$H_\infty$ Control of System With I/O Delay: A Review of Some Problem-Oriented Methods

LEONID MIRKIN
Faculty of Mechanical Engineering, Technion — IIT, Haifa 32000, Israel

GILEAD TAMOR
Electrical & Computer Engineering, Northeastern University, Boston, MA 02115
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Systems with input or output delays form the simplest, and yet one of the most widely applied classes of distributed parameter models. This is a review of some problem-oriented $H_\infty$ methods for that class, with an emphasis on computational simplicity. Reviewed methods include operator interpolation, game-theoretic state-space treatments in the time domain, a $J$-spectral factorization approach, and methods exploiting various ideas from sampled-data theory. Some interesting properties of the (sub)optimal solutions are discussed as well.

1. Introduction

This article reviews several $H_\infty$ methods for rational systems cascaded with input or output delays, see Fig. 1. Those are perhaps the simplest distributed parameter systems, and yet the most widely used as models in industrial applications. Hence the interest in methods developed specifically for that class. In view of the voluminous literature on robust distributed systems design, articles cited here are intended merely as samples, and by no means provide an exhaustive, or even a representative list.

Historical perspectives. Solutions of $H_\infty$ problems in distributed systems trace back to the mid 1980’s. Some early work [21, 17, 18, 64, 82] appealed to operator interpolation/commutant lifting theorem methods [61, 63]. One major outgrowth,

\[ e^{-\tau s} \]

Fig. 1. General LFT setup with single delay
with later impact on nonlinear robust control, was the development of the Skew-Toeplitz methodology [20, 4, 36, 52, 51, 53, 49, 19]. Yet another approach followed early on, was the use of reduction to a finite dimensional (rational) problem [10, 33, 24, 25, 56, 83, 29]. The introduction of time domain/game theoretic state space methods [66, 59, 80, 81, 68, 3] was immediately followed by extensions to a general framework of distributed parameter systems [67, 9, 70, 76, 32, 37]. A feature that facilitates design in systems with input (or output) delays is that their distributed component has very simple dynamics (essentially, an FIR). Indeed, in “classical” and some robust stabilization problems, this feature allows a reduction to a purely rational problem [35, 1, 71], using a Smith predictor.

**Scope and organization of the paper.** The common thread in the several reviewed methods is the utilization of the special structure and time domain representations of systems with I/O delay. The review does not cover the use of tools of broader scope, such as low order approximations, or general factorization techniques, as well as a variety of interesting ideas leading to simplified transform-domain solutions (such as elegant reproducing kernels methods [16], or the simplified computation of Hankel singular values [50], which requires an inner/outer factorization).

Some early work meet the set criterion in the context of operator interpolation methods and will be discussed in Section 2. Here, time domain analysis is used to characterize the interpolation conditions for one-block problems. Sections 3 and 4 are devoted to time- and frequency-domain treatments of the “standard problem” in Fig. 1. §3.1 reviews the use of a “lifting” trick, borrowed from work in sampled data control. It transforms the delay problem to a discrete time problem with distributed I/O operators, but with no delays. The game theoretic solution leads to a periodic, time varying compensator. Realizing that periodicity is clearly a technical artifact of “lifting”, immediate followups - reviewed in §3.2 - removed the resulting time variation. Here one considers the interplay between abstract evolution models over the product space $M_2 = \mathbb{R}^n \times L_2[-\tau, 0]$ and the original delay equation, in the associated differential games. A J-spectral factorization based approach is reviewed in §4.1. Here the Smith predictor trick is used to reduce the problem to an equivalent rational problem. The extraction of dead-time controllers from delay-free parameterizations is discussed in §4.2. This approach also has its roots in the sampled-data theory and leads to a simple and intuitively appealing solution. Finally, the controller structure and the achievable cost in the $H_\infty$ delay problems are discussed in Section 5.

