Optimal Hold Functions for MDCS Sampled-Data Problems

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Abstract

In this paper the $H^2$ and $H^\infty$ sampled-data problems with mixed discrete/continuous specifications (MDCS) are considered. The hold function is not fixed a-priory but rather is the design parameter. The (sub)optimal controller obtained is of the form of a serial interconnection of a (sub)optimal digital controller and a (sub)optimal hold function.

1 Introduction

The increasing use of digital equipment in control in the last half century made a tremendous impact on controller design methodology, see, e.g., (Åström and Wittenmark, 1989; Chen and Francis, 1995) and the references therein. On the other hand, it had a little affect on the control problem formulation: the control goals still remain to a large extent those of the analog world. Yet digital equipment used for control and information processing may give rise to various discrete-time disturbances and lead to situations where some of the continuous-time regulated signals are of interest mainly at discrete-time instances. Those discrete disturbances cannot always be modeled in continuous time and, if included in a continuous-time criterion, the discrete requirements get negligible weights. These observations motivated the introduction of the MDCS (mixed discrete/continuous specifications) sampled-data control by Mirkin and Palmor (1995, 1997). The MDCS approach is based on the simultaneous treatment of both discrete- and continuous-time disturbances and regulated signals (see also (Hagiwara and Araki, 1995; Toivonen, 1998), where sampled-data control problems with MDCS are also treated, though with different motivations). The MDCS framework thus enables the natural modeling of discrete-time disturbances as well as the achievement of a suitable tradeoff between the continuous- and discrete-time requirements.

Consider the latter point in more detail. Assume that we have a control problem where some performance specifications are in discrete time. Let the discrete-time requirements be reflected by the discrete regulated output $z_d[k] = z_d(kh)$, where $z_d(t)$ is some continuous-time signal and $h$ is the sampling period. It is clear that a “small” $z_d(t)$ causes a “small” $z_d[k]$. This observation suggests that discrete-time requirements may simply be embedded into the continuous-time ones.

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Such a strategy, however, may prevent the exploitation of the opportunities offered by the fact that $z_d$ is of interest at the discrete time instances $k_h$ only. The latter, in turn, might lead to a conservative design. Indeed, the achievable performance of $z_d$ is limited by properties of the continuous-time plant dynamics, say $P(s)$, whereas the achievable performance of $\bar{z}_d$ is limited by the dynamics of the discretized plant, say $\bar{P}(z)$. Although some properties of $P(s)$ and $\bar{P}(z)$ are closely connected, some are not (for instance, it is well known (Åström and Wittenmark, 1989) that the location of some zeros of $\bar{P}(z)$ is not related with those of $P(s)$). This means that the discrete-time performance obtained by a direct treatment of $z_d$ ($J_{d,\bar{z}}$) may be considerably better than that obtained by embedding discrete requirements into the continuous-time ones ($J_{d,z}$).

It is worth stressing at this point that the improvements in the discrete-time performance can only be achieved at the expense of a degradation of the continuous-time one. Thus, a design based on $\bar{z}_d$ might lead to unacceptably poor $z_d(t)$. An MDCS treatment, however, may allow a reasonable tradeoff between the improvement of the discrete-time performance and the deterioration of the continuous-time one (intersample tradeoff). In fact, the larger the gap between $J_{d,\bar{z}}$ and $J_{d,z}$, the more one can gain from the MDCS treatment of sampled-data systems. Consequently, the MDCS approach may be of value for those systems, for which the discretized plant $\bar{P}(z)$ is “easier” to control than its continuous-time counterpart $P(s)$. In this respect, the use of generalized sampled-data hold (GSDH) devices seems to be an attractive possibility.

As opposed to the conventional zero-order hold, a GSDH is a device that corrects the control signal in a pre-specified manner between sampling instances, see (Kabamba, 1987; Araki, 1993) and the references therein. Kabamba (1987) showed that by an appropriate choice of the GSDH the zeros of the discretized systems can be shifted arbitrarily. Hence, the use of GSDH enables one to circumvent the limitations associated with nonminimum-phase zeros (as long as the discrete-time performance is considered, of course). This property of the GSDH devices has been extensively used in the literature to solve problems of simultaneous and decentralized stabilization, tracking and the like, for possibly nonminimum phase systems. It is not surprising, however, that the intersample behavior of such systems tends to be extremely poor. As a consequence, the idea of using GSDH has found numerous opponents who questioned its practical applicability, see (Feuer and Goodwin, 1994; Braslavsky, 1995)).

It seems, however, that the GSDH’s can be advantageously exploited within the MDCS framework. In view of the discussion above, the employment of GSDH enables one to get $\bar{P}(z)$, which is indeed “easier” to control than its continuous-time counterpart $P(s)$, especially if the latter is nonminimum-phase. This, in turn, leaves more freedom for achieving a reasonable intersample tradeoff. A possible design strategy then may be to find a GSDH that maximize the improvements in the discrete-time performance while keeping an acceptable level of the continuous-time one. To this end, design methods for generalized holds in the MDCS framework should be developed.

In this paper such a problem is considered for the $H_\infty$ and $H_2$ performance measures. In particular, we consider the MDCS problems where not only the discrete-time part of the sampled-data controller, but also the hold device are the design parameters. Using the framework recently developed by Mirkin et al. (1997a) the problems are reduced to LTI discrete-time $H_\infty$ and $H_2$ problems in the lifted domain. Although the parameters of the plant and the controller are infinite-dimensional in the lifted domain, it is shown that these problems can be solved using only finite dimensional computations. The optimal hold function is found to have an interesting interpretation: it attempts to reconstruct the optimal control signal (subject to the “worst-case” disturbance in the $H_\infty$ case) for the continuous-time state-feedback optimization problem on the time interval $[0,h]$, where the discrete-time performance specifications affect only the terminal
Figure 1: MDCS sampled-data setup

state penalty. In the $H^2$ case the achievable discrete- and continuous-time performance is also quantified.

The paper is organized as follows. In Section 2 the MDCS $H^2$ and $H^\infty$ problems are formulated. Section 3 presents the lifted solutions to those problems, whereas in Section 4 the lifted solutions are "peeled-off" to the time domain giving the complete solutions to the MDCS problems considered in the paper. In Section 5 a simple illustrative example is presented. Section 6 contains all the proofs of the results presented in Section 4 and, finally, concluding remarks are given in Section 7.

