Computation of the Frequency-Response Gain of Sampled-Data Systems via Projection in the Lifted Domain
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Abstract—This paper addresses the computation of the frequency response gain of sampled-data systems with their intersample behavior taken into account. The proposed method is based upon a novel necessary and sufficient condition for the frequency-response gain to be less than a given $\gamma > 0$ and thus can be used as a basis for a bisection algorithm. Unlike all currently existing results, the proposed procedure requires the operator $\gamma^2 I - \mathcal{D}_{11} \mathcal{D}_{11}^*$ neither to be positive nor even to be invertible, thus eliminating unnecessary restrictions of the existing approaches (here $\mathcal{D}_{11}$ stands for the feedthrough term of the lifted system).

Index Terms—Sampled-data systems, frequency response, lifting technique.

I. INTRODUCTION AND PROBLEM STATEMENT

This paper is devoted to the computation of the frequency response gain of sampled-data (SD) systems, i.e., systems which consist of a continuous-time “plant” and a discrete-time “controller” connected by A/D (sampler) and D/A (hold) devises. Fig. 1(a) shows a general sampled-data feedback setup, where the generalized plant $\mathcal{P}$ and the controller $\mathcal{K}$ are time invariant with the transfer matrices

$$P(z) = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \hat{K}(z) = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}, \quad (1)$$

where $S_h$ is the ideal sampler and $H_h$ is the zero-order hold. Note that the matrices $D_{21}$ and $D_{22}$ above are both taken zero to guarantee the boundedness of the sampling operation, see [1], and $D_{11}$ is taken zero to simplify the exposition (otherwise, the standard loop-shifting in the continuous time can be applied, so that $D_{11}$ can be taken zero without loss of generality).

It is well known [1] that from the continuous-time behavior point of view the system in Fig. 1(a) is periodically time varying. This fact, together with the aliasing phenomenon due to the sampling operation, complicates the frequency-domain analysis of sampled-data systems. In particular, there appear to be no comprehensive generalization of the phase concept to such systems. On the other hand, the notion of the gain can be generalized to the system in Fig. 1(a) in several apparently equivalent ways, see [2], [3] and the references therein. Below, the notion of the frequency response gain of sampled-data systems introduced in the second reference above is briefly reviewed.

A. Frequency-response gain (FRG) of SD systems

The definition of Yamamoto and Khargonekar [3] is based on the fact that the hybrid and periodically time-varying sampled-data system in Fig. 1(a) can be equivalently described in the so-called lifted domain [1], where it becomes pure discrete-time and time invariant albeit with infinite-dimensional input and output spaces. More specifically the system in Fig. 1(a) can be equivalently presented in the form depicted in Fig. 1(b), where $\tilde{w}$ and $\tilde{z}$ (both are $\mathcal{L}^2[0,h]$-valued sequences) are the lifted versions of $w$ and $z$, respectively, and $\mathcal{P}$ is the lifting of $\mathcal{P}$ together with $S_h$ and $H_h$. Now, the lifted generalized plant $\hat{P}$ is time invariant with the transfer function

$$\hat{P}(z) = \begin{bmatrix} A_{\hat{P}} & B_{\hat{P}} \\ C_1 & D_{12} \end{bmatrix}. \quad (2)$$

The expressions for the parameters of $\hat{P}(z)$ are postponed to Section III. Just note that the following operator notation is adopted hereafter: a bar indicates an operator $\bar{\mathcal{O}}$ with both input and output spaces finite-dimensional; grave accent — $\hat{\mathcal{O}}$, when the input space is finite dimensional and the output is infinite dimensional (i.e., $\mathcal{L}^2[0,h]$); acute accent — $\check{\mathcal{O}}$, when the input space is infinite dimensional and the output is finite dimensional; and finally breve — $\check{\mathcal{O}}$, when both input and output spaces are infinite dimensional. Note also that whereas the ranks of the (infinite-dimensional) operators $\hat{B}_1$, $\hat{C}_1$, and $D_{12}$ are finite, the rank of $D_{11}$ is not.

