$H^\infty$ Control and Estimation with Preview—Part I: Matrix ARE Solutions in Continuous Time

Gilead Tadmor and Leonid Mirkin

Abstract—Preview control and fixed-lag smoothing allow a non-causal component in the controller/estimator. Time domain variational analysis is used in a reduction to an open loop differential game, leading to a complete, necessary and sufficient characterization of suboptimal values and an explicit state space design, in terms of a parameterized (non-standard) algebraic matrix Riccati equation in a general continuous time linear system setting. The solution offers insight into the appropriate structure of the associated Hamiltonian, where the state and co-state are not the usual state of the original dynamic system and that of its adjoint. Rather, the state and co-state are selected to capture the respective lumped effects of initial data and future input selection in the allied game.

Index Terms—Preview control; fixed-lag smoothing; differential games; $H^\infty$ optimization.

I. INTRODUCTION

Preview tracking and fixed-lag smoothing are extensions of standard full-information and filtering problems, with relaxed causality constraints (preview availability). Numerous control and estimation problems fall into the category of problems with information preview. In some tracking problems, e.g., those arising in aircraft control [1], robotics [2], and vehicle suspension control [3], previewed commands or disturbances may be available. Such problems are referred to as preview tracking. Similarly, in many communication systems, even interactive ones, a small amount of delay or latency can be tolerated. Some delay is often permissible in speech coding [4], multirate filter banks design [5], multi-target tracking of a maneuvering target [6], etc. Such problems can be formulated as estimation problems with a constant preview window and they are referred to as fixed-lag smoothing.

The $H^2$ (LQ) theory of the preview tracking and estimation (smoothing) is currently well developed, see e.g. [3], [7], [8]. Apparently, the first mention of preview tracking in the $H^\infty$ (game-theoretic) context appeared in [9]. Yet in that paper, only unmeasured disturbances were included in the game. Thus, preview tracking is actually treated in [9] in the $H^2$, rather than in the $H^\infty$ setting. The same approach was also adopted in [10].

Indeed, the $H^\infty$ preview theory is yet considerably less mature. $H^\infty$ control and estimation with preview proved to be a challenge already in the current, continuous-time setting, and even more so, in discrete-time and in sampled-data systems (stated as Open Problem 51, in [11]). Most of the existing results resort to strictly sufficient conditions, system restrictions, iterative approximations and dimension increase. For example, the solution of the continuous-time $H^\infty$ preview tracking, in [12], is derived in terms of the standard $H^\infty$ algebraic Riccati equation (ARE), that is associated with the tracking problem without preview. That equation, however, might not admit a stabilizing solution under some performance levels $\gamma$ for which the preview problem is solvable. In other words, the solvability condition in [12] is only sufficient. For some other (discrete-time) examples see the discussion in the companion paper [13]. To the best of our knowledge, the only complete solution of $H^\infty$ preview problems is the solution of the continuous-time $H^\infty$ fixed-lag smoothing problem in [14]. The approach in [14] is based on $J$-spectral factorization arguments (and a transformation introduced in [15]) and the solution there is formulated in terms of a modified $H^\infty$ ARE, the Hamiltonian matrix associated with which is similar to that associated with the filtering $H^\infty$ ARE.

The purpose of this, and of the companion paper [13], is to provide a cohesive solution to the $H^\infty$ preview control and estimation problems, in both the continuous-time and the discrete-time settings\(^1\). For notational simplicity, the exposition is made in a time invariant setting. Our results, however, and the variational, game theoretic arguments employed here, readily extend to time-varying systems, which is a main advantage over transform domain methods, as those in [14].

Linear systems with a single, pure, input lag are well recognized as the simplest among the various classes of distributed parameter systems. Custom design methods aim to exploit and match that simplicity. The Smith predictor, which reduces stabilization to an equivalent problem with no delay, is an early example. From the technical perspective of an adaptation of the variational / game theoretic analysis methods used here, the difficulty in the preview setting, is twofold. One issue is borne out by the mere fact that, with the presence of delay, this is a distributed parameter system. The complete state, including a relevant disturbance history, is embedded in the Hilbert space $M_2 := \mathbb{R}^n \times L_2[-h,0]$, as explained below, leading to infinite dimensionality. Following the example of earlier game theoretic solutions of $H^\infty$ problems in systems with control delay (see e.g., [17], [18] and the review [19]), this issue is addressed by a reduction to a non-distributed differential game and the utilization of an interpolation between a distributed parameters model and the original system.

The second difficulty, which is new in the current setting, concerns the utilization of Hamilton-Jacobi characterization of

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\(^1\)A solution to the sampled data problem can then be obtained, combining the discrete time solution with the $H^\infty$ discretization technique of [16].
optimal solutions of the underlying game, and the derivation of a Riccati equation, thereof. Specifically, here the initial state of the standard Hamiltonian depends, both on initial data and on future values of the disturbance. The key solution step is a change of state, whereby the new Hamiltonian state reflects only initial data contributions, and the co-state — only future disturbance effects. This leads to a complete (necessary and sufficient) non-standard ARE solution, in a general linear system. Our solution is then stated in terms of a combination of one $H^2$ and one, parameterized $H^\infty$ ARE’s.

In closing, it is interesting to view the developed approach against the backdrop of work dating to the mid 1970’s, on efficient abstract evolution models for retarded and neutral systems (i.e., with state and derivative delays). The so-called structural F-operator (see, e.g., [20]–[22]) reduces the standard state space $M_2$ (or variants thereof) to a smaller quotient space, factoring out state histories that have no long-term impact on the evolution. In the present case, the $\mathbb{R}^n$ state of the associated Hamiltonian can be viewed as belonging to an appropriate quotient space of the complete state space $M_2$, representing that complete state component that is relevant to the solution of the differential game.

The paper is organized as follows. The main results are formulated in Section II: preview control is addressed in §II-A and fixed-lag smoothing – in §II-B, followed by an illustrative example, in Section III. The proof of the preview control result is given in Section IV. The smoothing proof follows by duality. Finally, concluding remarks are provided in Section V.

A. Notations

All signal and lumped vector inner products and norms are understood in the standard $L_2^{}$ and Euclidean ($\mathbb{R}^n$) frameworks. The time interval over which signals are defined is included when not obvious (e.g., $[\|x\|_{L_2^2[0, h]}$). The default time interval is $[0, \infty)$, so that $L_2^{}$ stands hereafter for $L_2^{}[0, \infty)$. To avoid cumbersome references to the (finite) dimensions of the several Euclidean spaces used (for $x, w, u, etc.$), norm and inner product notations without an accompanying subscript will stand for Euclidean space norms and inner products. Matrix norms are defined by the maximal spectral value, $\|A\| = \sigma_{\text{max}}(A)$, which is the induced norm relative to the Euclidean vector norm. Functional operators are evaluated by the induced $L_2^{}$ norm. The notation $\tilde{w}_t \in L_2^{}[-h, 0]$ stands for the relevant, finite window history of an $L_2^{}[a, b]$ trajectory $w$ at the time $t$: $\tilde{w}_t(r) = w(t + r), r \in [-h, 0]$. The symbol $d_h$ stands for the $h$-time units delay operator: $d_h \phi(t) = \phi(t - h)$. Similar to earlier discussions of delay systems, we refer to the Hilbert space $M_2^{} := \mathbb{R}^n \times L_2^{}[-h, 0]$, as will be elaborated below.