The notation used throughout the paper is mainly standard. $\| \cdot \|$ denotes the $L^2[0, h] \oplus \mathbb{R}$ induced operator norm, while $\| \cdot \|_{\mathcal{H}}$ denotes the Hilbert-Schmidt operator norm. $S \in \text{dom Ric}_D$ implies that $S \in \mathbb{R}^{2n \times 2n}$ is symplectic and there exists a (unique) $X = X' \in \mathbb{R}^{n \times n}$ such that $\text{Im} \left[ I \ X \ X' \right]'$ is the $n$-dimensional spectral subspace of $S$ corresponding to eigenvalues in the unit disk $D$. In this case we write $X = \text{Ric}_D (S)$. $(S_1, S_r) \in \text{dom Ric}_D$ implies that the matrix pair $(S_1, S_r)$ $(S_r \in \mathbb{R}^{(2n+m) \times (2n+m)}$) is an extended symplectic pair and there exist (unique) $X = X' \in \mathbb{R}^{n \times n}$ and $F \in \mathbb{R}^{n \times n}$ such that $\text{Im} \left[ I \ X \ F' \right]'$ is the $n$-dimensional deflating subspace of $(S_1, S_r)$ corresponding to eigenvalues in $D$. In this case we write $(X, F) = \text{Ric}_D (S_1, S_r)$. The Redheffer star product of two suitably partitioned $2 \times 2$ block operators $O$ and $P$ is denoted as

$$O \star P \triangleq \begin{bmatrix} \mathcal{F}_t(O, P_{22}) & O_{12} (I - P_{22} O_{22})^{-1} P_{21} \\ P_{12} (I - O_{22} P_{22})^{-1} O_{21} & \mathcal{F}_t(P, O_{22}) \end{bmatrix},$$

where $\mathcal{F}_t(\cdot, \cdot)$ denotes the lower linear fractional transform. Finally, since operator compositions involving operators over infinite dimensional spaces (such as $L^2[0, h] \oplus \mathbb{R}$) are extensively used throughout the paper, the following notation is aimed to simplify the readability of the formulae: bar above a variable denotes an operator $\bar{O} : \mathbb{R}^n \mapsto \mathbb{R}^n$; grave accent — $\bar{O} : \mathbb{R}^n \mapsto L^2[0, h] \oplus \mathbb{R}^m$; acute accent — $\hat{O} : L^2[0, h] \oplus \mathbb{R}^n \mapsto \mathbb{R}^m$; and breve — $\breve{O} : L^2[0, h] \oplus \mathbb{R}^n \mapsto L^2[0, h] \oplus \mathbb{R}^m$.

2 Problem formulation

Consider the sampled-data setup depicted in Fig. 1, where $\mathcal{P}$ is a continuous-time generalized plant, $w$ is an exogenous plant input, $u$ is a control signal, and $y$ is a measured signal. The regulated output is split into two parts: $z_c$, which reflects the continuous-time performance requirements, and $z_d$, which reflects the discrete-time ones. The sampled-data controller consists of a discrete-time part $\mathcal{K}$, a sampler $\delta_h$, and a hold $\mathcal{H}_h$, assumed to be synchronized and with a sampling period $h$. Throughout this paper the sampler $\delta_h$ is assumed to be the standard ideal (instantaneous) sampler, while the generalized plant $\mathcal{P}$ is assumed to be LTI with the following
state-space realization:

\[
P(s) = \begin{bmatrix}
A & B_1 & B_2 \\
C_{1c} & 0 & D_{12c} \\
C_{1d} & 0 & 0 \\
C_2 & 0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
A & B \\
C_{1c} & D_{11c} \\
C_{1d} & 0 \\
C_2 & 0 \\
\end{bmatrix},
\]  

(1)

where the partitioning is compatible with that in Fig. 1. Since the ideal simpler is unbounded in the \(L^2\) sense (Chen and Francis, 1995), the matrices \(D_{11c}, D_{12d}, D_{21},\) and \(D_{22}\) are taken to be zero. This ensures the boundedness of the sampling operations. The matrix \(D_{11c}\) is assumed to be zero in order to simplify the derivations and obtain more transparent results.

In this paper we assume that both the discrete-time part of the controller \(\bar{K}\) and the hold \(H_h\) are available for the design. The hold \(H_h\) is assumed to belong to the class of zero-order generalized hold, that is it acts on the output of the controller \(\bar{u}\) to generate (Kabamba, 1987; Araki, 1993)

\[
(H_h \bar{u})(kh + \tau) = \phi_{H}(\tau)\bar{u}[k], \quad \forall \tau \in [0, h)
\]

for some generalized hold function \(\phi_{H}(\tau)\) defined on the interval \([0, h)\). During the inter-sample, the hold function shapes the form of the control signal.

Define by \(T_{zw}\) the closed-loop operator from \(w\) to \(z = [z_c' \ z_d']'\). The MDCS sampled-data \(H^\infty\) problem to be treated in the paper is as follows:

**OP\(_{H^\infty}\):** Given a scalar \(\gamma > 0\), find the controller \(\bar{K}\) and the hold function \(\phi_{H}(\tau)\) (if they exist), which internally stabilize the plant \(P\) and make the \(H^\infty\) norm of \(T_{zw}\) less than \(\gamma\).

We also consider the MDCS \(H^2\) problem:

**OP\(_{H^2}\):** Find the controller \(\bar{K}\) and the hold function \(\phi_{H}(\tau)\), which internally stabilize the plant \(P\) and minimize the \(H^2\) norm of \(T_{zw}\).

Although \(T_{zw}\) is \(h\)-periodic and hybrid, the notions of both \(H^\infty\) and \(H^2\) operator norms can naturally be extended to such operators (Chen and Francis, 1995). In particular, the \(H^\infty\) norm is just the induced norm of \(T_{zw}: L^2 \rightarrow L^2 \oplus \ell^2\), while the \(H^2\) norm is the time average of responses to impulses applied over the first time interval \([0, h)\). Note, that the deterministic interpretation of the \(H^2\) norm is chosen in the paper. To perform the \(H^2\) optimization consistent with the stochastic interpretation of the \(H^2\) norm (Bamieh and Pearson, 1992b) one needs to scale \(z_d\) by the factor \(\sqrt{h}\) (Hagiwara and Araki, 1995; Mirkin and Palmor, 1997).

The treatment of \(OP_{H^\infty}\) and \(OP_{H^2}\) is complicated owing to their hybrid continuous/discrete nature and inherent periodicity. To circumvent these difficulties the so-called lifting technique of Yamamoto (1994); Bamieh et al. (1991); Bamieh and Pearson (1992a) (see also (Toivonen, 1992; Tadmor, 1992)) can be applied.