Let $\check{G} \hat{=} \mathcal{F}_{\hat{I}}(\hat{P}, \hat{K})$ be the closed-loop operator from $\tilde{w}$ to $\tilde{z}$ for the lifted system in Fig. 1(b). The operator $\check{G}$ is time-invariant and its transfer function $\check{G}(z)$ is given by

$$\check{G}(z) = \begin{bmatrix} A + \hat{B}_2 D_{12} C_2 & \hat{B}_2 C_K \\ C_1 + D_{12} D_{12} C_2 & D_{12} D_{12} C_K \end{bmatrix} \begin{bmatrix} 0 \\ \hat{B}_1 \end{bmatrix}. \quad (3)$$

The frequency response of the SD system in Fig. 1(a) at a frequency $\omega \in [0, 2\pi/h]$ can then be defined as the (infinite-dimensional) operator $\check{G}(e^{j\omega h}) : \mathcal{L}^2[0,h] \mapsto \mathcal{L}^2[0,h]$. As argued in [3], the $\mathcal{L}^2[0,h]$-induced norm of $\check{G}(e^{j\omega h})$, $\|\check{G}(e^{j\omega h})\|$, is indeed the natural choice for the frequency-response gain (FRG) of the sampled-data system. Such a definition agrees well with the interpretation of the FRG in terms of the steady-state response to a sinusoidal input at $w$ and also takes into account the aliasing (frequency folding) phenomenon. Moreover, when $\check{G}$ is stable its $H^\infty$ norm ($\mathcal{L}^2$ induced norm in time domain) is equal to $\sup_{\omega \in [0,\omega_1]}\|\check{G}(e^{j\omega h})\|.

B. Computing $\|\check{G}(e^{j\omega h})\|$

Since the definition of the FRG of SD systems involves an infinite-rank operator, $\|\check{G}(e^{j\omega h})\|$ cannot be computed directly. Instead, an iterative procedure based on verifying whether

$$\|\check{G}(e^{j\omega h})\| < \gamma \quad (4)$$

or not for a given $\gamma > 0$ can be used. The verification of (4), however, appears to be nontrivial too. The main reason lies in the fact that the achievable $\gamma$ can in principle be smaller than $\|\hat{D}_{11}\|$. This, in turn, does not allow one to use the now-standard loop-shifting procedure of [4] to reduce the problem to an equivalent one involving only finite-rank operators.

In [3] a closed-form expression for the SD FRG which can be used for the computational purposes is derived. This, however, requires an infinite-rank condition to be checked at each iteration for every frequency, that makes the computational procedure extremely time consuming. Moreover, the procedure explicitly inverts the operator

$$\gamma^2 I - \hat{D}_{11} \hat{D}_{11}^*, \quad (5)$$

which is numerically unreliable when this operator is ill-conditioned. These difficulties motivated the publication of several approximation-based procedures, like the fast sampling approximation of [5] or frequency-sampling approximation of [2]. The approximations, however, might require manipulations of very high dimensional matrices and, consequently, might also be time consuming and computationally unreliable.

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Recently, a new procedure was proposed in [6]. The procedure is based on the computation of several singular values of \( \dot{D}_{11} \) (which is frequency independent) with the subsequent counting up the number of positive and negative eigenvalues of finite-dimensional matrices to verify (4). The approach of [6], however, still requires the inversion of (5). This might be unreliable numerically, since during the bisection search (5) almost certainly becomes ill-conditioned.