**Preliminaries and notation.** Given an LTI system $G(s) = C(sI - A)^{-1}B$, its conjugate is defined as $G^- = -B'(sI - A')^{-1}C'$. The completion of $e^{-\tau s}G$ to $[0, \tau]$ is defined as follows: $\pi_\tau\{e^{-\tau s}G\} = \int C(e^{-\tau s} - e^{-s I})(sI - A)^{-1}B$. It can be seen that $\pi_\tau\{e^{-\tau s}G\}$ is an entire function whose impulse response has support on $[0, \tau]$ (FIR system).
2. Time-domain methods in operator interpolation

Arguably the most significant early development in robust $H_\infty$ control was the connection made with operator interpolation/commutant lifting problems [61, 63], involving the following general setting. Given are $w, m \in H_\infty$, with $m(s)$ inner. One defines a subspace $K := H_2 \odot m H_2$, the orthogonal projection $\pi : H_2 \mapsto K$, and an operator $T : K \mapsto K$, via $Tf = \pi(wf)$, $f \in K$. A function $w_0 \in H_\infty$ is said to interpolate the operator $T$ if it satisfies: (i) $Tf = \pi(w_0f)$, $f \in K$, and (ii) \( \|T\| = \|w_0\|_\infty = \|A_{w_0}\| \), where $A_{w_0} : H_2 \mapsto H_2$ is the multiplication operator associated with $w_0$. In the context of control design, “$m$” is usually the inner part of the open loop plant, whereas “$w$” is a frequency shaping weight function [14, 22]. In the interest of simplicity, $w$ can be selected rational, yet non-rational, non-minimum phase elements in a distributed plant, such as delays, cannot be removed from $m$. The following is a key result -

\textbf{Lemma 1} ([61, Proposition 5.1]) Suppose that $f \in K$ is a maximal function for the operator $T$. Then $w_0 = Tf/f \in H_\infty$ is an all-pass function, and is the unique solution of the interpolation problem.

The main solution steps are thus the characterization of $K$, $\pi$, and $T$. Early case studies of distributed systems [21, 17, 18, 82, 25] concerned the example of scalar systems with several, commensurate input delays, and neither an inner/output plant factorization, nor a characterization of $K$ in terms of the plant’s inner part, would then be as simple. The solution outlined in [65, 64] utilizes the fact that the plants outer part has no effect on the definition of $K$, and on Fredholm’s alternative, whereby $K = H_2 \odot e^{-Ts}H_2$ is isometric $L_2(0, \tau), \pi : L_2[0, \tau] \mapsto L_2[0, \tau]$ is the truncation and $T$ is the restriction to $[0, \tau]$ of convolution with the inverse Laplace transform of $w$. Thus, $T$ is realized by a causal ordinary differential equation (ODE), its adjoint has an anti causal ODE realization, and Schmidt pairs of $T$ are solutions of a Hamiltonian, two point boundary value problem. Once that observation is made, the step leading to a solution based on transcendental equation characterization of Schmidt pairs, is immediate.

The pleasant feature of an inner distributed component fails in the next simplest example of a scalar system with several, commensurate input delays, and neither an inner/output plant factorization, nor a characterization of $K$ in terms of the plant’s inner part, would then be as simple. The solution outlined in [65, 64] utilizes the fact that the plants outer part has no effect on the definition of $K$, and on Fredholm’s alternative, whereby $K = H_2 \odot e^{-Ts}H_2 = N(A_\ell')$, with $g$ being the plant’s transfer function. The main point is that the adjoint multiplication operator, $A_\ell'$, has a system realization and can be characterized in terms of that realization. In a plant with multiple commensurate input delays and an outer rational part, it suffices to consider $g$ as the delay polynomial. Then $K$ is the stable component of the solution space of a difference equation and is isomorphic to initial $L_2[0, l\tau]$ portions of trajectories. A new challenge is to define the right Hilbert space structure in $K$, to make this isomorphism an isometry, as is needed in order to compute the orthogonal projection $\pi$, and adjoint of operators over $K$, as follows.

Let $S$ be the $l\tau$-time units shift along trajectories in $K$. Identifying trajectories $f \in L_2[0, \infty) \mapsto \{f^i\}_{i=0}^{\infty} \in \ell_2(L_2([0, \tau], \mathbb{R}^l))$, where $f^i(s) = [f((i\tau + s), f((i\tau + 1)s), \ldots, f((i+1)\tau + s)]^T$, $s \in [0, \tau], i = 0, 1, \ldots$, the shift $S$ admits a
representation by an \( l \times l \) matrix \( E \) with eigenvalues magnitude smaller than one. Let \( Q > 0 \) solve \( Q - E'QE = I \). Then, inner products between trajectories in \( K \) satisfy

\[
\langle x, y \rangle_{L^2([0, \infty))} = \sum_{k=0}^{\infty} \langle x^k, y^k \rangle_{L^2[0, r]} = \sum_{k=0}^{\infty} \langle E^k x^0, E^k y^0 \rangle_{L^2[0, r]} = \langle x^0, Qy^0 \rangle_{L^2[0, r]}
\]