### 3 Lifted solution

The notion of lifting is based on the conversion of real valued signals in continuous time into functional space valued sequences, that is sequences that take values not from \(\mathbb{R}\) but rather from some general Banach space (\(L^2[0, h]\) in this paper). Formally, let \(\ell_{L^2[0, h]}\) be the space of sequences, each element of which is a function from \(L^2[0, h]\), that is

\[
\ell_{L^2[0, h]} = \{ \tilde{x} : \tilde{x}[k] \in L^2[0, h] \quad \forall k \in \mathbb{Z}_+ \}.
\]
Then given any \( h > 0 \), the lifting operator \( W_h : L^2_e \rightarrow \ell_{L^2}L^2[0,h] \) is defined such that:

\[
\tilde{\xi} = W_h \xi \iff (\tilde{\xi}[k])(\tau) = \xi(kh + \tau) \quad \forall \tau \in [0,h].
\]

It is easy to see that the lifting operator is a linear bijection between \( L^2_e \) and \( \ell_{L^2}L^2[0,h] \). Moreover, if we restrict the domain of \( W_h \) to the Hilbert space \( L^2_e \), then by an appropriate choice of a norm on \( \ell_{L^2}L^2[0,h] \) the lifting operator can be made an isometry. Hence, treating a system \( \zeta = g\omega \) not as a mapping from \( \omega \) to \( \zeta \) but rather as a mapping from \( \tilde{\omega} \) to \( \tilde{\zeta} \) gives essentially the same system (as an input-output mapping). Indeed, lifting preserves system stability and system induced norms. This allows one to conclude that \( G \) and \( \tilde{G} = W_hGW_h^{-1} : \ell_{L^2}L^2[0,h] \rightarrow \ell_{L^2}L^2[0,h] \), which is called the lifting of \( G \), are equivalent. The advantage of treating systems in the lifting domain stems from the fact that \( \tilde{G} \) is time-invariant in discrete time even if \( G \) is \( h \)-periodic in continuous time. Hence, any periodic problem in continuous time can be reduced to a time-invariant one in discrete time.

The application of this idea to the sampled-data setup in Fig. 1 is straightforward. In order to convert this setup to a pure discrete time-invariant one the continuous-time signals \( w, z_c, \) and \( u \) should be lifted to \( \tilde{w}, \tilde{z}_c, \) and \( \tilde{u} \), respectively. Since \( z_d \) is just a linear combination of the state vector of \( P \) with zero initial conditions, one can replace \( z_d \) with \( \bar{U}\tilde{z}_d \), where \( \bar{U} \) denotes the backward shift operator. Then we no longer need to separate the regulated signal into continuous- and discrete-time parts. We thus have the regulated signal

\[
\tilde{z} = \begin{bmatrix} \tilde{z}_c \\ \bar{U}\tilde{z}_d \end{bmatrix} \in \ell_{L^2}L^2[0,h] \oplus R.
\]

With respect to these signals all the blocks of the system in Fig. 1 become pure discrete LTI. Now, following Mirkin et al. (1997a), one can form the lifted “standard problem” in Fig. 2. To this end, all fixed parts (i.e., the lifted plant and the lifted sampler) should be absorbed into the generalized plant \( \hat{P} \), while all parts to be designed (i.e., \( \hat{K} \) and the lifted hold) should be absorbed into the controller \( \hat{\theta} \). The only fixed part that will be absorbed into the controller is the backward shift \( z^{-1} \) of the lifted sampler. The reason for this is twofold: first, the state-space dimension of \( \hat{P} \) is preserved, and second, the characterization of \( \hat{K} \) is considerably simplified if it is strictly causal, see (Mirkin and Rotstein, 1997). The lifted generalized plant \( \hat{P} \) can be presented as follows:

\[
\hat{P}(z) = \begin{bmatrix} A & \tilde{B}_1 & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} = \begin{bmatrix} A & \tilde{B}_1 & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{C}_2 & \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} = \begin{bmatrix} A & \bar{B} \\ \bar{C}_1 & \bar{D}_{11} \\ \bar{C}_2 & \bar{D}_{21} \end{bmatrix}, \tag{3}
\]
where

\[
\begin{bmatrix}
\hat{C}_1 & \hat{D}_{i\ast}
\end{bmatrix} = \begin{bmatrix}
\hat{C}_{1c} & \hat{D}_{i\ast c}

\hat{C}_{1d} & \hat{D}_{i\ast d}
\end{bmatrix}.
\] (4)

The expressions for the parameters of \( \hat{K} \) can be derived using standard lifting arguments. They, however, are not essential for the discussion in this section and hence are postponed to §6.1. What is important is that any designed \( \hat{K} \) with finite-dimensional state space has the form

\[
\hat{K}(z) = \begin{bmatrix}
\hat{A}_K & \hat{B}_K \\
\hat{C}_K & 0
\end{bmatrix}
\]

and thus can always be factorized as \( \hat{K}(z) = z^{-1}\Phi_H \hat{K}(z) \) for some \( \Phi_H : \mathbb{R} \to L^2[0, h] \) and a finite-dimensional \( \hat{K}(z) \). This yields both the discrete-time part of the controller and the hold function \( \phi_H(\tau) \).

Having the lifted setup in Fig. 2, the \( OP_{H^2} \) and the \( OP_{H^\infty} \) can be reformulated in the lifted domain as follows:

\( OP_{H^2} \): Find the strictly causal controller \( \hat{K} \), which internally stabilizes the plant \( \hat{P} \) and minimizes \( \|F_P(\hat{P}, \hat{K})\|_{H^2} \).

\( OP_{H^\infty} \): Given a scalar \( \gamma > 0 \), find the strictly causal controller \( \hat{K} \) (if it exists), which internally stabilizes the plant \( \hat{P} \) and guarantees \( \|F_P(\hat{P}, \hat{K})\|_{H^\infty} < \gamma \).

Now, both the \( OP_{H^2} \) and the \( OP_{H^\infty} \) are standard discrete-time LTI optimization problems with a strictly proper controller and a finite-dimensional state, exactly like those solved by (Mirkin et al., 1997a, §4). Below, the solution to \( OP_{H^\infty} \) is presented. As shown in (Mirkin et al., 1997a), the solution to \( OP_{H^2} \) is just a special case of the solution to \( OP_{H^\infty} \) when \( \gamma \to \infty \).

We start by imposing the following assumptions on the generalized plant \( \hat{P} \):

(A1): The operator \( [\hat{A} - \lambda I \quad \hat{B}_2] \) is right invertible \( \forall \lambda \geq 1 \);

(A2): The matrix \( [\hat{A} - \lambda I \quad \hat{C}_2] \) is left invertible \( \forall \lambda \geq 1 \);

(A3): The operator \( [\hat{A} - e^{i\theta}I \quad \hat{B}_2 \quad \hat{D}_{12}] \) is left invertible \( \forall \theta \in [0, 2\pi) \);

(A4): The operator \( [\hat{A} - e^{i\theta}I \quad \hat{B}_1 \quad \hat{D}_{21}] \) is right invertible \( \forall \theta \in [0, 2\pi) \).