C. Contributions of this paper

In this paper a new approximation-free approach for the computation of the FRG of SD systems is proposed. Instead of inverting (5), the inversion of the operator

\[
\gamma^2 I - \dot{D}_{11}(I - \dot{B}_1^* (\dot{B}_1 \dot{B}_1^*)^{-1} \dot{B}_1) \dot{D}_{11}^{-1}
\]

is used. The clear advantage is that this operator turns out to be positive definite for all \( \gamma \) satisfying (4). In other words, the quantity

\[
\gamma_L \doteq \| \dot{D}_{11}(I - \dot{B}_1^* (\dot{B}_1 \dot{B}_1^*)^{-1} \dot{B}_1) \|
\]

is a (frequency-independent) lower bound for the FRG of (3). Note that (6), which involves both finite- and infinite-rank operators, looks considerably more complicated than (5). This, probably, was the main obstacle preventing so far the use of the former for the computation of the FRG. The main technical contribution of this paper is in showing that the operator (6) can actually be handled as easy as (5) using the technique proposed in [7]. We show also that the computation of (7) is of the same complexity as the computation of \( \| \dot{D}_{11} \| \) (the former just uses a different sub-block of the same matrix exponential). It is further shown that given any \( \gamma > \gamma_L \), inequality (4) holds iff the spectral norm of a finite-dimensional matrix, say \( \dot{G}_\omega(\omega^{j\omega}) \), is smaller than 1. This makes the proposed scheme well-suited for the bisection algorithm. Note also that the matrix \( \dot{G}_\omega(\omega^{j\omega}) \) is computed using the same matrix exponential as in the case of the computation of the \( H^\infty \) norm of SD systems, see [1], [7].

Comparing with [6], the proposed approach has several advantages. First, it is not based on the inversion of potentially ill-conditioned operators and thus is more reliable numerically. Second, the “infinite-dimensional” part, i.e., the computation of \( \gamma_L \), of the proposed procedure is simpler than the corresponding part in [6]: the latter requires the computation of several (depending on the plant dimension) singular values of an infinite-rank operator, while the former requires the computation of the maximal singular value only. Third, the proposed approach leads also to a simpler and numerically more stable bisection procedure: in each iteration the matrix norm (i.e., the maximal singular value) has to be calculated, whereas in [6] the full eigenvalue problem is required to be solved.

II. Solution

We start with the following assumption:

(A1): The pair \( (A, B_1) \) is controllable.

This assumption is made to guarantee that the matrix \( \dot{B}_1 \dot{B}_1^* \) is positive definite and can easily be omitted by extracting the uncontrollable subspace of \( (A, B_1) \), which does not contribute to the FRG of \( \dot{G} \) in (3).

A. The idea

Let us rewrite (3) in the form

\[
\dot{G}(z) = \dot{D}_{11} + \dot{C}_G \Phi(z) \dot{B}_1,
\]

where \( \dot{C}_G \doteq [\dot{C}_1 \ \dot{D}_{12}] \) and

\[
\Phi(z) \doteq \begin{bmatrix}
\dot{A} + \dot{B}_2 \dot{D}_{12} C_2 & \dot{B}_2 C_K \\
\dot{B}_K C_2 & \dot{A}_K & I \\
I & 0 & 0 \\
\dot{D}_{12} C_2 & 0 & 0
\end{bmatrix}.
\]

Then the standard completing to square arguments yield:

\[
\dot{G} \dot{G}^\sim = \dot{D}_{11} \dot{B}_1 \dot{D}_{11} + \dot{G}_\omega \dot{G}_\omega^*.
\]

where \( \dot{B}_1 \doteq I - \dot{B}_1^* (\dot{B}_1 \dot{B}_1^*)^{-1} \dot{B}_1 \) is the orthogonal projection operator onto the null space of \( \dot{B}_1 \),

\[
\dot{G}_\omega(z) \doteq \dot{D}_{11} \dot{B}_1 \dot{B}_1^* + \dot{C}_G \Phi(z) \dot{B}_1
\]

and \( \dot{B}_1 = (\dot{B}_1 \dot{B}_1^*)^{1/2} \). Hence, for each frequency \( \dot{G} \dot{G}^\sim(\omega^{j\omega}) < \gamma^2 I \) iff the following two conditions hold:

\[
\begin{align}
(\text{a}) & \quad \dot{D}_{11} \dot{B}_1 \dot{D}_{11} < \gamma^2 I, \\
(\text{b}) & \quad \dot{G}_\omega(\omega^{j\omega}) < \gamma^2 I - \dot{D}_{11} \dot{B}_1 \dot{D}_{11}.
\end{align}
\]

