On the Sampled-Data $H^\infty$ Filtering Problem*

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Abstract

In this paper the $H^\infty$ filtering problem of a continuous-time process is considered under the assumption that the measurement is sampled by a generalized sampling device. Both cases of known and unknown initial condition are considered. The problem is treated via the lifting transformation, which converts the problem to a pure discrete-time setting. It is shown that if a suboptimal filter exists, it can always be presented in the form of a finite dimensional discrete-time (possibly time-varying) filter followed by a stationary generalized hold function. This extends previous results in the literature, where a suboptimal filter of this structure (with an LTI discrete-time part of the filter) was obtained for the case of known initial conditions only. Also, the existence conditions of a suboptimal filter derived in this paper are simpler than those known in the literature.

1 Introduction and problem formulation

The $H^\infty$ filtering theory has been extensively developed for the last few years, see [10, 11] and the references therein. The $H^\infty$ filtering is a minimax problem where the maximum $L^2$ norm of the estimation error is minimized over all $L^2$ disturbances and uncertain initial conditions. Such an approach is useful if statistic properties of the system disturbances are unknown or known only partially. As argued in [11], the $H^\infty$ filter may also be less sensitive to the uncertainties in the plant parameters than its $L^2$ counterpart.

Consider a continuous-time LTI process described by the following state space equations:

$$\begin{align}
\dot{x}(t) &= Ax(t) + B_c w(t), \\
y(t) &= C_2 x(t) + D_{2c} w(t),
\end{align}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^r$ is the measured output, $w(t) \in \mathbb{R}^{m_c}$ is the disturbance vector, which includes both a process disturbance and a measurement noise, and $A$, $B_c$, $C_2$, and $D_{2c}$ are matrices of appropriate dimensions. Assume that it is required to estimate by a causal filter the following linear combination of the state vector:

$$z(t) = C_1 x(t),$$

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where $C_1 \in \mathbb{R}^{n_x \times n}$ is of full row rank. Then the continuous-time $H^\infty$ filtering problem may be formulated as follows:

Given the process (1a), the measurements (1b), a matrix $\Pi = \Pi' \geq 0$, and a constant $\gamma > 0$, find a causal filter generating an estimate $\zeta(t)$ of the signal $z(t)$ in (2) such that

$$\|z - \zeta\|^2_{L^2} < \gamma^2 (x_0' \Pi^T x_0 + \|w\|^2_{L^2}).$$

for all $x_0 \in \text{Im} \Pi$ and $w \in L^2$ which do not vanish simultaneously.

State-space solutions to such a problem for the cases where $\Pi > 0$ (i.e., $x_0$ is completely unknown) and $\Pi = 0$ (i.e., $x_0$ is known) were derived in [10].

In this paper we consider the $H^\infty$ filtering problem assuming that the measurements and the information processing are performed by digital hardware. We assume that the measurement equation is of the form

$$\tilde{y}_k = \int_{0}^{h_0^-} \phi_S(\tau) y(\tau^- - \tau) \, d\tau + D_{2d} \tilde{v}_{k-1}, \quad k = 1, 2, \ldots,$$  \hspace{1cm} (3)

where $\tilde{v}_k \in \mathbb{R}^{m_d}$ is a discrete-time measurement noise, $h$ is the sampling period, and the sampling function $\phi_S$ is defined on the interval $[0, h]$ as follows:

$$\phi_S(\tau) = D_S \delta(\tau) + C_S e^{A_S^T} B_S$$

for some finite dimensional matrices $A_S, B_S, C_S,$ and $D_S$. The ideal sampler corresponds in this setup to the case where $A_S = B_S = C_S = 0$ and $D_S = I$, while the averaging sampler [12] — to $A_S = D_S = 0$, $B_S = \frac{1}{h} I$, and $C_S = I$. Since the ideal sampling is an unbounded operator as a mapping $L^2 \mapsto l^2$ [3], we make the following assumption

$$D_S D_{2c} = 0,$$

which states that the sampler operates over “proper” signals. By “proper” it is meant here that pre-filtering by an anti-aliasing filter is provided if necessary. Hence, the equality above guarantees boundedness of the sampling operations. Note also, that for technical reasons we assume that there is no measurements available at $k = 0$. The results, however, can be modified to include the general case.

The sampled-data $H^\infty$ filtering problem to be considered in this paper is as follows:

FP$_{H^\infty}$: Given the process (1), the measurement (3), a matrix $\Pi = \Pi' \geq 0$, and a constant $\gamma > 0$, find a causal filter generating an estimate $\zeta(t)$ of the signal $z(t)$ in (2) such that

$$\|z - \zeta\|^2_{L^2} < \gamma^2 (x_0' \Pi^T x_0 + \|w\|^2_{L^2} + \|\tilde{v}\|^2_{l^2}).$$

for all $x_0 \in \text{Im} \Pi$, $w \in L^2$, and $\tilde{v} \in l^2$ which do not vanish simultaneously.

The causality of the filter is understood here as the dependence of the estimate $\zeta(t)$ only on the measurements $\tilde{y}_i$, $0 < i \leq k$, where $k$ is the largest integer such that $t \geq kh$. Also note that since the measurement $\tilde{y}_i$ depends on $y(t)$ on the interval $[kh - h, kh)$ only, the causality of the filter implies actually that the state estimation is performed by a strictly causal system (including the sampler (3)).

The sampled-data $H^\infty$ filtering problem for the ideal sampler was recently considered by Sun, Nagpal, and Khargonekar [13], where necessary and sufficient conditions for the existence of a
\(\gamma\)-suboptimal sampled-data filter were given in terms of the stabilizing solution to a differential Riccati equation with jumps at \(t = kh\). The suboptimal filter presented in [13] is also of the form of a continuous-time filter with jumps. In addition, it was shown in [13] that if the initial state \(x_0\) is known, then there exists a \(\gamma\)-suboptimal sampled-data filter, which has the structure of a discrete-time LTI filter followed by a generalized hold function. Also, in this case the conditions for the existence of a solution to the FP\(_{FP}\) is stated in terms of the existence of the stabilizing solution \(Q^0\) to a discrete-time algebraic Riccati equation and the non-singularity of the matrix \(M_{SNK}(t) = \Gamma_{11}(t) + \Gamma_{12}(t)Q^0, \forall t \in [0, h]\), where the matrices \(\Gamma_{11}(t)\) and \(\Gamma_{12}(t)\) are sub-blocks of a matrix exponential of the form \(e^{Mt}\).