These assumptions are the counterparts of the standard assumptions imposed on a discrete-time generalized plant, to guarantee input-output stabilizability and non-singularity of the \( H^\infty \) problem. Define also the following two \( H^\infty \) Dare’s

\[
\bar{X} = \bar{A}'\bar{X}A + \bar{C}_1\bar{C}_1 - (\bar{D}_{1\ast}\bar{C}_1 + B^*\bar{X}A)'(\bar{D}_{1\ast}\bar{D}_{1\ast} - \gamma^2E_{11} + B^*\bar{X}B)^{-1}(\bar{D}_{1\ast}\bar{C}_1 + B^*\bar{X}A) \quad (5a)
\]

and

\[
\bar{Y} = \bar{A}\bar{Y}A' + \bar{B}_1\bar{B}_1^* - (\bar{B}_1\bar{D}_{1\ast} + \bar{A}\bar{Y}\bar{C}^*)'(\bar{D}_{1\ast}\bar{D}_{1\ast} - \gamma^2E_{11} + \bar{C}\bar{Y}\bar{C}^*)^{-1}(\bar{B}_1\bar{D}_{1\ast} + \bar{A}\bar{Y}\bar{C}^*) \quad (5b)
\]

where \( E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) (these Dare’s reduce to the corresponding \( H^2 \) ones as \( \gamma \to \infty \)). Using (5), the solution to \( OP_{H^\infty} \) is as follows:
Theorem 1 (Mirkin et al. 1997a). Let (A1)-(A4) hold. Then the following statements are equivalent:

i) There exists a controller $\hat{K}$ which solves the $\text{OP}^\infty_{H^\infty}$.

ii) The DARE’s (5) have stabilizing solutions $\bar{X} \geq 0$ and $\bar{Y} \geq 0$ such that

$$\left\| \begin{bmatrix} \bar{X}^{1/2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{A} & \hat{B}_1 \\ \hat{C}_1 & \bar{D}_{11} \end{bmatrix} \begin{bmatrix} \bar{Y}^{1/2} & 0 \\ 0 & 1 \end{bmatrix} \right\| < \gamma.$$  

(6)

Given that the conditions of part ii) hold, then the matrix $\bar{Z} \doteq (1 - \gamma^{-2}\bar{Y}\bar{X})^{-1}$ is well defined and one controller which solves the $\text{OP}^\infty_{H^\infty}$ is given by

$$\hat{K}(z) = \left[ \begin{array}{c} \bar{A} + \bar{B} \hat{F} + \bar{Z}\bar{L}_2(\bar{C}_2 + \hat{D}_2 \hat{F}) \\ \hat{F}_2 \\ 0 \end{array} \right],$$

where

$$\hat{F} \doteq - (\bar{D}^*_1 \hat{D}^*_1 - \gamma^2 \bar{E}_{11} + \bar{B}^* \bar{X} \bar{B})^{-1} (\bar{D}^*_1 \hat{C}_1 + \bar{B}^* \bar{X} \bar{A}) = \left[ \begin{array}{c} \hat{F}_1 \\ \hat{F}_2 \end{array} \right],$$  

(7a)

$$\hat{L} \doteq -(\bar{B}_1 \hat{D}_1^* + \bar{A} \bar{Y} \hat{C}^*) (\hat{D}_1 \hat{D}_1^* - \gamma^2 \bar{E}_{11} + \hat{C} \bar{Y} \hat{C}^*)^{-1} = \left[ \begin{array}{cc} \hat{L}_1 & \hat{L}_2 \end{array} \right].$$  

(7b)

Remark 3.1. As was mentioned above, the solution to $\text{OP}^\infty_{H^2}$ can be derived from that of $\text{OP}^\infty_{H^\infty}$ by the substitution $\gamma = \infty$. For completeness, note that the optimal $H^2$ norm is then (Mirkin et al., 1997a)

$$j^2_{\text{opt}} = \frac{1}{\bar{n}}(\|\bar{D}_{11}\|_{HS}^2 + \text{tr}(\bar{X}\bar{B}_1 \bar{B}_1^* + \hat{C}_1 \hat{C}_1 \bar{Y} + (\bar{A}' \bar{X} \bar{A} - \bar{X} \bar{Y})).$$  

(8)

4 Main result

The solution of the MDCS $H^\infty$ (and consequently the $H^2$) problem obtained in the previous section is not readily implementable and has to be “peeled off” in order to be useful. Such a “peeling off” is the subject matter of this section. In particular, here the solutions of the $\text{OP}^\infty_{H^\infty}$ and $\text{OP}^\infty_{H^\infty}$ are presented. All proofs can then be found in Section 6.

Define the following symplectic matrix:

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \doteq \exp \left( \begin{bmatrix} A & -B_1 B_1' \\ \gamma^{-2} C_1 c & -A' \end{bmatrix} \right).$$

Then the following assumptions are the counterparts of (A1)-(A4) whenever $\gamma > \rho(\bar{D}_1^* \bar{D}_1)$:

(A1'): The pair $(A, B_2)$ is stabilizable;

(A2'): The pair $(C_2, \Sigma_{11})$ is detectable;

(A3'): The matrix

$$\begin{bmatrix} \Lambda - j \omega I & B_2 \\ C_{1c} & D_{12c} \end{bmatrix}$$

is left invertible $\forall \omega \in \mathbb{R}$ and $D_{12c} D_{12c} > 0$;

(A4'): The matrix

$$\begin{bmatrix} \Sigma_{11} - e^{i \theta} I & \Sigma_{12} \\ C_2 & 0 \end{bmatrix}$$

is right invertible $\forall \theta \in [0, 2\pi)$. 
Note, that \((A2')\) is equivalent to \((A2)\) only when \(\gamma\) is such that the “output injection” problem (Zhou et al., 1995, §12.2) associated with the \(\mathcal{OP}_{\mathcal{H}_\infty}\) has a solution. The solvability of the latter, however, is necessary for the solvability of the \(\mathcal{OP}_{\mathcal{H}_\infty}\). Hence, the replacement of \((A2)\) with \((A2')\) is justified. Moreover, in the \(\mathcal{H}_2\) case, that is when \(\gamma \to \infty\), \((A2')\) is completely equivalent to \((A2)\).