Note that condition (a) is frequency independent and equivalent to the positiveness of (6). Hence, \( \gamma_L \) defined by (7) is indeed a lower bound for the FRG of the SD system in Fig. 1(a) for every frequency. Then for every \( \gamma > \gamma_L \) condition (b) is equivalent to \( \| \dot{G}_\omega(\omega^{j\omega}) \| < 1 \), where

\[
\dot{G}_\beta(z) \doteq (\gamma^2 I - \dot{D}_{11} \dot{B}_1 \dot{D}_{11})^{-1/2} \dot{G}_\omega(z)
\]

The important point here is that \( \dot{G}_\beta \) is a finite-rank operator. Hence, the computation of \( \| \dot{G}_\beta(\omega^{j\omega}) \| \) can be reduced to the matrix norm computation in a straightforward manner. More specifically, let \( \dot{D}_r \) and \( \dot{C}_r \) be any matrices satisfying

\[
\begin{bmatrix}
\dot{D}_r \\
\dot{C}_r
\end{bmatrix} \begin{bmatrix}
\dot{D}_\gamma & \dot{C}_\gamma
\end{bmatrix} = M_\gamma = \begin{bmatrix}
M_{r1} & M_{r2} \\
M_{r3} & M_{r4}
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\dot{D}_r \\
\dot{C}_r
\end{bmatrix} \begin{bmatrix}
\dot{B}_1 \dot{D}_{11} \\
\dot{C}_G
\end{bmatrix} = \dot{D}_{11} \dot{B}_1 \dot{D}_{11}^{-1} \dot{C}_G
\]

(8)

Then the following result can be formulated:

**Theorem 1:** Let (A1) hold. Then for every \( \gamma > \gamma_L \)

\[
\| \dot{G}(\omega^{j\omega}) \| < \gamma \iff \| \dot{G}_\gamma(\omega^{j\omega}) \| < 1, \quad \forall \omega \in \mathbb{R},
\]

where \( \dot{G}_\gamma(z) \doteq \dot{D}_r \dot{B}_1 \dot{B}_1^* + \dot{C}_r \Phi(z) \dot{B}_1 \).

**Proof:** Follows from \( \| \dot{G}_\beta(\omega^{j\omega}) \| = \| \dot{G}_\gamma(\omega^{j\omega}) \| \) which, in turn, is the consequence of \( \dot{G}_\gamma \dot{G}_\beta(z) = \dot{G}_\gamma(\omega^{j\omega}), \forall \omega \in \mathbb{C} \).

The notable consequence of Theorem 1 is the extraction of the infinite-dimensional frequency/independent condition (a) from the overall computational procedure. Thus, having the quantity \( \gamma_L \), a bisection algorithm for verifying (4) based on finite-dimensional manipulations only can easily be organized.

Although the proof of Theorem 1 is simple, the whole procedure depends heavily on the ability to perform the nontrivial calculations (7) and (8). The main difference here from the conventional “\( H^\infty \)” discretization” lies in the presence of the projection operator \( \Pi_B \), which is rather cumbersome. It appears that the difficulties in handling this operator prevented so far the use of the arguments above in SD computations.

Nevertheless, we will demonstrate in the rest of this section that the use of the machinery developed in [7] enables one to carry out all steps required to convert (7) and (8) to finite-dimensional matrix expressions in a straightforward manner. Surprisingly, the final formulas turn out to be no more complicated than those used in the SD \( H^\infty \) control.
\begin{align*}
M_{22} &= \left[ (I - \Lambda_{22}^{-1} \Gamma_{33}) \Gamma_{23}^{-1} (\Lambda_{22} - \Gamma_{22}) - \Lambda_{32}^{-1} \Gamma_{32} \right] \\
&\quad \left[ (I - \Lambda_{22}^{-1} \Gamma_{33}) \Gamma_{23}^{-1} (\Lambda_{24} - \Gamma_{24}) - \Lambda_{34}^{-1} \Gamma_{34} \right] \\
&\quad (\Gamma_{13} - \Lambda_{13} \Lambda_{33} \Gamma_{33}) \Gamma_{23}^{-1} (\Lambda_{24} - \Gamma_{24}) + \Gamma_{14} - \Lambda_{13} \Lambda_{34}^{-1} \Gamma_{34} \right]
\end{align*}