The purpose of this paper is to show that even in the more general case where the measurement is given by equation (3) and the initial state is unknown, a solution to FP\(_{FP}\) exists iff there exists a solution in the form of a discrete-time filter followed by a stationary generalized hold function. Moreover, we will show that the necessary and sufficient conditions for the existence of a solution to FP\(_{FP}\) can be given in terms of a discrete-time finite-dimensional Riccati recursion. It will also be shown, that the non-singularity of the matrix \(M_{SNK}(t)\) for all \(t \in [0, h]\) is equivalent to the non-singularity of \(\Gamma_{11}(t), t \in [0, h]\) and to the condition \(\rho(\Gamma_{11}(h)^{-1}\Gamma_{12}(h)Q^0) < \gamma^2\). Furthermore, we consider the problem FP\(_{FP}\) under milder assumption than in [13]: we neither require \(D_d\) to be right invertible nor \(\Pi\) to be nonsingular. Finally, following [10] we will derive sufficient conditions for the existence of a stationary solution to FP\(_{FP}\), that is, a solution of the form of a discrete-time LTI filter followed by a stationary generalized hold function.

The paper is organized as follows. In Section 2 it is shown how the hybrid problem FP\(_{FP}\) can be reduced to an equivalent pure discrete-time H\(^\infty\) filtering problem, FP\(_{eq}\), with an LTI process by means of the lifting transformation. We show that the lifted problem always falls into the category of the so-called a-priori filtering problems and present solutions (both time varying and time invariant) to the FP\(_{eq}\). The latter solutions, however, are not readily implementable and have to be converted back ("peeled-off") into continuous time. That conversion is the subject matter of the rest of the paper. Section 3 contains the main results of the paper, i.e., solutions (both time-varying and stationary) to the FP\(_{FP}\), while in Section 4 the equivalence between the solutions to the FP\(_{eq}\) and the FP\(_{FP}\) is proven. Finally, in Section 5 some concluding remarks are given.

### 1.1 Notation

The notation throughout the paper is fairly standard. For a real valued matrix \(M\), \(M'\) denotes its transpose and \(\rho(M)\) denotes the spectral radius (if \(M\) is square). The notation \(M^{1/2}\) is adopted for the square root of a matrix \(M = M' \geq 0\), whereas \(M'\) — its pseudo-inverse.

Discrete-time signals in the time domain are highlighted by a bar (like \(\bar{z}\)), while in the lifted domain — by a breve (like \(\breve{z}\)). Finally, since we will extensively use operator compositions involving operators over infinite dimensional spaces (such as \(L^2[0, T] \oplus \mathbb{R}\)), the following notation is aimed to simplify the readability of the formulae: bar above a variable denotes an operator \(\bar{O}\) both from and to finite dimensional spaces; grave accent — \(\grave{O}\), from a finite dimensional space to an infinite dimensional one; acute accent — \(\acute{O}\), from an infinite dimensional space to a finite dimensional one; and breve — \(\breve{O}\), both from and to infinite dimensional spaces.

### 2 Lifted domain solution

The treatment of the FP\(_{FP}\) is complicated owing to its hybrid continuous/discrete nature and inherent periodicity. To circumvent these difficulties the so-called lifting technique of [16, 2, 1]...
and the equivalent lifted signal to be estimated.

The notion of lifting is based on a conversion of real valued signals in continuous time into functional space valued sequences, that is sequences that take values not on $\mathbb{R}$ but rather on some general Banach space ($L^2[0, h]$ in this paper). Formally, let $\ell_{L^2[0, h]}$ be the space of sequences, each element of which is a function from $L^2[0, h]$, that is

$$\ell_{L^2[0, h]} = \left\{ \tilde{\xi} : \tilde{\xi}[k] \in L^2[0, h], \forall k \in \mathbb{Z}_+ \right\}.$$

Then given any $h > 0$, the lifting operator $\mathcal{W}_h : L^2 \rightarrow \ell_{L^2[0, h]}$ is defined [2, 1] such that:

$$\tilde{\xi} = \mathcal{W}_h \xi \iff (\tilde{\xi}[k])(\tau) = \xi(kh + \tau) \quad \forall \tau \in [0, h].$$

It is easy to see that the lifting operator is a linear bijection between $L^2$ and $\ell_{L^2[0, h]}$. Moreover, if we restrict the domain of $\mathcal{W}_h$ to the Hilbert space $L^2$, then by an appropriate choice of a norm on $\ell_{L^2[0, h]}$ the lifting operator can be made an isometry. Hence, treating a system $\zeta = \mathcal{G}_w$ not as a mapping from $\omega$ to $\zeta$ but rather as a mapping from $\bar{\omega}$ to $\tilde{\zeta}$ gives essentially the same system (as an input-output mapping). Indeed, lifting preserves system stability and system induced norms. This allows one to conclude that $\mathcal{G}$ and

$$\tilde{\mathcal{G}} = \mathcal{W}_h \mathcal{G} \mathcal{W}_h^{-1} : \ell_{L^2[0, h]} \rightarrow \ell_{L^2[0, h]},$$

which is called lifting of $\mathcal{G}$, are equivalent. The advantage of treating systems in the lifting domain stems from the fact that $\tilde{\mathcal{G}}$ is time-invariant in the discrete time even if $\mathcal{G}$ is $h$-periodic in the continuous time. Hence, any periodic problem in continuous time can be reduced to a time-invariant one in discrete time.

The application of this idea to FP$_{H\infty}$ is straightforward. In order to convert this problem to a pure discrete time-invariant one we need to lift the signals $\bar{w}$, $z$, and $\bar{y}$ and consider process (1) and equation (2) as mappings from $\bar{w}$ to $\tilde{y}$ and $\tilde{z}$. This yields the equivalent lifted process

$$\dot{x}_{k+1} = A\bar{x}_k + B_c\bar{w}_k, \quad \bar{x}_0 = x_0, \quad (1a')$$

$$\bar{y}_k = C_2\bar{x}_k + D_2e\bar{w}_k \quad (1b')$$

and the equivalent lifted signal to be estimated

$$\tilde{z}_k = \tilde{C}_1\tilde{x}_k + \tilde{D}_1e\tilde{w}_k. \quad (2')$$

Notice that although $\tilde{A}$ is a square matrix with the same dimensions as $A$, the remaining parameters in this state-space representation are operators acting from or/and to infinite dimensional spaces. The measurement equation (3) can then be written as

$$\tilde{y}_k = \tilde{\Phi}_s\tilde{y}_{k-1} + D_{2d}\tilde{v}_{k-1}, \quad k = 1, 2, \ldots \quad (3')$$