Now, assuming that \(D_{12c}\) has a full column rank, define the Hamiltonian matrix

\[
H \doteq \begin{bmatrix}
A & \gamma^{-2}B_1B_1' \\
-C_1'C_{1c} & -A'
\end{bmatrix} + \begin{bmatrix}
-B_2' \\
C_1'C_{12c}
\end{bmatrix} (D_{12c}D_{12c})^{-1} \begin{bmatrix}
D_{12c}C_{1c} & B_2'
\end{bmatrix}
\]

and the symplectic matrix

\[
\Gamma \doteq \begin{bmatrix}
I & 0 \\
-C_1'dC_{1d} & I
\end{bmatrix} e^{H_h} = \begin{bmatrix}
\Gamma_{11} & \Gamma_{12} \\
\Gamma_{21} & \Gamma_{22}
\end{bmatrix}.
\]

Form also the matrices

\[
\Delta_l \doteq \begin{bmatrix}
0 & C_2 \\
\Sigma_{11}' & \Sigma_{21}' \\
\Sigma_{12}' & \Sigma_{22}'
\end{bmatrix} \begin{bmatrix}
I & \gamma^{-2}C_1'dC_{1d} & C_2'
0 & I & 0
\end{bmatrix}
\]

and

\[
\Delta_r \doteq \begin{bmatrix}
0 & 0 & 0 \\
I & 0 & 0 \\
0 & I & 0
\end{bmatrix}.
\]

The main result of the paper is then formulated as follows:

**Theorem 2.** Let assumptions \((A1')-(A4')\) hold and \(\gamma > \rho (\tilde{D}_{11}\tilde{D}_{11})\). Then the following statements are equivalent:

i) There exists a discrete-time controller \(\bar{K}\) and a hold function \(\mathcal{H}_h\) which solve the \(\mathcal{OP}_{\mathcal{H}_\infty}\).

ii) \(\Gamma \in \text{Ric}_D\) and \((\Delta_l, \Delta_r) \in \text{Ric}_D\) and the following conditions hold:

(a) \(X \succeq 0\) and \(\rho (X_d\Sigma_{12}\Sigma_{22}^{-1}) < \gamma^2\);

(b) \(Y \succeq 0\) and \(\rho (\Sigma_{22}^{-1}\Sigma_{21}Y) < 1\);

(c) \(\rho (Y(\Sigma_{22}^{-1}\Sigma_{21}Y)^{-1}X_d(\gamma^2\Sigma_{22}^{-1}\Sigma_{12}X_d)^{-1}) < 1\),

where \(X_d \doteq X + C_1'dC_{1d}, \ X = \text{Ric}_D(\Gamma)\) and \((Y, L_2') = \text{Ric}_D(\Delta_l, \Delta_r)\).

Furthermore, if the conditions of part ii) hold, then the matrix \(Z \doteq (I - \gamma^{-2}YX)^{-1}\) is well defined and one controller which solves the \(\mathcal{OP}_{\mathcal{H}_\infty}\) consists of the discrete part \(\bar{K}\) having the transfer function

\[
\bar{K}(z) = z \begin{bmatrix}
(I + ZL_2C_2)(\Gamma_{11} + \Gamma_{12}X) & -ZL_2 \\
I & 0
\end{bmatrix} \begin{bmatrix}
1 \\
X
\end{bmatrix}
\]

and the hold \(\mathcal{H}_h\) with the hold function

\[
\phi_{\mathcal{H}}(\tau) = -(D_{12c}D_{12c})^{-1} \begin{bmatrix}
D_{12c}C_{1c} & B_2'
\end{bmatrix} e^{H_h} \begin{bmatrix}
1 \\
X
\end{bmatrix}.
\]

When \(\gamma^{-2} = 0\), the conditions in ii) always hold true and \(\bar{K}\) and \(\mathcal{H}_h\) above are the unique solutions to the \(\mathcal{OP}_{\mathcal{H}_2}\).
Remark 4.1. It can be shown that $\phi_H(\tau)$ can be expressed equivalently as

$$\phi_H(\tau) = F_Q(\tau)\Phi_Q(\tau,0),$$

where $F_Q(\tau) = -(D'_{12c}D'_{12c})^{-1}(D'_{12c}C_{1c} + B'_2Q(\tau))$ and $\Phi_Q(\cdot, \cdot)$ denotes the transition matrix of $A + \frac{1}{\tau}B_1B'_1(t) + B_2F(t)$, where $Q(t)$ is the solution to the following differential Riccati equation:

$$\dot{Q} + A'Q + QA + C_{1c}C_{1c}$$

$$+ \frac{1}{\tau}QB_1B'_1 - (D'_{12c}C_{1c} + B'_2Q)'(D'_{12c}D_{12c})^{-1}(D'_{12c}C_{1c} + B'_2Q) = 0$$

with the initial condition $Q(0) = X$ or, equivalently, with the final condition $Q(h) = X_d$. Remarking 4.2. The previous remark prompts an interesting interpretation of the hold function $\phi_H(\tau)$ defined in Theorem 2. Indeed, in the $H^\infty$ case this $\phi_H(\tau)$ attempts to reconstruct the continuous-time state feedback control law for the $H^\infty$ problem

$$x'(h)X_dx(h) + \int_0^h z_c'(\tau)z_c(\tau)d\tau < \int_0^h w'(\tau)w(\tau)d\tau$$

subject to the “worst-case” disturbance $w_{wors}(t) = \frac{1}{\tau}B_1Q(t)x(t)$, where $x(t)$ is the state vector of $P$. Comparing the above interpretation with that of the PCTS hold proposed by (Mirkin et al., 1997a, §61), one can see that the MDCS hold takes the discrete-time requirements into account by adding a penalty on the final state $x(h)$.

4.1 Optimal $H^2$ cost

The optimal MDCS $H^2$ performance (8) can be peeled-off using standard lifting arguments (Bamieh and Pearson, 1992b). It may, however, be more informative to know how the cost is distributed between the discrete- and continuous-time components. This would allow an explicit accounting for the tradeoff between the discrete- and continuous-time performances. Thus, below we extract these components from $J_{opt}$.