II. MAIN RESULTS

A. Complete information preview control

The preview problem concerns a time invariant\footnote{The restriction to time invariant systems is made for simplicity; all arguments and results in this paper readily generalize to time varying systems.} system:

$$
\begin{align*}
\dot{x}(t) &= A x(t) + B_1 w(t - h) + B_2 u(t) \\
\dot{z}(t) &= C x(t) + D_1 w(t - h) + D_2 u(t)
\end{align*}
$$

with the usual interpretation of $x \in \mathbb{R}^n, w, u$ and $z$ as the state, exogenous input (here, representing disturbances and tracking reference), control and controlled output. The following are standard hypotheses:

- **$A_1$**: the pair $(A, B_1)$ is stabilizable;
- **$A_2$**: the realization $[A, B_2, C, D_2]$ has no invariant zeros on the imaginary axis and is left invertible.

Note that $A_2$ guarantees that $D_2$ is nonsingular.

The complete state (in the sense of Poincaré) at the time $t$ comprises the pair $(x(t), \tilde{w}_t) \in M_2^{}$. Given initial data $(x(0), \tilde{w}_0) \in M_2^{}$ and an exogenous input $w \in L_2^{}$, an admissible control is $u \in L_2^{}$ for which $x \in L_2^{}$. The set of admissible controls is non-empty by assumption $A_1$. Complete information feedback controllers in (1) comprise admissible feedback control policies that determine the control value at a time $t$ in terms of past and current values of $(x(t), \tilde{w}_t)$. Thus the delay in $w$ reflects the ability to preview the disturbance/reference.

The notation $\gamma_{\text{opt}}$ will be used for the infimal attainable induced $L_2^{}$ norm of the closed loop mapping $T_{\{w\}} : w \mapsto z$, here, under complete information feedback. The definition of $\gamma_{\text{opt}}$ presumes zero initial data or, equivalently, processes starting at $t_0 = -\infty$.

Remark 2.1: Note that the advantage afforded to the control decision maker over the disturbance in this problem statement is manifested by the fact that while decisions by $u$ and $w$ are made concurrently, the effect of the decision by $w$ on state evolution and the controlled output is delayed by $h$ time units. Thus, at the time $t$, $u$ is able to utilize its input along $[t, t + h)$ to preempt the effect of $w(t)$, which occurs only at $t + h$. It is worth emphasizing that the assumption that $w(t)$ affects the process only at $t + h$ constitutes the main difference between our formulation (see also [12]) and that discussed in [23], §3.3, where $w(t)$ is assumed to act already at $t$. The latter assumption changes the situation dramatically and is at the root of the difference between the problem addressed here and the case addressed in [23], where the additional information available in the previewed signal can be used by the controller but it will not improve the performance according to the performance criterion used in [23].

In preparation for the statement of our main result, in Theorem 2.1 below, we introduce some notations. First, define the following LQ-type control ARE

$$
X_A^{} A + A^T X_A^{} C^T C^{-1} (X_A^{} B_2^{} + C^T D_2^{}) (D_2^T D_2^{})^{-1} (B_2^T X_A^{} + C^T D_2^{} C) = 0,
$$

which, by assumptions $A_1$ and $A_2$, admits a stabilizing solution $X_A^{} \succeq 0$. For future reference we denote

$$
A_x^{} := A - B_2^{} (D_2^T D_2^{})^{-1} (B_2^T X_A^{} + C^T D_2^{} C),
$$

$$
E_x^{} := B_1^T X_A^{} - D_1^T (D_2^T D_2^{})^{-1} (B_2^T X_A^{} + C^T D_2^{} C),
$$

$$
G_x^{}(t) := \int_0^t e^{A_x^{} r} B_2^{} (D_2^T D_2^{})^{-1} B_2^T e^{A_x^{} r} dr.
$$

\footnote{It is a well known and easily established fact that, given any stabilizing state feedback $u = -K x$, the collection of admissible controls in (1) is completely parameterized by the response of the system $[A - B_2^{} K, [B_1, d_h B_2^{}], -K, [0, I]]$ to inputs $\bar{u} \in L_2^{}$.}
\[ u(t) = -(D_2^2 D_2)^{-1} \left( \left( D_2^2 C + B_2^*(X_x + e^{A_\gamma}x) e^{A_\gamma} \right) x(t) + D_2^2 D_1 w(t-h) + B_2^* \int_0^h e^{A_\gamma} \left( e^{A_\gamma} R_x + e^{A_\gamma} X_x e^{A_\gamma} (b-r) B_2(r) \right) w(t-h+r)dr \right). \] (4)

\[ B_\gamma(t) := B_2 - B_2(D_2^2 D_2)^{-1} D_2 D_1 - G_\gamma E_x, \]
and note that \( A_\gamma \) is Hurwitz. To simplify the exposition, in the sequel we will simply write \( G_\gamma \) and \( B_\gamma(t) \) instead of \( G_\gamma(t) \) and \( B_\gamma(h) \) (i.e., when \( t = h \)). For \( \gamma > \| D_1 (I - D_2(D_2 D_2)^{-1} D_2^2) \| \) we shall also use the notations
\[ \Gamma_\gamma := \gamma^2 I - D_1^2 (I - D_2^2 D_2^2)^{-1} D_2^2 D_1, \]
\[ A_\gamma := A + B_\gamma \Gamma_\gamma^{-1} E_x, \]
and
\[ R_\gamma := E_x A_\gamma + A_\gamma^* E_x \gamma - X_x R_\gamma X_x + E_x \Gamma_\gamma^{-1} E_x = 0 \]
(3) admits a stabilizing solution \( X_\gamma \geq 0 \) (so that \( A_\gamma - R_\gamma X_\gamma \)

Furthermore, if \( \gamma > \gamma_{\text{opt}} \) then one stabilizing, strictly \( \gamma \)-suboptimal feedback control is given by equation (4) at the top of this page.

