = kh$; and ii) the disturbance $\hat{w}[i] = \hat{F}_1 \hat{x}[i]$ is the worst case one. In other words, the $H_\infty$ suboptimal piecewise constant hold, tries to compensate for the deterioration in the system performance due to the insufficiently fast sampling rate, by imitating the control law of a faster $H_\infty$ suboptimal sampled-data controller. This is in contrast to an early design (Kabamba, 1987) where the hold device was designed to outperform single-rate sampled-data controllers with faster sampling rates. This property is similar to that of the generalized unconstrained hold developed in (Mirkin et al., 1999a) which tries to imitate the continuous-time $H_\infty$ suboptimal state-feedback control law.

6 Numerical example

Consider the robust stability setup in Fig. 3, where $P_0 = \frac{1}{(s-1)(s+2)}$ is the nominal plant, $F = \frac{1}{s+1}$ is the antialiasing filter, the uncertainty weighting function $W_\Delta = \frac{s}{s+1}$, and uncertainty $\Delta$ is an arbitrary possibly time-varying bounded operator. The control goal here is to maximize the robust stability radius $\alpha$, i.e., to find a maximum $\alpha > 0$ so that the closed-loop system is stable for all $\|\Delta\|_{L_2} < \alpha$.

It is well known (Dullerud, 1996) that the closed loop system in Fig. 3 is stable for all $\|\Delta\|_{L_2} < \alpha$ iff the $H_\infty$ norm of the closed-loop system in Fig. 1 is less than $1/\alpha$ for the generalized plant

$$\mathcal{P} = \begin{bmatrix} 0 & 1 \\ \mathcal{F} W_\Delta & \mathcal{F} P_0 \end{bmatrix}.$$ 

We solve the latter problem assuming that the hold $H_h$ is the zero-order hold and the sampler $S_h$ is the generalized piecewise-impulse sampler with the constraint division $\mu$. Three cases will be considered: $\mu = 1$, $\mu = 2$, and $\mu = 10$.

The simulation results (the robust stability radius $\alpha$ versus the sampling period $h$) are shown in Fig. 4. As expected, when $h \to 0$, the stability radius approaches that of the continuous-time control $\alpha_c \approx 1.83$, while $\alpha \to 0$ as $h \to \infty$ (since the nominal plant $P_0$ is unstable) for all $\mu$.

As seen from Fig. 4, even when $\mu = 2$ the optimal design of the hold function enables one to increase the stability radius considerably (up to 1.65 times larger than for the ideal sampler). For $\mu = 10$ the improvements are almost of the factor 4.5 (at $h = 0.35$).
Our goal in this example is to demonstrate, that by the proper design of $S_h$ the requirements on the sampling rate can be relaxed. To see this, let us compare sampling periods $h$ that guarantee equivalent stability radii for various $\mu$. The table below shows the comparison results for $\alpha$ equal to 1, 0.43, and 0.17.

One can see, that by processing one additional intersample measurement during the A/D conversion (i.e., $\mu = 2$) the required sampling rate can be relaxed by $60 \div 80\%$. When $\mu = 10$, the discrete-time part of the controller and the hold can work at up to 6 times slower rate than the sampling rate required in the case of the standard ideal sampler. Moreover, as the constraint division $\mu$ becomes larger, the requirements on the antialiasing filter can be relaxed. This, in turn, can lead to a further relaxation of the “fast sampling” requirement.

Note, that for this example the optimal design of the hold function has almost no affect on the robust stability radius.

### 7 Conclusions

In this paper the $H^2$ and the $H^\infty$ sampled-data control problems have been treated assuming that not only the digital controller but also the sampler and the hold are design parameters. Taking into consideration implementation requirements, the sampler and hold have been treated subject to waveform constraints. In particular, the hold has been assumed to belong to the class of piecewise-constant hold functions with a given number $\nu$ of intersample corrections of the control signal. The sampler has been assumed to average a given number $\mu$ of weighted measurements, equally spread within the intersample (thus, piecewise-impulse waveform of

<table>
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<th>$\alpha$</th>
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the sampling function). For the $H^\infty$ control problem, necessary and sufficient conditions of the existence of a $\gamma$-suboptimal sampled-data controller have been obtained and explicit formulae for the suboptimal sampler, hold, and the discrete-time part of the controller have been derived. The $H^2$ optimal solution can be obtained from the $H^\infty$ suboptimal one by the simple substitution $\gamma^{-1} = 0$. It is believed that these results will be helpful in many applications where the available sampling rate is not sufficiently fast.

Note, that the formula for the discrete-time part of the controller is presented only for the case where $\nu = \kappa \mu$ for a natural $\kappa$ (the existence conditions as well as the formulae for the sampler and hold are valid for arbitrary $\nu$ and $\mu$). The reason is that the formula for the general case turns out to be quite complicated. It is believed, however, that a simpler expression exists and its derivation is currently investigated.

References