B. Computing the lower bound $\gamma_L$

Define the matrix function

\[ \Sigma(t) = \exp \left( \begin{bmatrix} A & B_1 B_1' \\ -\frac{1}{2} C_1' C_1 & -A' \end{bmatrix} t \right) \]

and partition it compatibly with the partition of the right-hand side above. It is well known [1] that the condition $\gamma > \| D_{11} \|$ is equivalent to the non-singularity of $\Sigma(t)$ for all $t \in (0, h)$. The lemma below establishes that the more complicated condition $\gamma > \| D_{11} \|_{\text{F}}$ turns out to be equivalent to the non-singularity of just another sub-block of $\Sigma(t)$:

**Lemma 1:** Let (A1) hold true. Then $\gamma > \gamma_L$ iff $\det(\Sigma(t)) \neq 0$, $\forall t \in (0, h]$.

**Proof:** See §III-B

**Remark 1:** It is worth stressing that the computation of $\gamma_L$ is probably most involved and numerically sensitive part of the proposed algorithm. Yet the same is true for the computation of $\| D_{11} \|$, which is a part of any existing algorithm. We therefore claim that the computation of $\| D_{11} \|$ to the computation of $\| D_{11} \|_{\text{F}}$ no additional complexity is added.

**Remark 2:** Note that if (A1) is violated, then $\Sigma(t)$ is singular for all $t$. Yet the “normal” null space of $\Sigma(t)$ does not depend on $t$ and coincides with the uncontrollable sub-space of $(A, B_1)$. Therefore, Lemma 1 remains almost unchanged in the general case modulo the replacement of the whole $\Sigma(t)$ with its restriction to the controllable subspace of $(A, B_1)$.

C. Computing $C_\gamma$ and $D_\gamma$

Introduce the following Hamiltonian matrix:

\[ H_\gamma = \begin{bmatrix} 0 & D_{12}' C_1 & B_2' \\ 0 & A & B_1 B_1' & B_2' \\ 0 & -\frac{1}{4} C_1' C_1 & -A' & -\frac{1}{4} C_1' D_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

and define the (symplectic) matrix exponential

\[ \Gamma = e^{H_\gamma h} = \begin{bmatrix} I & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ 0 & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ 0 & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} \\ 0 & 0 & 0 & I \end{bmatrix}. \]

Also, the notation $\Lambda$ stands for $\lim_{t \to \infty} \Gamma$ (note that $\Lambda_{12}$, $\Lambda_{14}$, $\Lambda_{32}$, and $\Lambda_{34}$ are all zero matrices and $\Lambda_{33} = \Lambda_{22}$).

Note that according to Lemma 1 the matrix $\Gamma_{23}$ is non-singular whenever $\gamma > \gamma_L$.

**Lemma 2:** Assume that (A1) holds. Then for every $\gamma > \gamma_L$ the matrix $\Gamma_{23}$ is non-singular and

\[ M_\gamma^1 = \begin{bmatrix} (I - \Lambda_{23}^{-1} \Gamma_{33} \Gamma_{23}^{-1} (\Lambda_{22} - \Gamma_{22})) \Lambda_{23} \Lambda_{33}^{-1} \\ (I - \Lambda_{23}^{-1} \Gamma_{33} \Gamma_{23}^{-1} (\Lambda_{24} - \Gamma_{24})) \Lambda_{23} \Lambda_{34}^{-1} \end{bmatrix}, \]

and $M_{\gamma^2}$ is as given at the top of this page.