Remark 2.2: The sign indefinite ARE (3) in Theorem 2.1 is different from the standard \( H^\infty \) control ARE,
\[ \dot{X}_\gamma A + A^* \dot{X}_\gamma + X_\gamma C' - (\dot{X}_\gamma B + C') D \]
\[ = \left( D' D' - \gamma^2 [I [0 I_0]]^{-1} (B' \dot{X}_\gamma + D' C) = 0 \] (5)
\[ \text{here } B := [B_1 B_2] \text{ and } D := [D_1 D_2], \]
which arises in the solution of the preview-free \( H^\infty \) problem. It can be shown (by inverting the change of variables (32) in the proof of Theorem 2.1, below) that, when both exist, \( \dot{X}_\gamma \) and \( X_\gamma \) are related by
\[ \dot{X}_\gamma = X_x + X_\gamma (I - G_\gamma X_\gamma)^{-1} \]
\[ X_\gamma = (\dot{X}_\gamma - X_x) (I + G_\gamma (X_x - X_x))^{-1}. \]

In particular, \( I - G_\gamma X_\gamma \) is non-singular iff \( I + G_\gamma (\dot{X}_\gamma - X_x) \) is non-singular. Thus, in those cases where the ARE (5) does posses a stabilizing solution \( X_\gamma \), the characterization of the parameter \( \gamma \) as a suboptimal value is in terms of the conditions that
\[ I + G_\gamma (\dot{X}_\gamma - X_x) \] be non-singular and that the self adjoint matrix \( (\dot{X}_\gamma - X_x) (I + G_\gamma (\dot{X}_\gamma - X_x))^{-1} \geq 0 \). This statement, however, lacks the authority of a complete parameterization in the sense that its starting point is a strictly sufficient condition.

Remark 2.3: It can be shown that as \( \gamma \rightarrow \infty \), the solution of Theorem 2.1 approaches the \( H^2 \) preview tracking solution, see, e.g., [9]. Indeed, it is readily seen that in the limit \( \Gamma_\gamma^{-1} \rightarrow 0 \) and \( A_\gamma - R_\gamma X_\gamma \) so that \( X_\gamma \) is the stabilizing solution of (3). Hence, the conditions of Theorem 2.1 hold, and (4) reduces then to the corresponding \( H^2 \) formula.

Comparing with the previously available solution of [12], the result of Theorem 2.1 has two advantages.

1) Theorem 2.1 offers conditions that are both necessary and sufficient. In comparison, a solution in terms of (5) rather than (3) (as in [12]) offers only a sufficient condition, as it rules out some values of \( \gamma > \gamma_{\text{opt}} \), for which a stabilizing solution of (5) does not exist, but a solution based on (3) does.

2) Since the matrix \( A_\gamma \) is Hurwitz, the control law (4) involves exponentials of strictly stable matrices, which are well posed as \( h \rightarrow \infty \). This is an improvement over the control law in [12], which involves the exponential of a Hamiltonian matrix, half of which eigenvalues are strictly positive.

Note also that we do not impose any simplifying assumptions on the parameters of the system (1).

B. Fixed-lag smoothing
The fixed-lag smoothing problem concerns the system:
\[ \dot{x}(t) = Ax(t) + Bw(t) \]
\[ z(t) = C_2 x(t) + D_1 w(t) \]
\[ e(t) = z(t-h) - \hat{z}(t) \]
\[ y(t) = C_2 x(t) + D_2 u(t). \] (6)
where the (stable, causal, linear) system \( f \) is defined by a full state Luenberger observer\(^4\), and is used to reconstruct the output \( z \) from the measured signal \( y \). The delay, \( (t-h) \), in the definition of the reconstruction error \( e(t) \), represents the available latency in the reconstruction problem. As usual, it is required that the closed loop system governing the propagation of state estimation errors, \( \Delta x = x - \hat{x} \), where \( \hat{x} \) is the state estimate, be stable. The purpose of design is to minimize the induced \( L_2 \) norm of the closed loop mapping \( T_{\text{wue}} : w \mapsto e \), and \( \gamma_{\text{opt}} \) stands, again, for the infimal attainable induced \( L_2 \) norm of that mapping.

The following standard hypotheses are counterparts of assumptions \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \), in the transposed system:
\[ \mathcal{A}_3: \text{ the pair } (A, C_2) \text{ is detectable;} \]
\[ \mathcal{A}_4: \text{ the realization } [A, B, C_2, D_2] \text{ has no invariant zeros on the imaginary axis and is right invertible}. \]

We continue with the introduction of counterparts of the notations used in Theorem 2.1. Those notations will be used
\(^4\)This can be justified by the form of the optimal solution in [14].
\[ \dot{x}(t) = A\dot{x}(t) + (BD_2 + (Ye^{\tilde{A}x_h}Y_te^{\tilde{A}x_h})C_2)(D_2D'_2)^{-1}\varepsilon(t), \]
\[ \dot{z}(t) = C_1\dot{x}(t-h) + \int_0^h (\dot{E}_r e^{\tilde{A}x_r} + C_4(r)e^{\tilde{A}(h-r)}Ye^{\tilde{A}x_h})C_2'(D_2D'_2)^{-1}\varepsilon(t-r)dr + D_1D'_1(D_2D'_2)^{-1}\varepsilon(t-h). \]  

in the statement of Theorem 2.2, below. Define the LQ-type observer ARE
\[ Y_xA' + Y_xB'B' - (Y_xC_2' + BD_2')(D_2D'_2)^{-1}(C_2Y_x + D_2B) = 0. \]  
As in the case of (2), by \(A_3\) and \(A_4\), the ARE (7) admits a stabilizing solution \(Y_x \geq 0\). Next we introduce the counterpart notations of II-A
\[ \tilde{A}_x := A - (Y_xC_2' + BD_2')(D_2D'_2)^{-1}C_2, \]
\[ \tilde{E}_x := Y_xC_2' + BD_2' - (Y_xC_2' + BD_2')(D_2D'_2)^{-1}D_2D'_1, \]
\[ G_o(t) := \int_0^t e^{\tilde{A}_x'}C_2'(D_2D'_2)^{-1}C_2e^{\tilde{E}_x'}dr, \]
\[ C_\tilde{x}(t) := C_1 - D_1D'_2(D_2D'_2)^{-1}C_2 - \tilde{E}_xG_o(t), \]
where \(\tilde{A}_x\) is Hurwitz. As in Section II-A, the notations \(G_o\) and \(C_\tilde{x}\) stand for \(G_o(h)\) and \(C_\tilde{x}(h)\), respectively. For \(\gamma > \| (I - D'_2D_2D'_2)^{-1}D_2D'_1 \|\) we shall use the notations
\[ Y_\gamma := Y - D_1(D'_1 - (I - D'_2D_2D'_2)^{-1}D_2D'_1)D'_1, \]
\[ \tilde{A}_\gamma := \tilde{A}_x + \tilde{E}_xY_\gamma^{-1}C_\tilde{x}, \]
and
\[ Q_\gamma := e^{\tilde{A}_\gamma}C_2'(D_2D'_2)^{-1}C_2e^{\tilde{A}_\gamma} - C_\gamma Y_\gamma^{-1}C_\gamma. \]

The main result concerning the smoothing problem is then formulated as follows:

**Theorem 2.2:** The following two statements are equivalent
1) \(\gamma > \gamma_{\text{opt}}\) in the fixed-lag smoothing problem
2) \(\| (I - D'_2D_2D'_2)^{-1}D_2D'_1 \| < \gamma\) and the ARE
\[ Y_\gamma \tilde{A}_\gamma + \tilde{A}_\gamma Y_\gamma - Y_\gamma Q_\gamma Y_\gamma + \tilde{E}_xY_\gamma^{-1}\tilde{E}_x = 0 \]  
admits a stabilizing solution \(Y_\gamma \geq 0\) (so that \(\tilde{A}_\gamma - Y_\gamma Q_\gamma\)

is Hurwitz).