The \( Q \)-weighted \( L^2([0, r], \mathbb{R}^l) \) inner product thus defines the isometric representation of \( K \) by \([0, r] \) trajectories. The projection \( \pi : L^2[0, \infty) \mapsto K \) is then \([\pi f]^t = E^t Q^{-1} \sum_{k=0}^{\infty} E^k f^k \). The operator \( T = \pi A_{w}|_K \) is the compression of a convolution operator; here too, it admits a causal ODE realization over \([0, r] \), albeit subject to certain non-trivial (integral) boundary conditions, its adjoint \( T' \) has an anti causal ODE realization and the equation \((\sigma^2 I - T'T) f = 0 \) amounts to a two point boundary value problem. Thus singular values are again solutions of a transcendental equation and Schmidt pairs are derived from algebraic conditions on admissible boundary values.

Stressed in closing is the key step of utilizing the time domain structure of the delay operator to provide a time domain realization of the interpolation space \( K \) as the solution space of a difference equation. This leads to a problem reduction to a solution of a two point boundary value problem.

3. Time-domain methods

By the late 1980's, state space / Riccati equations methods became dominant, beginning with the finite dimensional LTI case \([2, 15]\), continuing with time varying (LTV) / finite time systems, \([67, 70, 3]\) and eventually allowing extensions to distributed systems. Practical impediments in the latter include the need to solve infinite dimensional operator Riccati equations, and compensator realizations in terms of abstract evolution models. Here we review methods utilizing the structure of systems with a pure input lag, to address both issues: operator Riccati equations are reduced to fixed size matrix equations and explicit delay system compensator realizations are provided. Early developments are reported in \([3]\). The focus of this review is on the subsequent results in \([46, 47]\), followed by \([34, 72, 74, 73, 71]\).

3.1 A sampled data method: matrix Riccati eqs. and “periodic” solutions

Several control and observation problems are considered in \([47]\), and reduced (using standard unitary loop transformations) to a single representative, akin to the “standard problem” in the context of Figure 1. Roughly, the idea here is to break the basic problem into two parts, one of which is just a delay-free estimation of \( z \) over \([r, \infty) \) and another one can be reduced to a Nehari-like problem defined over the finite horizon \([0, r] \). Nagpal and Ravi \([47]\) observed an opportunity in adapting the so-called “lifting” technique similar to that used for some sampled-data control problems \([8]\). In one guise or another, the various approaches to sampled data design built on the “lifting” trick (already used in in \( \S 2 \)) where, in I/O trajectories one identifies \( f \in L^2[0, \infty) \leftrightarrow \{ f^k \} \in \ell^2(L^2[0, r]), \) with \( f^k(s) = f(kr + s) \). This reduces
the original (analog) system to a discrete time system with the same state space, say $\mathbb{R}^n$ and distributed I/O signals. Two algebraic Riccati equations (AREs) determine a discrete time design, which leads to periodic actuation between samples. Since the system acts in open loop between successive decisions, an added differential Riccati equation (DRE) is also needed, in some variants, to evaluate the I/O gain during these uncontrolled periods.

The adaptation of these ideas to the context of the basic delay problem required sophisticated and elaborate computations, and we shall be content with highlights of some important characteristics of [46, 47]. First, as in sampled data control, simple solvability conditions are derived, comprising the AREs for the delay-free problem and a DRE over $[0, \tau]$. The latter quantifies the extra cost for tolerating delay. In contrast, the controller structure is cumbersome, even in the simplest case. It is derived in terms of the “lifted” representation, and, as mentioned earlier, its “unlifted” form is periodic, time varying. The challenge of “peeling-off” a time-invariant representation is addressed next.

3.2 A semigroup based reduction: matrix Riccati equations and LTI solutions

Inspired by [46, 47], the studies [34, 72, 74, 73, 71] searched for an alternative reduction that would lead to a similar computational component (i.e., the combining AREs and DRE), but that avoids compensator periodicity. Following are the guiding principles, underlying these developments.

The first is the use abstract evolution models for the delay system [30, 11, 5, 13, 77, 6, 7, 60, 12, 75]. A natural state in the current setting would be $f(t) = (x(t), u_t) \in \mathbb{R}^n \times L_2[-\tau, 0]$, where $u_t(s) = u(t+s)$, $s \in [-\tau, 0]$, is the relevant control history at the time $t$. Abstract state space/operator Riccati equations solutions are well known for the linear quadratic (LQ) optimal control problem [78, 31, 57, 58] and have been established also for $H_\infty$ problems [67, 9, 70, 76, 32, 37]. As in the non-distributed case, the solution of the operator Riccati equation (ORE) is completely characterized by the quadratic form for the optimal cost in terms of the initial state. In the $H_\infty$ case, that is the cost of an associated differential game.