The expressions for the parameters of the lifted process $(1')$, $(2')$ as well as the lifted sampler $\tilde{\Phi}_s$ can be derived using standard lifting arguments [1, 7]. They, however, are not essential for the discussion in this section and hence are postponed to the proofs (see §4.1). What is important in the discussion to follow are the following two facts. First, in the lifted domain the system to be considered is pure discrete time-invariant. Among other advantages this fact implies that we no longer need to distinguish between $\bar{w}$ and $\bar{v}$ and can deal with the single disturbance

$$\bar{\omega} = \begin{bmatrix} \bar{w} \\ \bar{v} \end{bmatrix} \in \ell^2_{L^2[0, h) \oplus \mathbb{R}}.$$
Second, the lifted measurement equation \(3'\) inherently contains a delay\(^1\). Consequently, the causality of the lifted filter as a mapping \(\bar{y} \rightarrow \bar{\zeta}\) is equivalent to its strict causality as a mapping \(\bar{\xi} \rightarrow \bar{\zeta}\), where

\[
\bar{\xi}_{k-1} = \bar{y}_k, \quad \forall k = 1, 2, \ldots
\]

Thus, \(\text{FP}_{H\infty}\) can always be reformulated in the lifted domain as the so-called a-priory filtering problem [11] that simplifies the treatment considerably. To this end, let us rewrite equations \((1')–(3')\) as follows:

\[
\begin{align*}
\bar{x}_{k+1} &= \bar{A}\bar{x}_k + \bar{B}\bar{\omega}_k, \\
\bar{\xi}_k &= \bar{C}_2\bar{x}_k + \bar{D}_2\bar{\omega}_k, \\
\bar{z}_k &= \bar{C}_1\bar{x}_k + \bar{D}_1\bar{\omega}_k,
\end{align*}
\]

where

\[
\begin{align*}
\bar{B} &\doteq [ \bar{B}_c \ 0 ], \\
\bar{C}_2 &\doteq \Phi_S\bar{C}_2, \\
\bar{D}_1 &\doteq [ \bar{D}_1c \ 0 ],
\end{align*}
\]

and

\[
\bar{D}_2 \doteq [ \bar{D}_{2c} \ D_d ],
\]

where \(\bar{D}_{2c} = \Phi_S\bar{D}_{2c}\). Let us impose the following assumptions on process \((4)\):

\[\text{(A1): The pair } (\bar{C}_2, \bar{A}) \text{ is detectable;}\]

\[\text{(A2): The operator } \begin{bmatrix} \bar{A} - e^{j\theta} & \bar{B} \\ \bar{C}_2 & \bar{D}_2 \end{bmatrix} \text{ is right invertible } \forall \theta \in [0, 2\pi).\]

Then the \(H\infty\) filtering problem can be reformulated in the lifted domain as follows:

\[\text{FP}_{eq}: \text{Given process } \((4a)\) \text{ and measurements } \((4b)\) \text{ such that } \text{(A1)} \text{ and } \text{(A2)} \text{ hold, a matrix } \Pi = \Pi' \geq 0, \text{ and a constant } \gamma > 0, \text{ find a strictly causal filter generating an estimate } \bar{\zeta} \text{ of the signal } \bar{z} \text{ in } \((4c)\) \text{ such that}

\[
\|\bar{z} - \bar{\zeta}\|^2 < \gamma^2(x_0'\Pi^tx_0 + \|\bar{\omega}\|^2)
\]

for all \(x_0 \in \text{Im } \Pi\) and \(\bar{\omega} \in \ell^2_{L^2[0,h] \oplus \mathbb{R}}\) which do not vanish simultaneously.

\[\text{Remark 2.1: The problem } \text{FP}_{eq} \text{ is in principle a standard discrete-time } H\infty \text{ a-priori filtering problem (see, e.g., [11, 5]). It differs from the problems considered in the literature basically in that the signal to be estimated and the disturbance operate not over } \mathbb{R}, \text{ but rather over } \text{infinite dimensional spaces } \ell^2_{L^2[0,h]} \text{ and } \ell^2_{L^2[0,h] \oplus \mathbb{R}}, \text{ respectively. The two spaces above, however, are Hilbert spaces and the dimension of the state vector of the process } \((4a)\) \text{ is finite. Hence, the solution to } \text{FP}_{eq} \text{ can be obtained using precisely the same reasoning as in the finite dimensional case.}\]

\(^1\)This fact does not mean that a computational delay is included in the measurement equation. Since \(\bar{y}_{k-1}\) contains all \(y(t)\) on the interval \([-((k-1)h), kh)\), \(\bar{y}_{k-1}\) depends on \(y(t)\) up to \(t = (kh)^-\).
Remark 2.2. Note also, that the assumptions imposed on the problem (4) are milder than those in the literature. We neither require \( D_2 \) to be right invertible nor \( \Pi \) to be nonsingular. The latter actually means that we allow initial conditions to be partly known. This is in contrast to the previous treatments, where initial conditions were assumed to be either completely unknown \([10, 13, 5]\) (\( \Pi > 0 \)) or completely known \([10, 13]\) (\( \Pi = 0 \)). Consequently, our treatment is unified and includes both of the cases above just as special cases.

To solve \( \text{FP}_{eq} \) the following discrete Riccati recursion is required:

\[
P_{k+1} = \bar{A}P_k\bar{A}' + \bar{B}\bar{B}' \left( \bar{A}P_k\bar{C} + \bar{B}\bar{D}' \right) \bar{H}(P_k)^{-1} \left( \bar{A}P_k\bar{C} + \bar{B}\bar{D}' \right)', \quad P_0 = \Pi,
\]

where

\[
\bar{H}(P) \doteq \bar{D}\bar{D}' - \gamma^2 \left[ \begin{array}{c} 0 \\ \bar{D} \end{array} \right] + \bar{C}\bar{P}\bar{C}'
\]

and

\[
\left[ \begin{array}{cc} \bar{C} & \bar{D} \end{array} \right] \doteq \left[ \begin{array}{cc} \bar{C}_1 & \bar{D}_1 \\ \bar{C}_2 & \bar{D}_2 \end{array} \right].
\]

A solution \( P_k = P_k', \ k \in \mathbb{Z}_+ \), to (5) is said to be \textit{stabilizing} if it is bounded, \( \det(\bar{H}(P_k)) \neq 0 \), and the homogeneous system

\[
\chi_{k+1} = (\bar{A} - (\bar{A}P_k\bar{C} + \bar{B}\bar{D}') \bar{H}(P_k)^{-1} \bar{C}) \chi_k
\]

is asymptotically stable.

The solution to \( \text{FP}_{eq} \) is then formulated as follows:

**Theorem 1.** Let the assumptions \((A1)\) and \((A2)\) hold, then the following two statements are equivalent:

i) There exists a filter which solves the \( \text{FP}_{eq} \).

ii) The Riccati recursion (5) has a stabilizing solution \( P_k \) such that \( \bar{D}_1\bar{D}_1' + \bar{C}_1P_k\bar{C}_1' < \gamma^2I \) \( \forall k \in \mathbb{Z}_+ \).