Furthermore, if \(\gamma > \gamma_{\text{opt}}\), then one stable, strictly \(\gamma\)-suboptimal smoother \(\tilde{F}\) is given by equation (9) at the top of this page, where \(\varepsilon(t) := y(t) - C_\tilde{x}(t)\).

As in standard \(H^2\) and \(H^\infty\) results, Theorem 2.2 is a transposed (or conjugate) restatement of Theorem 2.1, and, as such, is an immediate corollary. To see that, note that the transposed (or reversed-time adjoint) of the system (6), with a selected \(\tilde{F}\), represents the closed loop input-out dynamics in a system of the form of (1), with a controller \(u = -\tilde{F}^Tw\). Transposition (or taking the adjoint) maintains the input-output induced norm as in (6). Since our analysis of (1), below, is based on an open-loop differential game, the necessary condition on the characterization of upper bounds, \(\gamma > \gamma_{\text{opt}}\), from Theorem 2.1, will thus remain valid, in the transposed system. It is easy to see that this is precisely the necessary condition of Theorem 2.2. Should that condition be satisfied, Theorem 2.1 provides (4) as one stabilizing, strictly \(\gamma\)-suboptimal compensator. Using the appropriate counterpart, in the transpose of (6), it is also easy to see that the resulting closed-loop mapping \(w \mapsto u\) is then realized by \(-\tilde{F}^T\), from (9). In short, Theorem 2.2 is indeed an immediate corollary of Theorem 2.1.

### III. Illustrative Example

To illustrate the proposed solution and its advantage over existing ones based on the standard \(H^\infty\) ARE (5), consider the preview tracking problem for the plant (1) with
\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]
\[ C = \begin{bmatrix} 1 - \alpha \\ 0 \\ 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \]
where \(\alpha \geq 0\). We are interested in studying the effect of the preview length \(h\) on the achievable \(H^\infty\) performance \(\gamma\).

Using Theorem 2.1, a standard \(\gamma\)-iteration is used to find the minimal achievable tracking performance \(\gamma_{\text{opt}}\), for each \(h\). The results for \(\alpha = 0.2\) are presented in Fig. 1. As seen, preview can improve tracking performance by up to a factor of almost two (from 1.75 in the preview-free problem to 0.8899 with \(h \approx 1.5\)). The optimal performance saturates as a function of the preview interval \(h\), in agreement with the analysis in [24]: the minimal level is achieved at \(h \approx 1.5\), and no improvement is achieved with higher values of \(h\).

For further reference we denote by \(\gamma_\infty\) the minimal achievable performance level for \(h \to \infty\) (though it can be achieved with a finite \(h\)). It is known [12], [14] that this quantity is the largest \(\gamma\) for which the Hamiltonian matrix associated with the standard \(H^\infty\) ARE (5) has \(j\omega\)-axis eigenvalues. It can be shown that in our case
\[ \gamma_{\infty} = \begin{cases} \gamma_a & \text{if } 0 \leq \alpha \leq \alpha_0 \\ \frac{\sqrt{2}}{2}(\alpha - 1) & \text{if } \alpha > \alpha_0, \end{cases} \]
where \(\alpha_0 \approx 1.885\) is the unique positive real solution of the equation \(\alpha^4 - 2\alpha^3 + 2\alpha - 3 = 0\) and
\[ \gamma_a^2 = \frac{1}{(\alpha - 1)(3 - \alpha) + 2\sqrt{2}(2 - \alpha)(\alpha^3 - \alpha + 1)}. \]

![Fig. 1. Tracking performance vs. the length of preview for \(\alpha = 0.2\)](image)
The solid line in Fig. 2 represents $\gamma_\infty$ as a function of $\alpha$. The area above this line therefore corresponds to the set of all $(\alpha, \gamma)$ combinations for which the tracking problem is solvable, for some $h \geq 0$. Yet while the solvability conditions of Theorem 2.1 are satisfied in that entire area, this is not the case for a solution in terms of the standard $H^\infty$ ARE (5), as in [12]. The latter fails at $\gamma_\infty$'s for which (5) has no stabilizing solution, represented in Fig. 2 by both the dashed and dot-dashed lines. While the dot-dashed line corresponds to the optimal preview-free ($h = 0$) performance $\gamma_0$, the dashed line represents merely a computational singularity in the criterion of [12]. It could certainly be argued (see [12, Remark 2]) that the singularity is insignificant, as it represents a $\gamma$ set of measure zero. Yet when the dashed “singularity” line in Fig. 2 is close to the solid line of $\gamma_\infty$ (especially when $\alpha \to 0$ and $\alpha \to 1.96$), this singularity is likely to give rise to numerical difficulties in reaching the correct value of $\gamma_\infty$. Clearly, this is an “academic” and a rather simple example, but it serves to illustrate the computational advantage of Theorem 2.1, where the only singularity is the one we seek, at $\gamma_\infty$.

IV. THE PROOF

Throughout the rest of the paper we use the following simplifying assumption:

$$\mathcal{A}_5: D'_2 \left[ \begin{array}{c} C \\ D_1 \\ D_2 \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right]$$

which means that $\|z\|_{L_2} = \|Cx + D_1w\|_{L_2}^2 + \|u\|_{L_2}^2$. This considerably reduces the complexity of required algebraic manipulations in the proof of Theorem 2.1. By standard procedure, $\mathcal{A}_5$ is imposed without loss of generality, via the change of variables:

$$u(t) \to (D'_2 D_2)^{-1} u(t) - (D'_2 D_2)^{-1} D'_2 (C x(t) + D_1 w(t - h)).$$

This transformation modifies the system coefficients as follows:

$$\begin{bmatrix} A & B_1 \\ C & D_1 \end{bmatrix} \to \begin{bmatrix} A & B_1 \\ C & D_1 \end{bmatrix} - \begin{bmatrix} B_2 \\ D_2 \end{bmatrix} (D'_2 D_2)^{-1} D'_2 \left[ \begin{array}{c} C \\ D_1 \end{array} \right],$$

$$\begin{bmatrix} B_2 \\ D_2 \end{bmatrix} \to \begin{bmatrix} B_2 \\ D_2 \end{bmatrix} (D'_2 D_2)^{-1}.$$
Substituting (12b) into (12a), terms in $K$ are eliminated, showing that the arbitrary selection of $K$ has no impact on the optimization. Leaving $K$ has the advantage of maintaining the boundedness of the (anti-causal) mapping $z^g \mapsto p$, hence of a linear operator $p^\theta(x(0), \bar{w}_0, w)$. Using $\mathcal{A}_S$, the optimal control is then of the form
\[ u^\theta = -B_2^* p^\theta. \] (13)