Second, as in [46, 47] the explicit solution of the said game is computed in the original setting, based on fixed size matrix AREs/DRE. A positive semidefinite quadratic form for the game’s optimal value is then derived as a function of the initial data $f(0) = (x(0), u_0) \in M_2$. Returning to the abstract model, that quadratic form provides the desired explicit expression for the solution of the ORE.

Third, in complete analogy to finite dimensional results, suboptimal compensators are parameterized by state space equations, in an abstract evolution model. These state space equations are based on perturbations of the infinitesimal generator and of input and output operators from the original model. As it turns out, these perturbed generators are, by themselves, abstract models of delay differential equations. The tools used here are provided by the detailed analysis of the interplay between functional differential equations and their abstract models, in [6, 75].

In summary, the family of solutions described in this section comprise an analysis along two parallel tracks. Along one track, a formal solution of the $H_\infty$
problem, including a parameterization of suboptimal compensators, is provided in terms of abstract evolution model counterparts of familiar state space solutions. Along a second track, explicit solution of operator Riccati equations is reduced to matrix computations, via a solution of a differential game in the original delay equation setting. Finally, the abstract evolution model realization of suboptimal compensators is reduced to a delay (neutral) differential equation, using established relations between such equations and their abstract models, and specific details of the computed solution of the operator Riccati equation.

In order to provide a flavor of this set of results, the following is an abbreviated version of the main result in [72].

**Theorem 1** Consider a system $G$ in minimal LTI realization $[A, B, C]$, cascaded with the input delay $e^{-\tau s}$. Let $X, Y \succeq 0$ be the stabilizing solutions of the algebraic Riccati equations

$$A'X + XA - XBB'X + C'C = 0, \quad AY + YA' - YC'C + BB' = 0. \quad (1)$$

Let $\gamma_0$ be the infimal achievable induced norm of $F_K = [I K] (I + KG)^{-1}[I K]$ over all stabilizing linear feedback compensators $K$. Then $\gamma_0 > 1$, and $\gamma > \gamma_0$ if and only if there exists a bounded, positive semidefinite solution of the following differential Riccati equation over $[0, \tau]$

$$\dot{Z} = ZA + A'Z + \frac{1}{\gamma \nu - 1} ZC'CZ + BB', \quad Z(0) = Y \quad (2)$$

such that $(\gamma^2 - 1)I > X^{1/2}Z(\tau)X^{1/2}$. Given such stabilizing $Z$, a parameterization of stabilizing compensators that guarantee that performance level is in terms of

$$\dot{x}_c(t) = Ax_c(t) - Bv(t - \tau) + YC'(e(t) - Cx_c(t))$$
$$v(t) = U^0 x_c(t) + \int_{-\tau}^0 U^1(s)v(t + s)ds + b(t)$$
$$c(t) = V^0 x_c(t) + \int_{-\tau}^0 V^1(s)v(t + s)ds - e(t); \quad b = K^\Delta e$$

where the compensator's input is the signal “$e$”, where $U^0, V^0, U^1(\cdot)$ and $V^1(\cdot)$ are defined in terms of the system matrices and the solutions to the said Riccati equations, as specified in the article, and where $K^\Delta$ is a stable system with an induced norm smaller than $\gamma^2 - 1$.

4. Frequency-domain methods

We review two transform-based methods, where state space machinery is restricted to computational purposes. Again, the emphasis is on exploitation of the special structure of the delay block to simplify solutions.

4.1 $J$-spectral factorization and the Smith predictor

A natural way to address delay is via the Smith predictor [62]. The predictor compensates for the delay with a specially constructed internal feedback loop in the
controller (see §5.1). Consequently, it reduces some dead-time problems to delay-free forms [54]. This idea, however, had little impact on $H_\infty$ methods. While a controller structure, reminiscent of the Smith predictor, can be recovered from some $H_\infty$ frequency-domain solutions (e.g., [19]), and the possibility to recast controllers in a generalized Smith predictor form was pointed out by Naimark and Cohen [48], the idea to use a special controller transformation to reduce a problem to an equivalent finite-dimensional one had not been used in the $H_\infty$ literature till the work of Meinsma and Zwart [38, 40, 39], where it was applied first to a mixed sensitivity (two-block) problem, and then generalized to the standard (four-block) $H_\infty$ problem with a single delay.