To this end, let $J_{zd,w}$ and $J_{zc,w}$ be the closed-loop operators from $w$ to $z_d$ and $z_c$, respectively. With slight abuse of notation, denote $J_{d,opt} := \| J_{zd,w} \|_{H^2}$ and $J_{c,opt} := \| J_{zc,w} \|_{H^2}$, both subject to the MDCS optimal controller given in Theorem 2. Thus, $J_{opt}^2 = J_{d,opt}^2 + J_{c,opt}^2$. Define also the following matrix:

$$\Delta = \exp\left(\begin{bmatrix} -A' & C'_1C_{1c} & 0 \\ 0 & A & B_1B'_1 \\ 0 & 0 & -A' \end{bmatrix}\right)$$

$$= \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ 0 & \Delta_{22} & \Delta_{23} \\ 0 & 0 & \Delta_{11} \end{bmatrix}. \tag{9}$$

The following Lemma yields the formulae for $J_{d,opt}$ and $J_{c,opt}$

**Lemma 1.** Let $Q_d$ be the solution to the following discrete Lyapunov equation:

$$Q_d = (\Gamma_{11} + \Gamma_{12}X)'Q_d(\Gamma_{11} + \Gamma_{12}X) + C_{1d}C_{1d}.$$  

Then

$$J_{d,opt}^2 = \frac{1}{n} \text{tr}(Q_d(\Delta_{23}\Delta_{22} + \Delta_{22}\Delta_{22}' - 2Y) + C_{1d}C_{1d}Y)$$

and

$$J_{c,opt}^2 = \frac{1}{n} \text{tr}((X + C_{1d}C_{1d} - Q_d)(\Delta_{23}\Delta_{22} + \Delta_{22}\Delta_{22}' - 2Y) + (\Delta_{13} + \Delta_{12}Y)\Delta_{22}')$$.
5 Example

Consider the following continuous-time plant:
\[ \ddot{y} + 2\dot{y} + y = -0.2\dot{u} + 0.6u + w, \]
where \( y, u, \) and \( w \) are the output, the control input, and the disturbance input, respectively. Let the continuous-time performance index be
\[ J_c^2 = \int_0^\infty (y^2(t) + 10^{-4}u^2(t))\,dt \]
and the discrete-time index be
\[ J_d^2 = \sum_{k=1}^{\infty} y^2(kh), \]
where \( h = 0.1 \) is the sampling period. The MDCS \( H^2 \) problem to be considered is to minimize
\[ J(\lambda) \triangleq (1 - \lambda)J_c^2 + \lambda J_d^2, \]
where the weighting parameter \( \lambda \), which varies in the interval \([0, 1]\), serves to achieve a reasonable tradeoff between the continuous- and discrete-time requirements. \( \lambda = 0 \) corresponds in this setting to the pure continuous-time specifications (PCTS), whereas \( \lambda = 1 \) corresponds to the pure discrete-time specifications (PDTS). Increasing \( \lambda \) thus improves \( J_d \) at the expense of deteriorating \( J_c \). As a measure for the former we chose the quantity
\[ \Delta J_d(\lambda) \triangleq \frac{J_d(\lambda)}{J_d(0)}, \]
while as a measure for the latter:
\[ \Delta J_c(\lambda) \triangleq \frac{J_c(\lambda)}{J_c(0)}, \]
where \( J_d(\lambda) \) and \( J_c(\lambda) \) mean the values of \( J_d \) and \( J_c \), respectively, for the controller minimizing \( J(\lambda) \).

Fig. 3 presents the simulation results of \( \Delta J_d(\lambda) \) versus \( \Delta J_c(\lambda) \) for two cases. In the first one, the hold function is assumed to be the standard zero-order hold (ZOH) and the results of Mirkin and Palmor (1997) are used. In the second one, the hold is the design parameter and the results of Theorem 2 are applied. For an easier comparison, the ZOH results are normalized to \( J_d(0) \) and \( J_c(0) \) obtained for the MDCS design with the optimal hold, rather than to those obtained from the design with the ZOH. Consequently, in that case \( \Delta J_d(0) > 1 \) and \( \Delta J_c(0) > 1 \).

It is seen that the design with the fixed hold function leaves almost no room for a tradeoff between \( J_d \) and \( J_c \) (the graph essentially reduces to a point for all values of \( \lambda \)). In other words, for this problem there is almost no difference between the minimization of \( J_d \) and \( J_c \). The situation is completely different when the hold function becomes the design parameter. The nonminimum-phase zero of the continuous-time plant still imposes hard limitations on the achievable continuous-time performance, yet it does not impose any limitation on the achievable discrete-time one. Consequently, the PDTS approach yields almost 10 times better \( J_d \) than the discrete-time performance under the PCTS approach! This improvement, however, is achieved at the expense of more than 80\% deterioration of the continuous-time performance. The latter might not be acceptable. In such a situation the MDCS approach offers an attractive opportunity to balance between the improvements in \( J_d \) and the degradations in \( J_c \). For example, keeping the continuous-time performance within 5\% of its optimal value, \( J_d \) can be reduced below 0.8 \( J_d(0) \).
6 Proofs

This section is devoted to the proofs of Theorem 2 (it will be shown that the latter is equivalent to Theorem 1) and Lemma 1.

6.1 Preliminary: a new representation of lifted systems

We start with a brief exposition of the results of (Mirkin and Palmor, 1998) concerning a new representation of the parameters of the lifted systems, which plays a central role in the reasoning to follow. Conventionally (Bamieh and Pearson, 1992a; Chen and Francis, 1995), the parameters of lifted systems are described by integral operators over $L^2[0, h]$. Such a representation, however, makes manipulations over these parameters quite a cumbersome, if not an impossible, problem. This fact has motivated a different representation of the parameters of the lifted systems as proposed by Mirkin and Palmor (1998), which considerably simplifies the manipulations over these parameters. The new representation is based upon three components: systems with two-point boundary conditions (STPBC) operating on the time interval $[0, h]$, the impulse operator $I_\theta$, and the sampling operator $I^*_{\theta}$. i) STPBC are linear continuous-time operators $\tilde{O} : L^2[0, h] \rightarrow L^2[0, h]$, which are described by the following state equations:

$$\tilde{O} : \begin{cases} \dot{x}(t) = Ax(t) + Bw(t), & \Omega x(0) + \Upsilon x(h) = 0, \\
\zeta(t) = Cx(t) + Dw(t), & \end{cases}$$

where the square matrices $\Omega$ and $\Upsilon$ shape the boundary conditions of the state vector $x$. The boundary conditions are said to be well-posed if $\det(\Omega + \Upsilon e^{A h}) \neq 0$ and in this case the map $\zeta = \tilde{O} w$ is well defined $\forall w \in L^2[0, h]$, namely,

$$\zeta(t) = Dw(t) + Ce^{At}(\Omega + \Upsilon e^{Ah})^{-1} \left( \Omega \int_0^t e^{-As}Bw(s)ds - \Upsilon \int_t^h e^{A(h-s)}Bw(s)ds \right).$$

It is worth stressing that $I^*_{\theta}$ is not the adjoint of $I_\theta$. Nevertheless, we will proceed with this abuse of notation for the reasons discussed in (Mirkin and Palmor, 1998).
We will denote the STPBC by the following compact block notation:

$$\tilde{\mathcal{O}} = \begin{pmatrix} A & \Omega \leftarrow \Upsilon & B \\ C & \end{pmatrix}$$

and use the term “STPBC” to denote systems with well-posed boundary conditions only. In the case where $\Omega = I$ and $\Upsilon = 0$ (which corresponds to a causal STPBC) the boundary condition “window” will be omitted, like $\begin{pmatrix} A & B \\ C & \end{pmatrix}$.