This leads to the following Hamilton-Jacobi system, whose unique $L_2$ solution is the solution of the optimal control problem
\[ \begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} A & -B_2^* B_2' \\ -C' C & -A' \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} + \begin{bmatrix} B_1 \\ -E' \end{bmatrix} d_hw. \] (14)

In the homogeneous case (where $w$ vanishes throughout) the state and co-state of the $L_2$ solution are related via $p = X_x x$, where $X_x \geq 0$ is the stabilizing solution of the ARE (2). One common and very useful observation concerning the inhomogeneous case is obtained by integrating $\frac{dx}{dt}(x, p)$ under optimal control. Invoking (13) and the two equations in (14) for mutual cancellation of most terms, this leads to
\[ \|z^g\|^2_{L_2(0,t_1)} = -\langle x, p \rangle^{(1)}_{t_1} + \{d_hw, B_1 p^\theta + D_1 z^g\}_{L_2(0,t_1)}. \] (15)

A trick facilitating an explicit solution of the inhomogeneous (14) is based on a change of variables $\zeta = p - X_x x$. Invoking (2), it transforms (14) into a cascade of an anti-causal anti-stable system, in $\zeta$, followed by a stable, causal system, in $x$:
\[ \begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} A & -B_2^* B_2' \\ 0 & -A' \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} + \begin{bmatrix} B_1 \\ -E' \end{bmatrix} d_hw. \] (16)

This formulation makes transparent the boundedness of the mapping from the data $(x(0), \bar{w}_0) \in M_2$ and $w \in L_2$ to the combined state in (16) (equiv., in (14)) and the decay to zero of the combined state, as $t \to \infty$.

Yet another consequence of (16) is that the mapping $w \mapsto (x, \zeta)$ is strictly proper. In particular, with the zero initial data, if $w(t)$ is selected as a sinuousoid over a compact support, increasing its frequency indefinitely will result with $\|z^g(0,0, w)\|_{L_2} \approx \|D_1 w\|_{L_2}$. This establishes, in the current setting, the fact that $\gamma_{\text{opt}} \geq \|D_1\| = \sigma_{\max}\{D_1\}$, which is a basic observation in the standard $H^\infty$ problem. Using the notations of §(II-A), $\Gamma_\gamma > 0$ for $\gamma > \gamma_{\text{opt}}$.

2) The solution over $[0,h]$ The selection and optimization of $w$ comes into effect only over the positive ray $[h, \infty)$. Therefore, there is an obvious interest in the evolution along the initial interval $[0, h]$, and in its contribution to the total value of (10). The analysis begins by highlighting the distinction between the contribution of initial data and that of the selected $w$. Indeed, along that interval
\[ \zeta(t) = e^{A_r (h-t)} \zeta(h) + \int_0^h e^{A_r (r-t)} E_x' w(r-h) dr \] (17a)
with the boundary condition
\[ \zeta(h) = \int_0^\infty e^{A_r s} E_x' w(s) ds. \] (17b)

Consequently, by direct computation
\[ \begin{align*}
    x(t) &= e^{A_r t} x(0) - G_c(t) e^{A_r (h-t)} \zeta(h) \\
    &\quad + \int_0^h e^{A_r (t-r)} (B_1 - G_c(r) E_x') w(r-h) dr \\
    &\quad - G_c(t) \int_t^h e^{A_r (r-t)} E_x' w(r-h) dr,
\end{align*} \] (18)
where $G_c(t)$ is the Gramian defined in §(II-A). In particular
\[ \zeta(h) = \xi(h) - G_c \zeta(h), \] (19)

where
\[ \begin{align*}
    \xi(h) &= e^{A_r h} x(0) \\
    &\quad + \int_0^h e^{A_r (h-r)} (B_1 - G_c(r) E_x') w(r-h) dr \\
    &\quad = e^{A_r h} x(0) + \int_0^h e^{A_r (h-r)} B_1 w(r-h) dr \] (20)

 captures the effect of initial data, and $\zeta(h)$ captures the contribution of the selection of $w(t)$ for $t > 0$.

3) The optimal $w$: The computation of an optimal $w$ in (10) is performed as part of the proof of necessity, in Theorem 2.1. That is, here we assume that $\gamma > \gamma_{\text{opt}}$. This assumption means that there exists $\epsilon > 0$ and an admissible, complete information control policy, subject to which the following equality holds for all $w \in L_2$ and with the zero initial data
\[ \gamma^2 \|w\|^2_{L_2} - \|z^g(0,0, w)\|_{L_2}^2 \geq \|w\|^2_{L_2}. \] (21)

This inequality can only be sharpened if the closed loop response $z$ is replaced by the optimal response $z^g(0,0, w)$. That is
\[ \gamma^2 \|w\|_{L_2}^2 - \|z^g(0,0, w)\|_{L_2}^2 \geq \epsilon \|w\|_{L_2}^2. \] (22)

We use this fact to appeal, once again, to Theorem 4.1. Here we identify $w \in L_2$ with the optimization variable “$u \in U$”; the space $V$ will be identified with $L_2 \times L_2$, containing pairs $(w, z^g)$; the operator $S\bar{w} = (w, z^g(0,0, w))$ accounts for the contribution of $w$ to such pairs; we identify “$w$” with $(0, z^g(0,0, w))$, which is the contribution of the initial data; finally, we set $J = d\int_0^h [\gamma^2 J, -I]$. Under these definitions, (22) means that $S'JS \geq \epsilon I$, satisfying the condition of Theorem 4.1. The conclusion is that there is a unique optimal $w^*(x(0), \bar{w}_0)$, in (10). For later use we introduce the notations $x^*(x(0), \bar{w}_0) = x^g(x(0), \bar{w}_0, w^*)$, $u^*(x(0), \bar{w}_0) = u^g(x(0), \bar{w}_0, w^*)$, and $p^*(x(0), \bar{w}_0) = p^g(x(0), \bar{w}_0, w^*)$, and note that $w^*$, $x^*$, $u^*$, $z^*$, and $p^*$ are all bounded linear operators from $M_2$ to $L_2$. By Theorem 4.1, $w^*$ is completely characterized by the condition $S'J(Sw^* - v) = S'J(w^*, z^*) = 0$. Our next task is to derive an explicit interpretation of this condition.