Here are some details of the developments in [40, 39]. The core step is a $J$-spectral factorization: given a self-adjoint transfer matrix $M$, find a (bi-proper, outer, etc) transfer matrix $L$ satisfying

$$M = \begin{bmatrix} M_1 & M_3^\tau \\ M_3 & M_2 \end{bmatrix} = L^{-1} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} L$$

(3)

Solutions are well understood in the rational case [27]. In dead-time problems the diagonal blocks remain rational, and $M_3 = e^{-\tau s} M_{r,3}$, with rational $M_{r,3}$. This structure can be exploited in solving (3), as follows. The transfer matrix $M_3$ is (non-uniquely) expanded as $e^{-\tau s}M_3^{-1}M_{r,3} = R - \Delta$, where $R$ is rational and $\Delta \in H_\infty$.

In these terms one has $M = \Psi^* M \Psi$ where $M_r$ (determined by the rational blocks of $M$ and by $R$) is a rational and self adjoint, and $\Psi = \begin{bmatrix} -\Delta, 0 \end{bmatrix}$ is invertible in $H_\infty$. Thus, a $J$-spectral factorization is provided by $L = L_r \Psi$, where $L_r$ is the rational $J$-spectral factorization of $M_r$. Necessary to complete the reduction is an additional (stability) test, involving $L$, which cannot be expressed in terms of rational matrices. That test, however, involves only solvability conditions and does not affect the resulting controller.

To see the connection with the Smith predictor, consider the $J$-spectral factorization controller. As is well known, proper compensators $C$, providing $\gamma$-suboptimal solutions of the corresponding dead-time $H_\infty$ problem are those satisfying $[I \ -C \ L^{-1} \begin{bmatrix} \Psi^* \\ \Psi \end{bmatrix} Q] = 0$, for some $Q \in H_\infty$, $\|Q\|_\infty < \gamma$. Substituting the explicit form of $L$ into this expression, $C = C_r(I + \Delta \Psi)^{-1}$, where the rational $C_r$ satisfies $[I \ -C_r \ L_r^{-1} \begin{bmatrix} \Psi^* \\ \Psi \end{bmatrix} Q] = 0$, i.e., it is the controller corresponding to the “rationalized” version of (3). Thus, the resulting controller has the Smith predictor structure where $\Delta_r$ serves as a dead-time compensator block (in fact, $C$ above can be presented in the form shown in Fig. 2 on p. 9, where $C_r$ depends on $L_r$, only).

Note, that the choice of $\Delta_r$ and $R$ is non unique. In [40] $\Delta_r$ was chosen in the form $\Delta_r = \pi_r \{e^{-\tau s} P_{a} \}$ for $P_{a} \eqsim M_{r,3}^{-1}M_{r,3}$. When $P_a$ is stable, $R$ can be chosen either as $R = 0$ (in the spirit of the Morari’s internal model control) or $R = P_2$ (in the spirit of the Smith predictor). In the latter case $M_r$ is equal to the delay-free ($\tau = 0$) version of $M$, which means that if $P_a$ is stable, then the dead-time problem is solvable iff so is its delay-free counterpart. Typically, however, $P_a$ is unstable.

Comparing with the time domain methods in Section 3, the clear advantage is in a simpler and more transparent controller structure. The disadvantage lies in a somewhat less transparent structure of the solvability conditions, which are based
upon two $H_{\infty}$ AREs, one of which depends on the delay $\tau$ (in [47] only the coupling condition depends on $\tau$).

4.2 Extraction of dead-time controller from delay-free parameterizations

Conventionally, the delay block $e^{-\tau s}$ is treated as a part of a plant. This results in an infinite-dimensional plant, and as seen above, typically, a rather involved controller structure. In [41, 42] it is proposed to treat the delay as a part of the controller. This leads to the design of a constrained-structure controller for a finite-dimensional plant. The challenge here is to extract admissible (delay) controllers from the well known parameterization of all suboptimal controllers for the rational plant.

The idea is borrowed from [45], where the structure of sampled-data controllers in the lifted domain (see §3.1) was studied. It was shown that a controller can be implemented as the cascade of a generalized sampler, a discrete-time controller and a generalized hold iff its lifted “transfer function” is strictly proper. Using this fact, the sampled-data $H_{\infty}$ standard problem was then solved by extracting strictly proper transfer functions from the lifted version of the DGKF [15] parameterization of all continuous-time $H_{\infty}$ controllers. This procedure yields simple formulae both for the digital part of the controller and for the generalized sampler and hold. Moreover, an attractive byproduct of this procedure is that the solvability conditions are naturally separated into the existence conditions for a continuous-time controller and an additional condition imposed by the sampled-data structure.