ii) The impulse operator $J_\theta$ transforms a vector $\eta \in \mathbb{R}^n$ into a modulated $\delta$-impulse as follows:

$$(J_\theta \eta)(t) = \delta(t - \theta)\eta.$$  

iii) The sampling operator $J_\theta^*$ transforms a function $\zeta \in C_n[0, h]$ into a vector from $\mathbb{R}^n$ as follows:

$$J_\theta^* \zeta = \zeta(\theta).$$

With the aid of these operators, the parameters of the lifted generalized plant $\tilde{\mathcal{P}}$ given by (3) can be represented as follows (Mirkin and Palmor, 1998):

$$\begin{bmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}}_{1c} & \tilde{\mathcal{D}}_{1*} \end{bmatrix} = \begin{bmatrix} J_\theta^* & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} A & I & B \\ I & 0 & 0 \\ C_{1c} & 0 & D_{1*} \end{pmatrix} \begin{bmatrix} J_\theta & 0 \\ 0 & 1 \end{bmatrix} \tag{11a}$$

and

$$\begin{bmatrix} \tilde{\mathcal{C}}_{1d} & \tilde{\mathcal{D}}_{1*d} \\ \tilde{\mathcal{C}}_{2} & \tilde{\mathcal{D}}_{2*} \end{bmatrix} = \begin{bmatrix} C_{1d} & C_2 \end{bmatrix} \begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \end{pmatrix}. \tag{11b}$$

The advantages of this representation are that a) the algebraic manipulations over STPBC can be easily performed in the state space, much like manipulations over standard LTI systems; and b) the operators $J_\theta$ and $J_\theta^*$ fit nicely into the state-space framework. In particular, the adjoint of any composition of STPBC, $J_\theta$ and $J_\theta^*$ can be computed component-wise, like

$$(J_\theta^* \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} J_\theta)^* = J_\theta^* \begin{pmatrix} -A' & 0 & 0 \\ -B' & \Omega \leftarrow \Upsilon & 0 \end{pmatrix} J_\theta.$$  

Moreover, $J_\theta$ and $J_\theta^*$ can be “absorbed” into the STPBC’s in an elegant manner. For instance, the matrix $\tilde{\mathcal{Y}}$ in the operators $\tilde{\mathcal{D}}_{11} \tilde{\mathcal{D}}_{11}^* + \tilde{\mathcal{C}}_1 \tilde{\mathcal{Y}} \tilde{\mathcal{C}}_1^*$ affects only the boundary conditions of $\tilde{\mathcal{D}}_{11} \tilde{\mathcal{D}}_{11}^*$ (Mirkin and Palmor, 1998).

In this paper the latter result is generalized and it is established that such an “absorption” property holds even in a more general framework. To this end define the $2 \times 2$ operator $\tilde{\mathcal{O}} : \ell_{L^2[0, h]} \oplus \ell_{L^2[0, h]} \mapsto \ell_{L^2[0, h]} \oplus \ell_{L^2[0, h]}$ as follows:

$$\tilde{\mathcal{O}} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{\mathcal{O}}_{\theta, h} \end{bmatrix} \begin{pmatrix} A & B_{\alpha} \\ C_{\alpha} & 0 \\ \tilde{\mathcal{C}}_{\alpha} & 0 \\ C_\beta & 0 \\ \tilde{\mathcal{C}}_\beta & 0 \end{pmatrix}.$$
and the matrix $\Psi$ as follows:

$$\Psi = \exp \left( \begin{bmatrix} A & -B \beta B' \\ C' \beta & C \beta \end{bmatrix} h \right).$$

The proposition below shows that $C_h$ affects $\tilde{\Omega}^* \star \tilde{\Omega}$ by “reshaping” its boundary conditions only:

**Proposition 1.** Given a matrix $C_h$, then

$$\tilde{\Omega}^* \star \tilde{\Omega} = \begin{pmatrix} A & -B \beta B' \\ C' \beta & C \beta \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & B \alpha \\ C' \alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and the star product exists iff $\det(\Psi_{22} + C' \beta C \beta \Psi_{12}) \neq 0$, where $\Psi_{ij}$ denote the sub-blocks of $\Psi$.

**Proof.** The formula for $\tilde{\Omega}^* \star \tilde{\Omega}$ follows by rather straightforward flow-tracing. Just note that the “discrete” component of $\tilde{\Omega}$ contributes to the state equation of $\tilde{\Omega}$ through the term $C' \beta C \beta x(h) \delta(t-h)$ (here $x$ stands for the state vector of $\tilde{\Omega}$). Hence, its affect consists only in the reshaping of the final conditions.

Now, repeating the arguments of (Gohberg and Kaashoeck, 1984, Thm. 2.1) yields that $\tilde{\Omega}^* \star \tilde{\Omega}$ exists iff it has well posed boundary conditions, from which the existence condition of the proposition follows.

Finally, the computations of matrices involving infinite-dimensional parameters of lifted systems can be reduced to the computation of matrix exponentials:

**Proposition 2 (Mirkin and Palmor 1998).** Let $A$, $B \alpha$, $B \beta$, $C \alpha$, and $C \beta$ be appropriately dimensioned matrices so that $C \alpha B \beta = 0$ and $B \beta B \alpha = 0$, then

$$\begin{pmatrix} J_0 & 0 \\ C' \alpha & C' \beta \end{pmatrix} \begin{pmatrix} A & B \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} J_h & 0 \\ 0 & J_0 \end{pmatrix} = \begin{pmatrix} C' \alpha \gamma e^{A h} & \Omega + \gamma e^{A h} \end{pmatrix}^{-1} \begin{pmatrix} -\gamma B \alpha & \Omega B \beta \end{pmatrix}.$$

### 6.2 Proof of Theorem 2

Theorem 2 is proven by showing the equivalence between all its statements and the corresponding statements of Theorem 1. In particular, in §6.2.1 the equivalence of assumptions (A1)–(A4) and their prime counterparts (A1’–A4’) is established; §6.2.2 and §6.2.3 are devoted to the DARE’s (5); in §6.2.4 the equivalence between condition (6) and conditions (a)–(c) of Theorem 2 is shown; and finally, §6.2.5 gives the formulae for the discrete part of the controller $\bar{K}$ and the hold function $\phi_H(\tau)$.