The computation of $S'$ is cumbersome, and we avoid the need to do so, using a trick from [25], [29]. Again, let $u = -Kx$ be a stabilizing feedback control. Define a bounded operator $\tilde{S}w = (w, \tilde{z})$, where $\tilde{z}$ is the response to $w$ and the zero initial data in $[A - B_2 K, B_1 d_2, C - D_2 K, D_1 d_2]$, and, in these notations, let $\tilde{z}_w = z^g(0,0, w) - \tilde{z}$. We return for a moment to the discussion of the optimization in $u$, using the
fact that \( z^* = z^b(x(0), \tilde{w}_0, w^*) \) is the LQ optimal output, given \( w^* \) and the initial data. By Theorem 4.1, this implies that \( z^* \) is orthogonal in \( L_2 \) to the response of \([A, B_2, C, D_2] \) to any admissible control and the zero initial state. One such response is \( \delta z \), hence \( \langle \delta z, z^* \rangle_{L_2} = 0 \). Now, we can interpret \( S' J(w^*, z^*) = 0 \) as follows: for any \( w \in L_2 \)

\[
0 = (Sw, (J(w^*, z^*))_{L_2} \times L_2
= \gamma^2(w, w^*)_{L_2} - (\delta z(0, 0, w), z^*)_{L_2}[b, \infty)
= \gamma^2(w, w^*)_{L_2} - (\delta z, z^*)_{L_2}[b, \infty) - \gamma^2 \delta z, z^*)_{L_2}[b, \infty)
= \gamma^2(w, w^*)_{L_2} - (\delta z, z^*)_{L_2}[b, \infty) - \gamma^2 \delta z, z^*)_{L_2}[b, \infty)
= (\tilde{S}w, (J(w^*, z^*))_{L_2} \times L_2
That is, the unique optimal \( w^* \) is completely characterized by the condition \( S' J(w^*, z^*) = 0 \). The explicit realization of this condition is in terms of an anti-causal system whose state equation is identical to, and thus coincides with, that in (12):

\[
\dot{p} = -(A - B_2K)'p - (C - D_2K)'z^*, \quad i > h.
\]

(23a)

(23b)

In particular, since (23a) coincides with (12a), the equality (12b) remains satisfied, and \((A - B_2K)' \) and \((C - D_2K)' \) can be replaced by \( A' \) and \( C' \), in the state equation.

One observation, based on (15) and (23b), provides an expression for the component of the value of the game due to the evolution over \([b, \infty)\)

\[
\|z^*\|_{L_2[b, \infty]} - \gamma^2\|w^*\|_{L_2}^2 = (x(h), p(h)).
\]

(24)

In closing §IV-A.1 we verified that the matrix \( \Gamma_y \) is positive definite, hence invertible, when \( \gamma > \gamma_0 \). Thus, (23b) and \( A_S \) imply that

\[
d_h w^* = \Gamma_y^{-1}(D'_1Cx + B'_1p) = \Gamma_y^{-1}(E_\infty x + B'_1z).
\]

(25)

which should hold for all \( i > h \). Over the ray \([b, \infty)\), the solution of (10) is therefore characterized by the unique \( L_2 \) solution of a homogeneous Hamilton-Jacobi system of a standard form

\[
\begin{bmatrix}
\dot{x}
\dot{p}
\end{bmatrix} = \begin{bmatrix}
\tilde{A}_y & B_1' - B_2B_2'
-C'\Delta_y & -A_y
\end{bmatrix}
\begin{bmatrix}
x
p
\end{bmatrix},
\]

(26)

where \( \tilde{A}_y := A + B_1\Gamma_y^{-1}D'_1 \) and \( \Delta_y := I + D_1\Gamma_y^{-1}D'_1 \).

Returning to the familiar, standard case, when \( h = 0 \) (i.e., with no preview), the initial state \( x(0) = x(h) \) in (1) can be arbitrarily assigned. In particular, each \( x(0) \in \mathbb{R}^n \) is associated with a unique \( p(0) \in \mathbb{R}^n \), so that the ensuing solution of (26) is in \( L_2 \). The linear dependence of \( p(0) \) on \( x(0) \) leads to the existence of a matrix \( \tilde{X}_y \), such that \( p = \tilde{X}_y x \). It is a simple procedure then to show that \( \tilde{X}_y \) is a stabilizing, self adjoint solution of the associated ARE

\[
\tilde{X}_y \tilde{A}_y + \tilde{A}_y' \tilde{X}_y + C'\Delta_y C
+ \tilde{X}_y (B_1\Gamma_y B'_1 - B_2B_2') \tilde{X}_y = 0.
\]

(27)

Moreover, (24) then means that the optimal value of the game is \( (x(0), \tilde{X}_y x(0)) \). Since \( w = 0 \) is one viable option in the search for an optimal \( w \), the conclusion is then that the optimal value of (10) is non-negative, meaning that \( \tilde{X}_y \geq 0 \). Thus, the existence of a positive semidefinite, stabilizing solution of (27) becomes a necessary condition for the sub optimality of \( y \).

This chain of arguments fails in the preview problem, at two critical points. First, as seen in (19), \( x(h) \) is determined by both the initial data and the optimization of \( w \). There is therefore no a priori assurance that any \( x(h) \in \mathbb{R}^n \) is an initial state in an \( L_2 \) solution of (26). Second, even if a stabilizing solution of (27) does exist, then \( (x(h), \tilde{X}_y x(h)) \) is only one component of the optimal value of the game, and the positivity requirement applies only when the contribution of the evolution along \([0, h]\) is included. The ARE (27) must therefore be replaced by an alternative equation in a characterization of \( y > y_0 \). In preparation, computing the optimal value of the game and deducing a positivity condition in that context, is our next task.

4) The optimal value of (10): Our starting point remains (15) and the fact that (23a) must be satisfied for \( i > h \). Setting \( t_0 = 0 \) and letting \( t_1 \to \infty \) in (15), we have

\[
J_y = \|z^*\|_{L_2[b, \infty]} - \gamma^2\|w^*\|_{L_2}^2 = (x(0), p(0)) + (d_h \tilde{w}_0, B'_1p + D'_1z^*). \]

(28)

An explicit expression for (28) is computed, using the derivations of §IV-A.2. To begin with,

\[
(x(0), p(0)) = \left\{ \begin{array}{c}
(x(0), X_\infty x(0) + e^{A_\infty h}z(h)
\end{array} \right.
\]

\[
+ \int_0^h e^{A_\infty (r-h)} E_\infty w(r-h)dr.
\]

(29)

Next, in analogy to the derivation of (25), one obtains

\[
(d_h \tilde{w}_0, B'_1p + D'_1z^*). \]

\[
= (d_h \tilde{w}_0, B'_1p + D'_1(Cx - D_2B_2p + D_1d_h \tilde{w}_0)). \]

\[
= (d_h \tilde{w}_0, B'_1p + D'_1(Cx + D_1d_1d_h \tilde{w}_0)). \]

(30)

Using the explicit expressions for \( z \) and \( x \), in (17) and (18), and also the definition of \( z(h) \), in (20), the combined expression for the cost is now computed as shown in equation (31) at the top of this page.