Given that ii) holds, then then one possible \( \gamma \)-suboptimal filter is

\[
\bar{\chi}_k = \bar{C}\bar{\eta}_k, \quad \bar{\eta}_{k+1} = \bar{A}\bar{\eta}_k - (\bar{A}P_k\bar{C} + \bar{B}\bar{D}') \bar{H}(P_k)^{-1} \left[ \begin{array}{c} 0 \\ \bar{D} \end{array} \right] \bar{\eta}_k - \bar{C}_2\bar{\eta}_k, \quad \bar{\eta}_0 = 0.
\]

**Proof.** As mentioned above, \( \text{FP}_{eq} \) is in principle a standard a-priory \( H^\infty \) filtering problem, which can be solved using known approaches, like in \([11, 5]\). Just note, that the reduced rank \( \bar{D}_2 \) and the singular \( \Pi \) can be easily incorporated into the proof (a possible approach is to perform an \( \epsilon \)-perturbation of \( D_2 \) and \( \Pi \) to make the data nonsingular). It is worth mentioning that the requirement \( P_k \geq 0 \) turns out to be redundant. To see this note (see the proof of Lemma 6 in \([6]\)) that whenever statement ii) above holds for some \( k \), \( P_{k+1} \) given by (5) is just the Schur complement of the 2,2-sub-block of the matrix \( S(P_k) \), where

\[
S(P) \doteq \left[ \begin{array}{cc} \bar{A}\bar{P}^{1/2} & \bar{B} \\ \bar{C}_2\bar{P}^{1/2} & \bar{D}_2 \end{array} \right] \left( I - \gamma^{-2} \left[ \begin{array}{cc} \bar{P}^{1/2} & \bar{D}_1' \end{array} \right] \left[ \begin{array}{cc} \bar{C}_1\bar{P}^{1/2} & \bar{D}_1 \end{array} \right] \right)^{-1} \left[ \begin{array}{cc} \bar{A}\bar{P}^{1/2} & \bar{B} \\ \bar{C}_2\bar{P}^{1/2} & \bar{D}_2 \end{array} \right]'.
\]

Since \( S(P_k) \geq 0 \) for any \( P_k \geq 0 \) such that \( \bar{D}_1\bar{D}_1' + \bar{C}_1P_k\bar{C}_1' < \gamma^2I \), \( P_{k+1} \geq 0 \) as well. \[\square\]
The solution to \( \text{FP}_{eq} \) given in Theorem 1 is in general time varying. In some situations, however, one is interested in a stationary solution. Sufficient conditions for the existence of such a solution are presented below. To this end the following discrete-time algebraic Riccati equation (DARE) is required:

\[
Y = \bar{A}Y\bar{A}^* + \bar{B}\hat{\bar{B}}^* - (\bar{A}Y\hat{C}^* + \hat{B}\bar{D}^*)\bar{H}(Y)^{-1}(\bar{A}Y\hat{C}^* + \hat{B}\bar{D}^*)^* \tag{9}
\]

Then we have:

**Theorem 2.** Let the assumptions \((A1)\) and \((A2)\) hold, then there exists a filter which solves the \( \text{FP}_{eq} \) only if the DARE \((9)\) has the stabilizing solution \( Y_\infty \) such that \( \hat{\bar{B}}\bar{D}_1^* \bar{C}_1^* + \hat{C}_1 Y_\infty \bar{C}_1^* < \gamma^2 I \).

Provided that the latter condition holds and that \( \Pi \leq Y_\infty \), the LTI observer

\[
\begin{align*}
\tilde{\xi}_k &= \hat{\bar{C}}_1 \tilde{\eta}_k, \\
\tilde{\eta}_{k+1} &= \bar{A} \tilde{\eta}_k - (\bar{A} Y_\infty \hat{C}^* + \hat{B}\bar{D}^*)\bar{H}(Y_\infty)^{-1}[0] (\tilde{\xi}_k - \hat{C}_2 \tilde{\eta}_k), \quad \tilde{\eta}_0 = 0
\end{align*}
\]

solves \( \text{FP}_{eq} \).

**Proof.** The first claim follows from the fact that if recursion \((5)\) has a stabilizing solution, then it asymptotically converges to the stabilizing solution \( Y_\infty \) of the DARE \((9)\). Then, it is obvious that when \( \Pi = Y_\infty \), \( P_k \equiv Y_\infty \) for all \( k \in \mathbb{Z}^+ \). Hence, the rest of the Theorem follows from the observation that if some observer solves \( \text{FP}_{eq} \) for \( \Pi = \Pi_0 \), then the same observer solves \( \text{FP}_{eq} \) for every \( 0 \leq \Pi \leq \Pi_0 \). \( \square \)

**Remark 2.3.** Theorem 2 actually implies that whenever \( \Pi \leq Y_\infty \), the problem \( \text{FP}_{eq} \) has a solution iff it has an LTI solution. In particular, the latter claim is always true for the \( H^\infty \) filtering problems with known initial conditions (\( \Pi = 0 \)).

### 3 Main results

The solutions to \( \text{FP}_{H^\infty} \) in the previous section are presented in the lifted domain. Hence, in order to be implemented the results of Theorems 1 and 2 have be translated back in time domain (“peeled-off”). This will be the subject matter of the rest of the paper. In particular, in this section we formulate the counterparts of the lifted results of Section 2, while the next section is devoted to proofs.

In order to represent the lifted solutions in Theorems 1 and 2 in implementable forms we need the matrix exponential

\[
\Sigma = e^{\Gamma_y h} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \Sigma_{34} \\ 0 & 0 & 0 & \Sigma_{44} \end{bmatrix}, \tag{10}
\]

where

\[
\Gamma_y = \begin{bmatrix} A_S & B_S C_2 & B_S D_2 c B'_c & B_S D_2 c D'_2 c B'_S \\ 0 & A & B'_c c & B'_S c D'_2 c B'_S \\ 0 & -\gamma^{-2} C'_1 C_1 & -A'_f & -C'_2 B'_S \\ 0 & 0 & 0 & -A'_f \end{bmatrix},
\]
and the constant
\[ \gamma_0 = \| P_1 \|_{L^2[0,h]}, \]

where \( P_1 \) is an LTI continuous-time system with the transfer function \( C_1 (sI - A)^{-1} B_c \). In other words, \( \gamma_0 \) is the \( L^2[0,h] \)-induced norm of the operator from \( w(t) \) to \( z(t) \) (see [1, 4] for various approaches to the computation of \( \gamma_0 \)). With these definitions, assumptions (A1) and (A2) can be rewritten as follows:

(A1'): The matrix \( \begin{bmatrix} \Sigma_{22} - zI \\ zDSC_2 + CS\Sigma_{12} \end{bmatrix} \) is left invertible for all \( |z| \geq 1 \);

(A2'): The matrix \( \begin{bmatrix} \Sigma_{22} - e^{j\theta}I & \Sigma_{24}\Sigma'_{11}C_S & \Sigma_{23} & 0 \\ e^{j\theta}DSC_2 + CS\Sigma_{12} & CS\Sigma_{14}\Sigma'_{11}C'_{S} & CS\Sigma_{13} & D_d \end{bmatrix} \) is right invertible for all \( \theta \in [0,2\pi] \) and for any \( \gamma > \gamma_0 \);

While (A2') is equivalent to (A2), (A1') is equivalent to (A1) only for those \( \gamma \), for which the DARE (9) has the stabilizing solution \( Y_\infty \) such that \( D_1 D_1^* + C_1 Y_\infty C_1^* < \gamma^2 I \) [9]. The latter, however, is necessary for \( FP_{H\infty} \) to have a solution. Thus, (A1) can be replaced with (A1') without loss of generality.

The Riccati recursion (5) can now be equivalently written as
\[ P_{k+1} = \Lambda_{11}(P_k) - \Lambda_{12}(P_k)\Lambda_{22}(P_k)^{-1}\Lambda_{21}(P_k), \quad P_0 = \Pi, \]

where
\[ \Lambda(P) = \begin{bmatrix} 1 & 0 & \Sigma_{23} + \Sigma_{22}P & \Sigma_{24} \\ DSC_2 & CS & \Sigma_{13} + \Sigma_{12}P & \Sigma_{14} \end{bmatrix}^{-1} \begin{bmatrix} 1 & C'_2D'_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D_dD'_d \end{bmatrix} \]

Then the time domain equivalent of Theorem 1 is:

**Theorem 3.** Let the assumptions (A1') and (A2') hold and \( \gamma > \gamma_0 \), then \( \det \Sigma_{33} \neq 0 \) and the following two statements are equivalent:

i) There exists a filter which solves the \( FP_{H\infty} \).

ii) The Riccati recursion (11) has a bounded solution \( P_k \) such that

(a) \( \det \Lambda_{22}(P_k) \neq 0 \) \( \forall k \in \mathbb{Z}_+ \);

(b) The homogeneous system
\[ \chi_{k+1} = (\Sigma_{22} - P_{k+1}\Sigma_{32} - \Lambda_{12}(P_k)\Lambda_{22}(P_k)^{-1}(DSC_2\Sigma_{22} + CS\Sigma_{12}))\chi_k \]

is asymptotically stable;

(c) \( \rho(\Sigma_{33}\Sigma_{32}P_k) < \gamma^2 \) \( \forall k \in \mathbb{Z}_+ \).

Given that ii) holds, then one possible \( \gamma \)-suboptimal filter is
\[ \zeta(kh + \tau) = C_1 e^{A\tau} \hat{n}_k, \]
\[ \hat{n}_{k+1} = e^{Ah}\hat{n}_k - \Lambda_{12}(P_k)\Lambda_{22}(P_k)^{-1}(g_{k+1} - C_2 e^{Ah}\hat{n}_k), \quad \hat{n}_0 = 0. \]
Some remarks are in order:

**Remark 3.1.** The \( \gamma \)-suboptimal filter \((13)\) has an interesting structure. It consists of a pure discrete-time finite-dimensional time-varying part \((13b)\) followed by a stationary *generalized hold* having the hold function \( \phi_\mathcal{H}(\tau) = C_1 e^{A_\tau} \). In \([13]\) a filter with the similar structure was obtained in the case of known initial conditions only (with the LTI discrete-time part of the filter).

**Remark 3.2.** It is worth stressing that the existence conditions in ii) consist only on verifying, whether the spectral radius of the finite-dimensional matrix \( \Sigma_{33}^{-1} \Sigma_{32} P_k \) is smaller than \( \gamma^2 \) in each iteration. This offers a clear advantage over the test in \([13]\), where one has to verify the non-singularity of the matrix function\(^2\) (in our notation) \( \Sigma_{22}(t) - \Sigma_{32}^2(t) P_k \) for all \( t \in [0, h) \). Actually, the requirement \( \gamma > \gamma_0 \) in our case is equivalent to the non-singularity of \( \Sigma_{22}(t) \) for all \( t \in [0, h) \). The latter, however, is to be verified only once, before the iteration starts, while the test in \([13]\) should be performed in each iteration.

**Remark 3.3.** The formulae of Theorem 3 can be substantially simplified if additional assumptions on \( D_d \) or \( \phi_S \) are imposed. For example, assume that \( D_d D_d' > 0 \) and define the matrices

\[
\begin{bmatrix}
\Psi_1 & \Psi_2 & \Psi_3
\end{bmatrix} = \begin{bmatrix}
\Sigma_{32} & \Sigma_{33} & \Sigma_{34}
0 & 0 & \Sigma_{44}
\end{bmatrix} + \begin{bmatrix}
C_1^d D_c^d & C_S^d
\end{bmatrix} (D_d D_d')^{-1} \begin{bmatrix}D_S C_2 & C_S \end{bmatrix} \begin{bmatrix}
\Sigma_{22} & \Sigma_{23} & \Sigma_{24}
\Sigma_{12} & \Sigma_{13} & \Sigma_{14}
\end{bmatrix}.
\]

Then the Riccati recursion \((11)\) can be written as

\[
P_{k+1} = \begin{bmatrix}
\Sigma_{23} + \Sigma_{22} P_k & \Sigma_{24}
\end{bmatrix} \begin{bmatrix}
\Psi_2 & \Psi_1 P_k & \Psi_3
\end{bmatrix}^{-1} \begin{bmatrix}I & 0
\end{bmatrix}.
\]

Further, if \( S_h \) is the ideal sampler (i.e., \( A_S = B_S = C_S = 0, \ D_S = I \)), then \((11')\) simplifies to:

\[
P_{k+1} = (\Sigma_{23} + \Sigma_{22} P_k)(\Sigma_{33} + \Sigma_{32} P_k + C_2 (D_d D_d')^{-1} C_2 (\Sigma_{23} + \Sigma_{22} P_k))^{-1},
\]

while if \( D_S = 0 \) and \( C_S = I \) (this case includes the average sampling as well as the \( H^2 \) and \( H^\infty \) optimal samplers \([8]\)), then the Riccati recursion becomes:

\[
P_{k+1} = \begin{bmatrix}
\Sigma_{23} + \Sigma_{22} P_k & \Sigma_{24}
\end{bmatrix} \begin{bmatrix}
\Sigma_{33} + \Sigma_{32} P_k & \Sigma_{34}
\Sigma_{13} + \Sigma_{12} P_k & D_d D_d' \Sigma_{14}
\end{bmatrix}^{-1} \begin{bmatrix}I & 0
\end{bmatrix}.
\]