The constraints imposed upon dead-time controllers in the lifted domain are more complicated than in the sampled-data case. For that reason, a direct extension of the technique of [45] to delay systems is much more involved. The extraction approach, however, can be adopted to dead-time systems by translating the dead-time constraints into the frequency domain. This approach was followed in [42] resulting in simple and intuitively appealing solvability conditions and the controller structure for the dead-time $H_{\infty}$ standard problem. Below, the solution procedure in [42] is outlined.

The standard (four-block) $H_{\infty}$ problem with a single delay $e^{-\tau s}$ in the loop is considered. The solution starts with the parameterization of all $\gamma$-suboptimal controllers $K$ for the rational part of the plant. This parameterization is well-known [15] and has the form of an LFT interconnection of a $2 \times 2$ block transfer matrix $G$ with an arbitrary $Q_a \in H_{\infty}$ such that $\|Q_a\|_{\infty} < \gamma$. The next step is then to extract all $Q_a$ which bring the LFT in the form $e^{-\tau s}K_{\tau}$ for a proper $K_{\tau}$. This can easily be done by the loop-shifting arguments of [44] yielding the following characterization of all admissible $Q_a$: $Q_a = \Delta_{22} + e^{-\tau s}Q_b$, where $Q_b \in H_{\infty}$ is arbitrary and $\Delta_{22}$ is the FIR truncation of the $(2,2)$ subblock of $G^{-1}$. Now, taking into account the norm constraint on $Q_a$ one can see that the original (four-block) problem is reduced to the (one-block) problem of the characterization of all $Q_b \in H_{\infty}$ such that:

$$\|\Delta_{22} + e^{-\tau s}Q_b\|_{\infty} < \gamma.$$  \hspace{1cm} (4)

According to the Nehari Theorem (see [19, Theorem 15]), (4) is solvable iff $\gamma$ is larger than the norm of the Hankel operator with symbol $e^{\tau s}\Delta_{22}$ that, in turn,
is equivalent to the $L_2[0,\tau]$ induced norm of $\Delta_{22}$. Thus, $\gamma > \|\Delta_{22}\|_{L_2[0,\tau]}$ is the additional solvability condition, as compared with the delay-free case, capturing the effect of delay on achievable performance. Note, that this condition can be expressed either in terms of the matrix exponential of a Hamiltonian matrix or in terms of the solvability of a differential Riccati equation defined over $[0,\tau]$, see [8, 28], where several computational schemes are discussed (as a matter of fact, when expressed in terms of a Riccati equation, these conditions coincide with those obtained by Nagpal and Ravi [47]). Then, for any admissible $\gamma$ the set of all solutions to (4) can be parameterized in terms of an arbitrary $Q \in H_{\infty}$ such that $\|Q\|_{\infty} < \gamma$. Finally, the latter is substituted into the original delay-free parameterization yielding, after some elaborate simplification steps, the characterization of all dead-time controllers solving the original problem.

5. Discussion

Perhaps the most serious disadvantage of the early $H_{\infty}$ methods for delay systems is the lack of transparency. Design procedures typically involved several intermediate steps, so the controller structures and the achievable cost are not easily recoverable. On the other hand, the methods described in Sections 3 and 4 enable one to compute both the (sub)optimal controllers and the achievable cost explicitly for general problems in terms of the original data. Among other advantages, this enables one to gain deeper insight into the controller structures and the limitations on performance imposed by delay elements. These issues will be discussed below.

5.1 Controller structure

One of the most widely used controller configurations to control delay systems is the so-called dead-time compensator (DTC) depicted in Fig. 2. DTC consists of a rational part $G_r$ and a stable internal feedback $\Delta_s$ containing the delay (the free parameter $Q$ is typically zero). This scheme was proposed by Smith [62], whose idea was to use the internal feedback with $\Delta_s = P_{22} - e^{-\tau s}P_{22}$ to convert dead-time problems to their delay-free versions. The rationale behind this approach becomes apparent when the signal $y_e = y + \Delta_s u$ entering the rational part of the controller
in Fig. 2 is considered. Indeed, it is readily seen that
\[ y_e = e^{-\tau s}(P_{21}w + P_{22}u) + (P_{22} - e^{-\tau s}P_{22})u = P_{21}e^{-\tau s}w + P_{22}u \]
and thus the resulting feedback loop \((u \odot y_e)\) does not contain any delay and is exactly the feedback loop for \(\tau = 0\). In other words, \(\Delta_s\) “compensates” the delay in the loop. The scheme of Smith can be applied to stable systems only. Yet it can easily be modified to cope with the general case by choosing \(\Delta_s = \tilde{P}_{22} - e^{-\tau s}P_{22}\) with any \(\tilde{P}_{22}\) making \(\Delta_s\) stable. In particular, it is always possible to make \(\Delta_s\) FIR [[79]]. Some other possible choices are discussed in [40]. Moreover, as shown in [44] the set of all stabilizing dead-time controllers can be parameterized in the DTC form shown in Fig. 2.