One basic fact revealed by this equality is that the contribution of \( w^* \) to \( J_y \) is captured by the single term \( \left\langle \xi(h), z(h) \right\rangle \). In particular, if \( (x(0), \tilde{w}_0) \) determine \( \xi(h) = 0 \), then the contribution of \( w^* \) to the game’s value is zero. That is, then the game’s optimal value is also achieved with \( w \equiv 0 \), hence \( \xi(h) = 0 \), \( x(h) = 0 \), \( u^*(t) = 0 \) for \( t > h \), and finally, \( x^*(t) = 0 \) for \( t > h \). The uniqueness of the optimal solution of (10) thus implies that, indeed, if \( \xi(h) = 0 \) then \( w^* = 0 \), \( z(h) = 0 \), and that \( u^*(t) = 0 \) and \( x^*(t) = 0 \), for \( t > h \).
Obviously, the collection of all possible quintuples \((\xi(h), \zeta(h), x^*, w^*, u^*)\) forms a linear manifold, parameterized by the initial data. The observation above thus demonstrates that \(\zeta(h), w^*, \) and the restrictions of \(x^*\) and \(u^*\) to \([h, \infty)\), depend linearly on \(\xi(h)\). In particular, there exists a matrix \(X_\gamma\) such that \(\xi(h) = X_\gamma \xi(h)\).

Should the optimization procedure be re-initiated at any time \(t > 0\), with the initial data \((x^*(t), w^*_t)\), it is clear that the solution over \([t, \infty)\) must coincide with the solution of the original problem. In particular, the relation \(\xi(t + h) = X_\gamma \xi(t + h)\) should prevail for all \(t > 0\), where \(\xi(t + h)\) is defined by \((x^*(t + h), w^*_t)\), as in (20).

5) An ARE for \(X_\gamma\): Solutions of LQ optimization problems, including both traditional optimal control, \(H_\infty\) and LQ differential game problems, are characterized by Hamilton Jacobi systems. Associated Riccati equations are used to characterize \(L_2\) solutions, as well as optimal feedback gains. The common situation is that where the state and co-state equations of the Hamilton Jacobi system are derived directly from the state equation of the original system, and that of its adjoint, as is the case in (14) and in (26). In the current problem, however, we have already noted the limited value of (26) and the associated ARE (27). Here we look for an alternative, where the roles of the state and co-state are played by the trajectories of \(\xi\) and \(\zeta\). The rationale is that the state of the Hamilton Jacobi system typically captures the contribution of past inputs / initial data, whereas the co-state captures the contributions of future inputs, which have been shown to be the respective roles of \(\xi\) and \(\zeta\).

The transformation from the state and co-state of (26) to the desired state and co-state is via

\[
\begin{bmatrix}
\xi \\
\zeta
\end{bmatrix} =
\begin{bmatrix}
I & G_c \\
0 & I
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
-X_c & I
\end{bmatrix}
\begin{bmatrix}
x \\
p
\end{bmatrix}.
\]

It is a somewhat laborious but straightforward algebra that transforms (26) into the following, equivalent system

\[
\begin{bmatrix}
\dot{\xi} \\
\dot{\zeta}
\end{bmatrix} =
\begin{bmatrix}
A_{\gamma} & -R_{\gamma} \\
-E'_x F^{-1} E_x & -A'_x
\end{bmatrix}
\begin{bmatrix}
\xi \\
\zeta
\end{bmatrix}.
\]

where, again, the definitions of §II-A are used. Our conclusion, in the context of (33), are summarized as follows: Since (20) allows to assign an arbitrary value to \(\xi(h)\), the HJB system (33) admits a unique \(L_2(h, \infty)\) solution for any selection of \(\xi(h)\). That solution is determined by the linear relation \(\zeta = X_\gamma \xi\). The fact that the ARE (3) is satisfied by the matrix \(X_\gamma\) in the relation \(\zeta = X_\gamma \xi\), readily follows from equating \(\zeta = X_\gamma \xi\) in the two equations, in (33). Since the evolution of \(\xi\) is generated by \(A_{\gamma} := A_\gamma - R_\gamma X_\gamma\), we conclude that \(A_{\gamma}\) is stable. To see that \(X_\gamma\) is self adjoint we rewrite (3) as a Lyapunov equation with a stable “A” matrix and a self adjoint free term

\[
X_\gamma A_{\gamma} + A'_x X_\gamma + X'_\gamma R_\gamma X_\gamma + E'_x F^{-1} E_x = 0.
\]

To see that \(X_\gamma\) is actually positive semidefinite, we consider (10) with \(\bar{w}_0 = 0\). Then the expression (31) for the optimal value reduces to \(f(t) = (x(t), \bar{w}_1) \in M_2\). Such models have been documented extensively (see [22], [30] and references therein), and have been used in the solution of \(H_\infty\) problems of delay systems (see, e.g., [17], [18] and the review [19]). We shall therefore be content with a brief review of the appropriate model and solution, for the preview problem. Details of technical aspects of treating a distributed system with unbounded input and output coefficients can be found in the references cited right above.

The abstract model is of the form

\[
\begin{align*}
\dot{f} &= A f + B_1 w + B_2 u, \\
\dot{z} &= C f + D_2 u,
\end{align*}
\]

where \(A\) is an infinitesimal generator of a \(c_0\)-semigroup over \(M_2\), defined by

\[
\mathcal{A}(\eta, \phi) = (A_\eta + B_1 \mathbb{I}(-\phi), \mathbb{I} \psi(\phi)),
\]

\[
D(\mathcal{A}) = \{ (\eta, \phi) : \phi(\theta) = \int_0^\theta \psi(\sigma) d\sigma, \psi \in L_2[-h, 0] \}.
\]

The other coefficients in (35) are defined via

\[
\begin{align*}
B_1 w &= (0, \delta_0(\cdot)) w, \\
B_2 u &= (B_2 u, 0),
\end{align*}
\]

\[
\mathcal{C}(\eta, \phi) = C \eta + D_1 \phi(-\cdot),
\]

\[
D_2 u = D_2 u,
\]

where \(\delta_0(\cdot)\) is Dirac’s delta function, centered at \(\theta = 0\).

The advantage in this formulation is that (35) involves no delay, whereby the well known and relatively simple formalism of the Riccati equation based, state space solution of the standard \(H_\infty\) problem applies [29], [31], [32]. For completeness we restate here the complete information result.

Theorem 4.2: The following two statements are equivalent.