**Proof:** See §III-C.

**Remark 3:** Note that the matrix $\Gamma$ is exactly the matrix exponential required in the $H^\infty$ control of SD systems [1], [7] and in the bisection scheme for the computation of the FRG of SD systems in [6]. The proposed approach requires also the matrix $\Lambda$ to be computed which seems to complicate the calculations. We argue, however, that the need to compute the latter matrix practically does not complicate the overall procedure. First, $\Lambda$ does not depend on the parameter $\gamma$, so it needs to be computed only once during the bisection search. Second, the matrix $\Lambda$ is the matrix exponential appearing in the SD $H^2$ control [1]. Since $H^2$-like computations are required to calculate upper and lower bounds of the FRG of SD systems, see [8] and the references therein, the matrix $\Lambda$ is to be computed anyway.

D. Computing $\tilde{A}$, $\tilde{B}_2$, and $\tilde{B}_o$

To complete the construction of $G_\gamma(z)$ in Theorem 1 one now only needs to compute $A$, $B_2$, and $B_o$. To this end note that these are actually the matrices involved in the SD $H^2$ control. Thus, the formulae below are standard and can be borrowed from [1], for instance:

\[ \tilde{A} = \Lambda_{22}, \quad \tilde{B}_2 = \Lambda_{24}, \quad \tilde{B}_o = (\Lambda_{23} \Lambda_{33}^{-1})^{1/2}. \]

This completes the construction of $G_\gamma(z)$.

III. PROOFS

A. Preliminary: STPBC representation of lifted parameters

We start with a brief review of the representation of the parameters of the lifted systems based on systems with two-point boundary conditions (STPBC) introduced in [7], see also [9] for more details.

Consider the lifted plant (2). As shown in [7],

\[ \begin{bmatrix} \tilde{A} & \tilde{B}_1 & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \end{bmatrix} = \begin{bmatrix} A & B_2 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{I}_0 & B_1 & 0 \\ 0 & 0 & I \end{bmatrix}. \]

Here the compact block notation $\tilde{O} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ is used to denote an operator $\tilde{O}: L^2[0, h] \to L^2[0, h]$ described by the following equations:

\[ \tilde{O} : \begin{cases} \tilde{x}(t) = A\tilde{x}(t) + B_\mu(t), \\ y(t) = C\tilde{x}(t) + D\mu(t), \end{cases} \]

the impulse operator $\tilde{I}_0$ transforms $\eta \in \mathbb{R}^n$ into a modulated $\delta$-impulse as $\tilde{I}_0 \eta(t) = \delta(t - \theta) \eta$; and the sampling operator $\tilde{T}_\delta$ transforms $\zeta \in C_n[0, h]$ into a vector from $\mathbb{R}^n$ as $\tilde{T}_\delta \zeta = \zeta(\theta)$. In the sequel, we shall also use operators $\tilde{O}: L^2[0, h] \to L^2[0, h]$ described by equations like (9) but with the 2-point boundary conditions $\Omega x(0) + \bar{\Omega} x(h) = 0$ for some square matrices $\Omega$ and $\bar{\Omega}$. Such operators, denoted as

\[ \tilde{O} = \begin{bmatrix} A & 0 \\ C \end{bmatrix}, \]

are well-posed (more precisely, have well-posed boundary conditions) iff the matrix $\Xi_\tilde{O} = \Omega + \bar{\Omega} e^{A h}$ is non-singular and in this case $y = \tilde{O}u$ implies that $y = Du + y_\mu + y_\alpha$, where

\[ y_\mu(t) = C e^{A t} \Xi_\tilde{O}^{-1} \Omega \int_0^t e^{-A s} B_\mu(s) ds, \]

\[ y_\alpha(t) = -C e^{A t} \Xi_\tilde{O}^{-1} \bar{\Omega} \int_t^h e^{A(t-s)} B_\mu(s) ds. \]

The advantage of the STPBC representation over the conventional one based on the integral operator description stems from the fact that the manipulations over STPBC can be performed in the state space,
much like the manipulations over standard finite-dimensional state-space systems, see [10]. Moreover, as shown in [7], the sampling and impulse operators fit well into the STPBC formalism. The reader is referred to the latter paper as well as to [11] for further details. We present here only the following result which will be used in §III-C:

**Proposition 1**: Let $A, B_1, B_2, C_1$, and $C_2$ be appropriately dimensioned matrices so that $C_0 B_0 = 0$ and $C_3 B_3 = 0$, then

\[
\begin{bmatrix}
T_0 C_0 \\
T_0 C_3
\end{bmatrix} \begin{bmatrix}
A & 0 \\
0 & A
\end{bmatrix} \begin{bmatrix}
T_1 \Omega \\
T_0 \Omega
\end{bmatrix} = \begin{bmatrix}
C_0 e^{A h} \\
C_3 \end{bmatrix} (\Omega + T e^{A h})^{-1} \begin{bmatrix}
-\Omega B_0 \\
0
\end{bmatrix}.
\]

**B. Proof of Lemma 1**

Introduce operators

\[
\begin{bmatrix}
\tilde{B}_r \\
\tilde{D}_r
\end{bmatrix} = \begin{bmatrix}
T_0 r \\
C_1
\end{bmatrix} \begin{bmatrix}
A & B_1 \\
I & 0
\end{bmatrix} : L^2[0, \tau] \mapsto \mathbb{R} \oplus L^2[0, \tau]
\]
and $\tilde{\Pi}_r \doteq I - \tilde{B}_r^* (\tilde{B}_r \tilde{B}_r)^{-1} \tilde{B}_r$. Obviously, $\tilde{D}_{11} = \tilde{B}_h$, $\tilde{B}_1 = \tilde{B}_h$, and $\tilde{B}_2 = \tilde{B}_h$. The following technical result plays a key role in the sequel:

**Proposition 2**: Define the $L^2[0, \tau] \mapsto L^2[0, \tau]$ operator

\[
\tilde{O}_r = \begin{bmatrix}
0 & B_1 B_1' \\
0 & -A'
\end{bmatrix}
\]

(\text{below $\Phi$ stands for $\Phi(\tau)$). STPBC $\tilde{O}_r$ is well-posed iff the matrix

\[
\Xi_{\tilde{O}_r} = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & I
\end{bmatrix} \Phi = \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix} \Phi_{11} \Phi_{12}
\]

is nonsingular, which is equivalent to the non-singularity of $\Phi_{12}$. On the other hand, it can be shown [1] that $\Phi_{12} = \int_0^\tau e^{A s} B_1 B_1' e^{A s} d s e^{-A' r}$, from which the equivalence between (A1) and the well-posedness of $O_r$ follows immediately.

Now, since

\[
\tilde{D}_r \tilde{D}_r^* = \begin{bmatrix}
A & B_1 B_1' \\
0 & -A'
\end{bmatrix}
\]

the causal and anti-causal parts of its response to $u$ are, by (10),

\[
y_{1c}(t) = -C_1 \Phi_{11}(t) \int_0^t \Phi_{12}(-s) C_1' u(s) d s,
\]
and hence

\[
\tilde{D}_r (I - \tilde{\Pi}_r) \tilde{D}_r^* = \tilde{D}_r \tilde{B}_r \cdot \Phi_{22} \tilde{B}_r' \cdot \tilde{B}_r \tilde{D}_r^*.
\]

Thus, the response of $\tilde{D}_r (I - \tilde{\Pi}_r) \tilde{D}_r^*$ to an input $u$ is

\[
y_{2c}(t) = -C_1 \Phi_{11}(t) \int_0^t \Phi_{12}(-s) C_1' u(s) d s
\]

Therefore, for every $u \in \tilde{\Pi}_r L^2[0, \tau]$ (i.e., $\tilde{B}_r u_r = 0$). For a given $\delta > 0$ define then $u_{\tau, \delta} \in L^2[0, \tau + \delta]$ as follows:

\[
u_{\tau, \delta}(t) = \begin{cases}
u_{\tau}(t) & \text{if } t \in [0, \tau], \\
0 & \text{otherwise}.
\end{cases}
\]