In the context of \(H_\infty\) control, the DTC controller structure was derived by Meinsma and Zwart [40] and Mirkin [41]. The version of the \(H_\infty\) DTC proposed in [41], in which the emphasis is on transparency, is discussed below (for more details see [43]).

The set of all \(\gamma\) suboptimal dead-time controllers is parameterized in [41, 42] in the form depicted in Fig. 2, subject to a rational \(G_r\), similar to that for the delay free problem, an arbitrary \(Q \in H_\infty\) such that \(\|Q\|_\infty < \gamma\), and the FIR \(\Delta_s = \pi \{e^{-\tau s}P_a\}\), where \(P_a = F_u(P, \gamma^{-2}P_{11})\). Unlike the Smith predictor and its modifications, the delay in the loop is now not canceled (since \(e^{-\tau s}(P_{22} - P_a) \neq 0\)). Yet it is easily seen that the relationship \(y = P_a u\) can be equivalently written as
\[ y = P_{21}w_* + P_{22}u, \]
where the “disturbance” \(w_*\) satisfies:
\[ w_* = \frac{1}{\gamma^2}P_{11}^{-1}(P_{11}w_* + P_{12}u). \]

Thus, if the disturbance \(w\) in Fig. 1 were equal to the \(w_*\) above, then the block \(\Delta_s = \pi \{e^{-\tau s}P_a\}\) would just compensate the dead time in both feedback and feedforward loops (“predict” \(y\) subject to a given \(u\)). This agrees well with the result of Palmor and Powers [55], who proposed to transmit measured disturbances into the Smith predictor to improve its disturbance attenuation properties (feedforward Smith predictor). Although the \(H_\infty\) DTC does not measure the disturbance, it generates it artificially.

Taking into account the worst-case nature of the \(H_\infty\) methodology, one would expect that the disturbance \(w_*\) is generated on the basis of a worst-case scenario. It turns out that this indeed happens. To see this, consider the relationship \(y = P_a u\) in the state-space setting. We have:
\[ \begin{cases} \dot{x} = Ax + B_1w_* + B_2u \\ y = C_2x + D_{21}w_* \end{cases} \quad \text{and} \quad \begin{cases} -\dot{\lambda} = A'\lambda + C'_1z \\ w_* = \frac{1}{\gamma^2}B'_1\lambda \end{cases} \]
where \(z = C_1x + D_{12}u\). Using the standard arguments from the calculus of variations [26], the \(w_*\) above can roughly be thought of as the maximizing disturbance for the index \(J = \int_0^\infty (z'z - \gamma^2w^2)dt\) subject to any fixed \(u\).
Thus, the $H_\infty$ DTC attempts to compensate the dead time $h$ assuming that the disturbance $w$ is the worst-case one for the open-loop problem. The fact that $w_*$ is worst-case for an open-loop problem agrees well with the open-loop nature of the dead-time compensation.

5.2 The “cost of delay”

Another important question raising in $H_\infty$ control of delay systems concerns the quantification of the effect of delay on achievable performance. Obviously, any delay problem is solvable only if so is its delay-free counterpart. One therefore would expect that the achievable performance level for any dead-time problem can be presented as a sum of the delay-free performance and a term that reflects the cost of delay. The knowledge of the latter may be of value in numerous applications where delay tolerance is important. Unfortunately, in most early treatments, this “cost of delay” cannot be easily recovered from the solvability conditions. This complicates any further analysis considerably. In this respect, [47, 74, 41] offer a clear advantage since the solvability conditions there are explicitly presented in such a two-term form with a transparent “cost of delay” term. In particular, the following result can be summarized:

**Lemma 2** Consider the system in Fig. 1 with $P_{11} = C_1(sI - A)^{-1}B_1$. The following three conditions are equivalent (below, $X, Y \succeq 0$ stand for the stabilizing solutions to the matrix Riccati equations associated with the delay-free problem):