1) \(\gamma > \gamma_{\text{opt}}\) in (35)

2) There exists a bounded, positive semidefinite operator \(X\) over \(M_2\) such that it is a weak solution of the operator Riccati equation

\[
X \mathcal{A} + \mathcal{A}' X + X \mathcal{C} \left( \frac{1}{2} B_1 B_1' - B_2 B_2' \right) X + C C' = 0
\]

and the generator \(\mathcal{A}_X = \mathcal{A} + (\frac{1}{2} B_1 B_1' - B_2 B_2') X\) gives rise to a uniformly exponentially stable \(c_0\)-semigroup over \(M_2\).

That solution of (36) is completely characterized by the fact that the optimal value of the game (10) is given by the quadratic form \(\langle x(0), \bar{w}_0\rangle X(x(0), \bar{w}_0) \rangle M_2\). Furthermore, assume that indeed, \(\gamma > \gamma_{\text{opt}}\) and \(X\) is as above. Then

\[
\dot{u}(t) = -B_2 X(x(t), \bar{w}_1)
\]

is a stabilizing, strictly \(\gamma\)-suboptimal control policy.

It is worth stressing that, by the structure of \(B_2\), the control law in (37) depends only upon the first \(\mathbb{R}^n\) “row” block of \(X\). This fact will be exploited in the derivation of the control law (4).
C. The proof of sufficiency

In this section it is assumed that \( \Gamma > 0 \) and that a stabilizing solution \( X_Y \geq 0 \) exists in the ARE (3). Our goal is to establish the fact that \( y > y_{\text{opt}} \) and that (4) is a stabilizing, strictly \( y \)-suboptimal control policy. The proof utilizes the interplay between (1) and the abstract model (35).

Specifically, let \( X \) be defined as the self adjoint operator over \( M_2 \), that serves as the kernel for the quadratic form (31) (we leave out the obvious details). Thus, the value of (31) is \( \langle (x(0), \tilde{w}_0), X(x(0), \tilde{w}_0) \rangle \), for any \((x(0), \tilde{w}_0) \in M_2\). We intend to show that \( X \) is the sought positive semidefinite, stabilizing solution of the operator Riccati equation (36), and appeal to Theorem 4.2. Since similar arguments were made in the articles cited earlier on LQ and \( H^\infty \) solutions in systems with control delay, we shall be content with a brief outline.

Indeed, let \( X_Y \) be the assumed solution of (3), fix initial data \((x(0), \tilde{w}_0) \in M_2\), let \( \xi(h) \) be defined by (20), let \( \xi(t) = e^{A_t - B_A t} \xi(h) \), \( \xi(t) = X_Y \xi(t) \), \( x(t) = \xi(t) + G_c \xi(t) \) and \( p(t) = X_x \xi(t) + \xi(t) \), for \( t \geq h \). Define \( \xi(t) \) and \( x(t) \), \( t \in [0, h] \), as in (17) and (18), respectively, and \( p(t) = X_{x \xi}(t) + \xi(t) \), over that interval, as well. Finally, let \( w \) and \( u \) be defined by the feedback formulae (25) and (13). Backtracking our computations heretofore, it follows that \( x, w \) and \( u \) are \( L_2 \) trajectories, satisfying (1). Moreover, with these selections, the associated value of \( \| z \|^2_{L_2} - y^2 \| w \|^2_{L_2} \) is given by (31), meaning that it is equal to \( \langle (x(0), \tilde{w}_0), X(x(0), \tilde{w}_0) \rangle \), albeit, so far, without any claim to optimality. Deriving the explicit form of \( X \) from (31) (with \( \xi(h) = X_Y \xi(h) \)), it is a matter of straightforward computation to verify that the combined equalities (25) and (13) are equivalent to \( w(t) = \frac{1}{\gamma} B'_L X(x(t), \tilde{w}_t) \) and \( u(t) = -B_L X(x(t), \tilde{w}_t) \). Therefore, the trajectory \( f(t) = (x(t), \tilde{w}_t) \) is generated by \( \mathcal{A}_X \). Let \( S_X(t) \) be the associated \( \epsilon_0 \) semigroup over \( M_2 \). That is, \( S_X(t) \) is defined by the relation \((x(t), \tilde{w}_t) = S_X(t)(x(0), \tilde{w}_0) \), where \( x \) and \( w \) are the trajectories defined above. Thus

\[
\langle f(t), X f(t) \rangle = \int_0^\infty \langle S_X(t) f(0), (C C + X(B' B_2' - \frac{1}{\gamma^2} B'_L B'_L)) X \rangle dt \times \langle S_X(t) f(0) \rangle dt. \tag{38}
\]

This last equation is equivalent to the fact that \( X \) is a weak solution of (36). The fact that \( X \) is a stabilizing solution then follows directly from the fact that \( X_Y \) is a stabilizing solution of (3) and the subsequent exponential decay of all the defined trajectories, relative to \( \| x(x(0), \tilde{w}_0) \|_{M_2} \). The fact that \( X \) is self adjoint follows from its definition, in (31), and the fact that \( X_Y = X' \geq 0 \).

We still need to establish that \( X \geq 0 \). Indeed, consider now (35) with \( u(t) = 0, t > 0 \), under the feedback policy (37). In analogy to the cited distributed parameters references, an integration by parts argument, extended to (35), yields

\[
\langle f(t), X f(t) \rangle = \langle f(t), X f(t) \rangle + \| x \|^2_{L_2(0, t)} \gamma^2 \| w \|^2_{L_2(0, t)} \tag{39}
\]

for any \( t \in (0, \infty) \), where \( w^\Delta = \frac{1}{\gamma^2} B'_L X f \). Since \( w \equiv 0 \), for \( t > h \) we have \( \tilde{w}_t = 0 = \langle f(t), X f(t) \rangle M_2 = \langle x(t), (X_x + e^{A_t - B_A t} X_x) X f(t) \rangle_{M_2} \geq 0 \). In particular, then the right hand side of (39) comprises three non negative terms. Thus, the left hand side is non negative, for any selection of \( f(0) \in M_2 \), and \( X \) must be positive semidefinite, as claimed.

Having established these properties of \( X \), the conclusion in Theorem 4.2 is that, indeed, \( y > y_{\text{opt}} \) and (37) is a stabilizing, strictly \( y \)-suboptimal feedback. Again, it is straightforward to verify that this feedback is equivalent to (4), in (1). The proof of Theorem 2.1 is complete.

V. Concluding remarks

We have addressed continuous-time \( H^\infty \) control and observation problems for systems with preview (preview tracking and fixed-lag smoothing). Using the game-theoretic approach, complete necessary and sufficient solvability conditions and corresponding sub-optimal solutions have been derived. The solvability conditions are formulated in terms of two, \( H^2 \) and \( H^\infty \), ARE’s having the same dimension as the delay-free plant. The advantages of the proposed solutions are that (a) they do not involve potentially ill-conditioned equations and (b) the resulting solutions remain well-posed when the preview (smoothing lag) increases.

While results have been formulated in a time-invariant, infinite-horizon setting, all our arguments can be extended, (almost) mutatis mutandis, to time-varying and/or finite-horizon systems.

References