It can be easily shown that $\tilde{B}_{r+\delta} u_{\tau, \delta} = e^{A\delta} \tilde{B}_r u_r = 0$, which means that $u_{\tau, \delta} \in \tilde{\Pi}_{\tau+\delta} L^2[0, \tau + \delta]$. Therefore, for every $u \in \tilde{\Pi}_r L^2[0, \tau]$.

\[
||\tilde{D}_r \tilde{B}_r u_r|| = ||\tilde{D}_r u_r|| \leq ||\tilde{D}_r u_{\tau+\delta}|| = ||\tilde{D}_r \tilde{B}_r u_{\tau+\delta}||.
\]

On the other hand, since

\[
\sup_{u \in L^2[0, \tau]} ||\tilde{D}_r u_r|| = \sup_{u \in L^2[0, \tau]} \frac{||\tilde{D}_r u_r||}{||u||} = \sup_{u \in L^2[0, \tau]} \frac{||\tilde{D}_r \tilde{B}_r u_r||}{||u||} = \sup_{u \in L^2[0, \tau]} \frac{||\tilde{D}_r u_{\tau, \delta}||}{||u||} = ||u_{\tau, \delta}|| + ||\tilde{D}_r||.
\]

Propositions 2 and 3 yield that $\gamma > \gamma_1$ iff $\gamma^2 I - \tilde{O}_r > 0$ for all $\tau \in (0, h]$. Yet the latter holds iff

\[
(I - \gamma^{-2} \tilde{O}_r)^{-1} = \begin{bmatrix}
A & B_1 B_1' \\
0 & -A'
\end{bmatrix}
\]
is well-posed for all $\tau \in (0, h]$ (since $\|O_2\|$ is continuous as a function of $\tau$) which, in turn, is equivalent to the non-singularity of

$$
\Xi(I - \gamma^{-2}O_2)^{-1} = \begin{bmatrix}
I & 0 \\
\Sigma_{11}(\tau) & \Sigma_{12}(\tau)
\end{bmatrix}.
$$

This completes the proof of Lemma 1.

**C. Proof of Lemma 2**

Having Proposition 2, the calculation of the matrix $M_\gamma$ is rather routine and follows the procedure proposed in [9] for the computation required for the "$H^\infty$" discretization. In particular, it is easily seen that

$$(I - \hat{D}_{11} \hat{\Pi}_B \hat{D}_{11}^T)^{-1} = \begin{pmatrix}
A & B_1B_1^T \\
C_1^T & -A^T
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
I & C_1^T
\end{pmatrix},$$

(here we assume that $\gamma = 1$ for simplicity). Further,

$$\hat{C}_G = \begin{pmatrix}
A & B_2 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & 0
\end{pmatrix}$$

and

$$\hat{D}_{11} \hat{B}_1^T = \begin{pmatrix}
A & B_1B_1^T \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
I & 0
\end{pmatrix}$$

from which the STPBC realization of $M_\gamma$ is obtained by straightforward state-space manipulations. Then, the formulae of Lemma 2 are obtained by application of Proposition 1. In fact, the only deviation from the procedure in [9, §4.3] is that the different boundary conditions of $(I - \hat{D}_{11} \hat{\Pi}_B \hat{D}_{11}^T)^{-1}$ and $\hat{D}_{11} \hat{B}_1^T$ do not enable to reduce the state-space realization of their product. This is the reason for the presence of both $\Gamma$ and $\Lambda$ in the final formulae.

**IV. Conclusion**

In this paper a new method for the computation of the frequency-response gain of sampled-data systems has been proposed. The method is based upon a novel necessary and sufficient condition for the frequency-response gain to be less than a given $\gamma > 0$ and thus can be used as a basis for a bisection algorithm. Unlike all currently existing results, the proposed procedure requires the operator $\gamma^2 I - \hat{D}_{11} \hat{D}_{11}^T$, neither to be positive nor even to be invertible, thus eliminating unnecessary restrictions of the existing approaches (here $\hat{D}_{11}$ stands for the feedthrough term of the lifted plant in (2)).

**References**


3Available at http://leo.technion.ac.il/publications