(a) There exists a stabilizing $K_\tau$ so that $\|F_\ell(P, e^{-\tau s}K_\tau)\|_\infty < \gamma$

(b) The problem is solvable for $\tau = 0$ and, in addition, $\Pi(t)$ defined on $[0, \tau]$ as

$$\dot{\Pi} + \Pi A + A'\Pi + C_1'C_1 + \frac{1}{\gamma^2}\Pi B_1B_1'\Pi = 0, \quad \Pi(\tau) = X,$$

exists and satisfies $\rho(Y\Pi(0)) < \gamma^2$;

(c) The problem is solvable for $\tau = 0$ and, in addition, $\|P_{11}\|_{L_2[0,\tau]} < \gamma$ and $\rho(X\Sigma_{12}\Sigma_{22}^{-1}) < 1$ and $\rho((\Sigma_{22} - X\Sigma_{12})^{-1}(\Sigma_{21} - X\Sigma_{11})Y) < \gamma^2$, where

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \exp \left( \begin{bmatrix} A & \frac{1}{\gamma^2}B_1B_1' \\ -C_1'C_1 & -A' \end{bmatrix} \tau \right).$$

Condition (b) above is from [47] while condition (c) is from [41]. Although the solvability conditions in [74] appear slightly different, they can be brought to a form equivalent to that in [47] using standard manipulations of Hamiltonian matrices.

Some observations are in order. First, it appears that in some situations (small enough) delay does not contribute to the final cost, i.e., any performance achievable with $\tau = 0$ can be achieved with “small” $\tau$ as well. Indeed, it is known [23] that the delay-free test fails if either (i) $\rho(YX) = \gamma^2$ or (ii) $X$ or $Y$ is no longer stabilizing. Assume for simplicity that the triple $(C_1, A, B_1)$ is observable and controllable. Then the solution $\Pi(t)$ of (5) is monotonically increasing as $t$ passes from $\tau$ to 0. Therefore, if (i) happens prior to (ii) in the delay-free case, then the condition
\( \rho(\Pi(0)) < \gamma^2 \) fails for all \( \tau > 0 \). Yet if the opposite happens for some \( \gamma = \gamma^* \), then there exists a \( \tau_0 > 0 \) such that for every \( \gamma > \gamma_0 \) condition (b) of Lemma 2 holds true \( \forall \tau < \tau_0 \). Moreover, if the observability or/and controllability of \((C_1, A, B_1)\) are relaxed, then small enough delay might not affect the achievable performance even when the spectral radius condition fails first. This, for instance, occurs in the case when \( A \) is Hurwitz and \( P_{11} = 0 \), see [44].

Another interesting observation is that exactly the same condition appears in the context of sampled-data control when both sampler and hold are the design parameters [69, 45]. In other words, it turns out that the cost of delay is equal to the cost of sampling. This fact is surprising since the constraints imposed by the delay are more restrictive, than those imposed by the sampling. Indeed, as was mentioned in §4.2, a controller \( K \) contains the delay \( e^{-\tau s} \) iff its kernel \( k(t, s) = 0 \) whenever \( t < s + \tau \). On the other hand, it follows from [45] that a controller is the cascade of a generalized sampler, a discrete-time controller and a generalized hold with the period \( \tau \) iff its kernel \( k(t, s) = 0 \) whenever \( t < \lceil \frac{s}{\tau} \rceil \tau + \tau \), where \( \lceil \cdot \rceil \) stands for the “floor” (round towards \(-\infty\)) operation. Thus, the sampled-data controller “receives odds” on the intervals \( \left\lceil \frac{s}{\tau} \right\rceil + 1 < \frac{s}{\tau} < \left\lceil \frac{s}{\tau} \right\rceil + \tau \) but is not able to take any advantage of it. A possible explanation may be that on this interval the sampled-data controller can act only in an open-loop fashion. Then, the disturbance is “smart enough” to outplay the control.

6. Conclusions

In this paper several problem-oriented methods for the \( H_\infty \) control of continuous-time systems with a single delay in the feedback loop have been reviewed. Reviewed methods include operator interpolation methods (Section 2) and game-theoretic state-space (Section 3) treatments in the time domain and a \( J \)-spectral factorization (Subsection 4.1) and extraction (Subsection 4.2) approaches in the frequency domain. Some properties of the resulting solutions and solvability conditions have also been discussed in Section 6.

References