Now, consider a stationary solution of \( \mathbf{FP}_{1\infty} \), that is the solution which involves an LTI discrete-time part. The following theorem gives sufficient conditions for the existence of such a solution and also the formulae for one possible stationary filter:

**Theorem 4.** Given a \( \gamma > \gamma_0 \), then \( \det(\Sigma_{33}) \neq 0 \) and an estimator which solves \( \mathbf{FP}_{1\infty} \) exists only if the DARE

\[
Y = \Lambda_{11}(Y) - \Lambda_{12}(Y) \Lambda_{22}(Y)^{-1} \Lambda_{21}(Y).
\]

has the stabilizing solution \( Y_\infty \) such that \( \rho(\Sigma_{33}^{-1} \Sigma_{32} Y_\infty) < \gamma^2 \). If in addition \( Y_\infty \geq \Pi \), then the LTI observer

\[
\zeta(kh + \tau) = C_1 e^{A_\tau} \bar{\eta}_k,
\]

\[
\eta_{k+1} = e^{A_h} \bar{\eta}_k - \Lambda_{12}(Y_\infty) \Lambda_{22}(Y_\infty)^{-1} (\bar{y}_{k+1} - C_2 e^{A_h} \bar{\eta}_k), \quad \bar{\eta}_0 = 0,
\]

solves \( \mathbf{FP}_{\text{eq}} \).
Remark 3.4. It can be shown [9] that equation (14) can be solved by finding the stable deflating subspace of the extended symplectic matrix pair \((\Lambda_l,\Lambda_r)\), where
\[
\Lambda_l \doteq \begin{bmatrix}
\Sigma'_{22} & \Sigma'_{32} & \Sigma'_{12} \\
\Sigma'_{23} & \Sigma'_{33} & \Sigma'_{13} \\
C_S\Sigma_{11}\Sigma'_{24} & C_S\Sigma_{11}\Sigma'_{34} & C_S\Sigma_{11}\Sigma'_{14}
\end{bmatrix}
\begin{bmatrix}
I & 0 & C'SD'S \\
0 & -1 & 0 \\
0 & 0 & C'
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & DSC_2 & D_dD_d'
\end{bmatrix}
\]
and
\[
\Lambda_r \doteq \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
In this case \(Y_\infty\) and \(\bar{L}_2 = -\Lambda_{12}(Y_\infty)\Lambda_{22}(Y_\infty)^{-1}\) satisfy the equation
\[
\Lambda_l \begin{bmatrix}
1 \\
Y_\infty \\
\bar{L}_2
\end{bmatrix} = \Lambda_r \begin{bmatrix}
1 \\
Y_\infty \\
\bar{L}_2
\end{bmatrix} A_{cl}
\]
for some Schur matrix \(A_{cl}\).

4 Proofs

This section is devoted to the proof of the results presented in the previous section. We prove Theorem 3 (4) by showing its equivalence to Theorem 1 (2). Also, we give the proof only for Theorem 3. The results for the stationary case follow by replacing the matrices \(P_k\) satisfying the Riccati recursion (11) with the matrix \(Y_\infty\), which satisfies the DARE (14).

4.1 A new representation of lifted systems

We start with a brief exposition of the results of [7] concerning a new representation of the parameters of the lifted systems, which plays a central role in the reasoning to follow. Conventionally [1, 3], the parameters of lifted systems are described by integral operators over \(L^2[0,h]\). Such a representation, however, makes the manipulations of these parameters quite a cumbersome, if not an impossible, problem. This fact has motivated a different representation of the parameters of the lifted systems as proposed in [7], which considerably simplifies the manipulations of these parameters. The new representation is based upon three components: systems with two-point boundary conditions (STPBC) operating on the time interval \([0,h]\), the impulse operator \(I_0\), and the sampling operator \(I^*_0\).

i) STPBC are linear continuous-time operators \(\bar{O}: L^2[0,h] \mapsto L^2[0,h]\), which are described by the following state equations:
\[
\bar{O} : \begin{cases}
\dot{x}(t) = Ax(t) + B\omega(t), \\
\z(t) = Cx(t) + D\omega(t),
\end{cases}
\quad \Omega x(0) + \Upsilon x(h) = 0,
\]
where the square matrices \(\Omega\) and \(\Upsilon\) shape the boundary conditions of the state vector \(x\). The boundary conditions are said to be well-posed if \(\det(\Omega + \Upsilon e^{Ah}) \neq 0\) and in this case

\[\text{It is worth stressing that } J^*_0\text{ is not the adjoint of } J_0.\text{ Nevertheless, we will proceed with this abuse of notation for the reasons discussed in [7].}\]
the map $\zeta = \hat{O}\omega$ is well defined $\forall \omega \in L^2[0,h]$. We will denote STPBC by the following compact block notation:

$$\hat{O} = \begin{pmatrix} A & \Omega \rightarrow \gamma & B \\ C \\ D \end{pmatrix}$$

and use the term “STPBC” to denote systems with well-posed boundary conditions only. In the case where $\Omega = I$ and $\gamma = 0$ (which corresponds to a causal STPBC) the boundary condition “window” will be omitted, like $\begin{pmatrix} A \mid B \end{pmatrix}$.

ii) The impulse operator $I_\theta$ transforms a vector $\eta \in \mathbb{R}^n$ into a modulated $\delta$-impulse as follows:

$$(I_\theta \eta)(t) = \delta(t - \theta)\eta.$$  

iii) The sampling operator $I^*_\theta$ transforms a function $\zeta \in C_n[0,h]$ into a vector from $\mathbb{R}^n$ as follows:

$$I^*_\theta \zeta = \zeta(\theta).$$

With the aid of these operators, the parameters of the lifted process (4) can be presented in the following form:

$$\begin{bmatrix} \bar{A} & \bar{B}_c \\ \bar{C}_1 & \bar{D}_{1c} \end{bmatrix} = I^*_h \begin{bmatrix} 0 & A & B_c \\ 0 & I & 0 \\ 0 & 0 & C_1 \\ 0 & 0 & 0 \end{bmatrix} I_h \begin{bmatrix} 0 & 0 \end{bmatrix}$$

and, taking into account the equality $D_S D_{2c} = 0$,

$$\begin{bmatrix} \bar{C}_2 & \bar{D}_{2c} \end{bmatrix} = I^*_h \begin{bmatrix} A_S & B_S C_2 & 0 & B_S D_{2c} \\ 0 & A & I & 0 \\ C_S & D_S C_2 & 0 & 0 \end{bmatrix} I_h \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

The advantages of this representation are twofold: a) the algebraic manipulations over STPBC can be easily performed in the state space, much like manipulations over standard LTI systems; and b) the operators $I_\theta$ and $I^*_\theta$ fit nicely into the state-space framework. See [7] for explanations and technical details. Below, we just present several important properties of the three operators above, which will be required used in the sequel. First, the following lemma is a part of [7, Lemma 2]:

**Lemma 1.** Let $\hat{O}$ be given by (15). Then

$$J_\theta^* \hat{O} J_0 = C e^{A_h}(\Omega + \gamma e^{A_h})^{-1}\Omega B$$

and if in addition $CB = 0$, then

$$J_\theta^* \hat{O} J_0 = -C e^{A_h}(\Omega + \gamma e^{A_h})^{-1}\gamma B.$$

Second, the adjoint of any composition of STPBC, $I_\theta$ and $I^*_\theta$ can be computed componentwise, for instance,

$$\begin{bmatrix} \bar{A} & \bar{B}_c \\ \bar{C}_1 & \bar{D}_{1c} \end{bmatrix}^* = \begin{bmatrix} J_0^* & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -A' & 0 & 0 & C_1' \\ -I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{O}^* & 0 \\ 0 & 1 \end{bmatrix}.$$

Finally, the operators $I_\theta$ and $I^*_\theta$ can be absorbed into STPBC in an elegant manner. In particular, the following lemma, which is a corollary of [7, Lemma 3], will be widely used in the sequel:
Lemma 2. Let

\[
\begin{bmatrix}
\hat{C}_i \\
\hat{D}_i
\end{bmatrix} = \begin{pmatrix}
\hat{A} & \hat{B} \\
\hat{C}_i & \hat{D}_i
\end{pmatrix} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad i = 1, 2.
\]

Then for any appropriately dimensioned matrix \(M\)

\[
\hat{D}_1\hat{D}_2^* + \hat{C}_1\hat{M}\hat{C}_2^* = \begin{pmatrix}
A & BB' & \begin{bmatrix}
1 & -M \\
0 & 0
\end{bmatrix} \\
0 & -A' & \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} \\
C_1 & D_1B' & \begin{bmatrix}
BD_1' \\
-C_2'
\end{bmatrix}
\end{pmatrix}.
\]

4.2 Proof of Theorem 3

First, note that the equivalence of the assumptions (A1), (A2) and their “prime” counterparts can be shown in a similar fashion as in [9]. The only difference between the results there and here is the presence of the matrix \(D_d\) in our case.

Now, for a given \(P = P' \geq 0\) introduce the following notation:

\[
\begin{bmatrix}
\hat{M}_x \\
\hat{M}_y \\
\hat{M}_z
\end{bmatrix} = \begin{pmatrix}
\hat{A} & \hat{B}_c \\
\hat{C}_2 & \hat{D}_{2c} \\
\hat{C}_1 & \hat{D}_{1c}
\end{pmatrix} \begin{bmatrix}
p^{1/2} & 0 \\
0 & 1
\end{bmatrix}
\]

and

\[
\hat{R} = (I - \gamma^{-2}\hat{M}_x\hat{M}_x^*)^{-1},
\]

where the latter operator is well defined if \(\hat{D}_{1c}\hat{D}_{1c}^* + \hat{C}_1P\hat{C}_1^* < \gamma^2I\). The latter inequality, which is required in Theorem 1, plays a central role throughout this subsection. Hence, we start with the following proposition:

Proposition 1. If \(\gamma > \|\hat{D}_1\|_2\), then \(\det(\Sigma_{33}) \neq 0\) and for any \(P = P' \geq 0\), \(\hat{D}_1\hat{D}_1^* + \hat{C}_1P\hat{C}_1^* < \gamma^2I\) iff \(\rho(\Sigma_{33}^{-1}\Sigma_{32}P) < \gamma^2\).

Proof. The non-singularity of \(\Sigma_{33}\) subject to \(\gamma > \|\hat{D}_1\|_2\) is well known (see, e.g., [1]). Further, it is a straightforward matter to verify that if the latter inequality holds, then

\[
\hat{D}_1\hat{D}_1^* + \hat{C}_1P\hat{C}_1^* < \gamma^2I \iff p^{1/2}\hat{C}_1(I - \gamma^{-2}\hat{D}_1\hat{D}_1^*)^{-1}\hat{C}_1^* < \gamma^2I.
\]

As shown in [7], \(\hat{C}_1(1 - \gamma^{-2}\hat{D}_1\hat{D}_1^*)^{-1}\hat{C}_1 = \Sigma_{33}^{-1}\Sigma_{32}\), which completes the proof.

The following proposition establishes the equivalence between the matrix functions \(S(P)\) and \(\Lambda(P)\) defined by (8) and (12), respectively:

Proposition 2. Whenever \(\hat{D}_1\hat{D}_1^* + \hat{C}_1P\hat{C}_1^* < \gamma^2I\), \(S(P) = \Lambda(P)\).

Proof. It can easily be shown that

\[
S(P) = S_c(P) + \begin{bmatrix}
0 & 0 \\
0 & D_dD_d^*
\end{bmatrix},
\]

where the function \(S_c(P)\) is produced form \(S(P)\) by replacing the operators \(\hat{B}, \hat{D}_1, \text{and } \hat{D}_2\) with \(\hat{B}_c, \hat{D}_{1c}, \text{and } \hat{D}_{2c}\), respectively. Now, \(S_c(P)\) can be written as

\[
S_c(P) = \begin{bmatrix}
\hat{M}_x \\
\hat{M}_y \\
\hat{M}_z
\end{bmatrix} \begin{bmatrix}
\hat{M}_x^* \\
\hat{M}_y^* \\
\hat{M}_z^*(\gamma^2I - \hat{M}_2\hat{M}_2^*)^{-1}\hat{M}_z
\end{bmatrix} \begin{bmatrix}
\hat{M}_x \\
\hat{M}_y \\
\hat{M}_z
\end{bmatrix}.
\]
Applying Lemma 2 to all the terms of the form $\tilde{M}_k\tilde{M}_k^*$ above (here the tilde stays for either the breve or the acute accent) and then performing appropriate “cancellations,” one can get that

$$S_c(P) = J_n\left( \begin{array}{cc} \Gamma_y - \Omega_a \gamma_a & B_a \\ C_a & 0 \end{array} \right) J_n,$$

where $B_a \doteq -\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & D_2S_2 & C_2 \end{bmatrix}'$, $C_a \doteq \begin{bmatrix} 0 & 1 & 0 & 0 \\ C_2 & D_2S_2 & 0 & 0 \end{bmatrix}$, and

$$\Omega_a \doteq \gamma_a \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -P & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \doteq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now, calculating $S_c(P)$ by Lemma 1 and the equality

$$\begin{bmatrix} 1 & -P \\ \Sigma_{32} & \Sigma_{33} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} P \\ \Sigma_{33} + \Sigma_{32}P \end{bmatrix}^{-1}[-\Sigma_{32} \ I],$$

one can see that $S_c(P)$ is equivalent to the first term in the right-hand side of (12). \hfill \Box

Taking into account that the solution $P_{k+1}$ of (5) is the Schur complement of the 2,2-subblock of the matrix $S(P_k)$ (see the proof of Theorem 1), Proposition 2 establishes, actually, that the solutions of (5) and (11) coincide. Therefore, to prove the equivalence of statements ii) in Theorems 1 and 3, we have only to prove that the asymptotic stability of (6) is equivalent to the asymptotic stability of the homogeneous system $\chi_{k+1} = A_{cl}(P_k)\chi_k$, where

$$A_{cl}(P) \doteq \Sigma_{32} - \Lambda_{11}(P)\Sigma_{32} - \Lambda_{12}(P)\Lambda_{22}(P)^{-1}(C_2\Sigma_{12} + D_2S_2\Sigma_{22} - \Lambda_{21}(P)\Sigma_{32}).$$

To this end the first step is to find an expression for the filter “gains” $\hat{L} \doteq -(\hat{A}\hat{P}\hat{C}^* + \hat{B}\hat{D}^*)\hat{H}(P)^{-1} = \begin{bmatrix} \hat{L}_1 \\ \hat{L}_2 \end{bmatrix}$. We have:

**Proposition 3.** Whenever $\bar{D}_1\bar{D}_1^* + \hat{C}_1\hat{P}\hat{C}_1^* < \gamma^2 I$, $\hat{L}_1 \doteq \gamma^{-2}(\hat{M}_x + L_2M_y)\check{R}\check{M}_z^*$

and

$\hat{L}_2 \doteq -\Lambda_{12}(P)\Lambda_{22}(P)^{-1}$.

**Proof.** It can be verified by a straightforward algebra that

$$\hat{H}(P)^{-1} = \begin{bmatrix} \hat{M}_x^* \\ \hat{M}_y^* \end{bmatrix} \left[ \begin{array}{cc} \hat{M}_z & \hat{M}_y \\ \hat{M}_y & \hat{M}_z \end{array} \right] + \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & D_dD_d' \end{bmatrix} \right]^{-1}$$

$$= \begin{bmatrix} \gamma^{-2}\hat{M}_x^*\check{R}\check{M}_y^* \\ I \end{bmatrix} S_{22}(P)^{-1} \begin{bmatrix} \gamma^{-2}\hat{M}_y\check{R}\check{M}_z^* \\ 1 \end{bmatrix} \left[ \begin{array}{cc} (\gamma^2 I - \hat{M}_y\check{R}\check{M}_z^*)^{-1} & 0 \\ 0 & 0 \end{array} \right].$$

Then

$$\hat{L} = -\hat{M}_x\begin{bmatrix} \hat{M}_z^* \\ \hat{M}_y^* \end{bmatrix} \hat{H}(P)^{-1}$$

$$= -S_{12}(P)S_{22}(P)^{-1} \begin{bmatrix} \gamma^{-2}\hat{M}_y\check{R}\check{M}_z^* \\ 1 \end{bmatrix} + \hat{M}_x\check{R}\check{M}_z^* \begin{bmatrix} \gamma^{-2}I \\ 0 \end{bmatrix}$$

from which the formulae for $\hat{L}$ follow by Proposition 2. \hfill \Box
Using Proposition 3 we can now prove the following result, which completes the proof of the first part of Theorem 3:

**Proposition 4.** Whenever \( \tilde{D}_1 \tilde{D}^*_1 + \tilde{C}_1 P_1^* < \gamma^2 I \),

\[
\tilde{A} + \tilde{L} \tilde{C} = A_{cl}(P).
\]

**Proof.** By Proposition 3

\[
\tilde{A} + \tilde{L} \tilde{C} = \begin{bmatrix} 1 & L_2 \end{bmatrix} \left( \begin{bmatrix} \tilde{A} \\ \tilde{C}_2 \end{bmatrix} + \begin{bmatrix} \tilde{M}_x \\ \tilde{M}_y \end{bmatrix} \tilde{M}_z^*(\gamma^2 I - \tilde{M}_Z \tilde{M}_Z^*)^{-1} \tilde{C}_1 \right).
\]

Using Lemma 2 one can get that

\[
\begin{bmatrix} \tilde{A} \\ \tilde{C}_2 \end{bmatrix} + \begin{bmatrix} \tilde{M}_x \\ \tilde{M}_y \end{bmatrix} \tilde{M}_Z^*(\gamma^2 I - \tilde{M}_Z \tilde{M}_Z^*)^{-1} \tilde{C}_1 = \mathcal{J}_h\left( \begin{bmatrix} \Gamma_x & \Omega_a & \gamma_a & B_p \\ C_a & 0 & 0 & 0 \end{bmatrix} \right),
\]

where \( C_a, \Omega_a, \) and \( \gamma_a \) are as in the proof of Proposition 2 and \( B_p \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}' \), from which the proof follows by applying Lemma 1.

From the definition of the operator \( \tilde{C}_1 \) it follows that (13a) is the time-domain equivalent of (7a). Hence, to complete the proof of Theorem 3 we need only to show the equivalence of the systems (7b) and (13b). But the latter follows directly from Proposition 3.

5 Conclusions

This paper considered the sampled-data H\(^\infty\) infinite horizon filtering problem. The problem stated and treated in this paper is more general than those previously treated in the literature. It includes both cases of completely known and completely unknown initial conditions as special cases and does not require the discrete-time measurements to be corrupted by a discrete measurement noise. It has been proven that whenever a sampled-data \( \gamma \)-suboptimal filter exists, there exists a \( \gamma \)-suboptimal filter of the form of a finite dimensional discrete-time (possibly time-varying) filter followed by a stationary generalized hold function. This extends previous results in the literature, where a suboptimal filter of this structure (with an LTI discrete-time part of the filter) was obtained for the case of known initial conditions only. Also, it has been shown that if uncertainties on the initial conditions of the continuous-time process are, in a sense, sufficiently "small," then there exists a suboptimal filter with an LTI discrete-time part. Furthermore, the existence conditions of a suboptimal filter derived in this paper are simpler than those known in the literature and involve in each iteration tests on finite-dimensional matrices only.

References