#### 6.2.1 Assumptions (A1’)–(A4’)

We start the proof of Theorem 2 by showing the equivalence between assumptions (A1)–(A4) and their prime counterparts (A1’–A4’). Actually, we only need to prove the equivalence of (A3) and (A3’), since the others are exactly as in (Mirkin et al., 1997a). Thus, we have:

**Proposition 3.** For every $\gamma > \| \bar{D}_{11} \|$ (A3) is equivalent to (A3’).
Proof. Taking into account the first row in (11b) and the fact that $e^{i\theta} \neq 0$, one can verify that

$$\ker\left(\begin{bmatrix} \hat{A} - e^{i\theta} & \hat{B}_2 \\ \hat{C}_1 & \hat{D}_{12c} \end{bmatrix}\right) = \ker\left(\begin{bmatrix} \hat{A} - e^{i\theta} & \hat{B}_2 \\ \hat{C}_{1c} & \hat{D}_{12c} \end{bmatrix}\right).$$

Hence, (A3) holds only if $\ker\left(\begin{bmatrix} \hat{B}_2^* & \hat{D}_{12c}^* \end{bmatrix}\right) = 0$, from which the necessity of $\hat{D}'_{12c}\hat{D}_{12c} > 0$ follows by the same arguments as in (Mirkin et al., 1997b, Lemma 2).

Now, assume that $\hat{D}'_{12c}\hat{D}_{12c} > 0$. Then, following the reasoning of (Mirkin et al., 1997b, Lemma 2), there exist $\bar{\eta}$ and $\bar{\eta}_\alpha$ such that

$$\begin{bmatrix} \bar{\eta} \\ \bar{\eta}_\alpha \end{bmatrix} = \begin{bmatrix} \hat{A} - j^{\theta+\gamma} & \hat{B}_2 \\ \hat{C}_{1c} & \hat{D}_{12c} \end{bmatrix}^{-1} \begin{bmatrix} \eta \\ \eta_\beta \end{bmatrix} = 0$$

iff there exist a vector $\eta_\beta$ and a $\kappa \in \mathbb{Z}$ such that

$$\begin{bmatrix} \hat{A} - j^{\theta+2\pi\kappa} & \hat{B}_2 \\ \hat{C}_{1c} & \hat{D}_{12c} \end{bmatrix} \begin{bmatrix} \eta \\ \eta_\beta \end{bmatrix} = 0.$$ 

This completes the proof.

6.2.2 Control Riccati equation

In this subsection we study the control $H^\infty$ DARE (5a). In particular, we are concerned with its stabilizing solution $X$, the closed-loop matrix $\hat{A}_{c1,X} = \hat{A} + \hat{B}\hat{F}$, where $\hat{F}$ is defined by (7a), and the $H^\infty$ state-feedback “gain” $\hat{F}_2$.

The following lemma plays a key role in studying (5a):

**Proposition 4.** The operator $\hat{D}'_{12c}\hat{D}_{12c} - \gamma^2 E_{11} + \hat{B}^*\hat{X}\hat{B}$ is invertible iff $\det(\hat{D}'_{12c}\hat{D}_{12c}) \neq 0$ and $\det(\Gamma_{22} - X\Gamma_{12}) \neq 0$. Moreover, if these two conditions hold true, then the right-hand side of (5a) becomes

$$J_0^{(12)} = \begin{bmatrix} H_{11} & H_{12} & I \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} J_0,$$

and

$$\hat{F} = \begin{bmatrix} \gamma^2 I & 0 & 0 \\ 0 & -\hat{D}'_{12c}\hat{D}_{12c} \end{bmatrix}^{-1} \begin{bmatrix} H_{11} & H_{12} & I \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} J_0,$$

where $X_d = X + \hat{C}'_1\hat{D}_{12c}$. 

**Proof.** Since $[\hat{C}_{1d} \hat{D}_{1d}] = \hat{C}_{1d}[\hat{A} \hat{B}_1]$, the right-hand side of (5a) with respect to the variable $X_d$ reduces to the right-hand side of the $H^\infty$ DARE arising in the sampled-data problem with pure continuous-time specifications. Thus, the proof is completed by substituting

$$[B_1 B_2] \rightarrow B, \quad \begin{bmatrix} \hat{C}_{1c} \\ 0 \end{bmatrix} \rightarrow C, \quad \begin{bmatrix} 0 \\ I \end{bmatrix} \rightarrow D,$$

and $J = [I \gamma^2 I]$ into (Mirkin and Palmor, 1998, Lemma 6). □
Assume that

Proposition 5. Whenever $\| \bar{D} \| < 1$, $M_{23}$ is nonsingular,

\[
\begin{bmatrix}
-M_{21} & I \\
M_{12} & M_{22} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
= 0,
\]

or, equivalently, to

\[
\begin{bmatrix}
\Gamma_{11} & \Gamma_{12} \\
\Gamma_{21} & \Gamma_{22}
\end{bmatrix}
\begin{bmatrix}
I \\
X
\end{bmatrix}
= \begin{bmatrix}
I \\
X
\end{bmatrix}(\Gamma_{11} + \Gamma_{12}X).
\]

As seen from (14'), Im$[X]$ is $\Gamma$-invariant. Moreover, one can verify (see (Mirkin and Palmor, 1998, Section 5)) that given any $X$ satisfying (14), then $A_{c1,X} = \Gamma_{11} + \Gamma_{12}X$ and the matrix $\Gamma_{22} - X\Gamma_{12}$ is nonsingular. These observations lead to the following

**Lemma 2.** There exists the stabilizing solution to DARE (5a) iff $\det(D'_{12c}D_{12c}) \neq 0$ and $\Gamma \in \text{dom} \text{Ric}_D$. Moreover, if these two conditions hold true, then $X = \text{Ric}_D(\Gamma)$ is the stabilizing solution to (5a), $A_{c1,X} = \Gamma_{11} + \Gamma_{12}X$, and

\[
\hat{F}_2 = -D'_{12c}D_{12c}^{-1} \begin{pmatrix}
H_{11} & H_{12} & I \\
H_{21} & H_{22} & X \\
D'_{12c}C_{12c} & B_2^* & 0
\end{pmatrix}.
\]

**6.2.3 Filtering Riccati equation**

In this subsection we study the filtering $H^\infty$ DARE (5b). In particular, we are concerned with its stabilizing solution $Y$, and the $H^\infty$ filter gain $L_2$.

We start with some technical results. Let $\hat{R} \doteq (I - \hat{D}_{11}'\hat{D}_{11})^{-1}$, $\hat{S} \doteq (I - \hat{D}_{11}'\hat{D}_{11})^{-1}$ and define the matrix

\[
M = \begin{bmatrix}
M_{11} & M_{12} & M_{13} \\
M_{21} & M_{22} & M_{23} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix} \doteq \begin{bmatrix}
\hat{A}_Y' & \hat{C}_Y' & \hat{C}_1^* \hat{S} \hat{C}_1 \\
\hat{B}_1 \hat{R} \hat{B}_1^* & \hat{B}_1 \hat{R} \hat{D}_{21}^* & \hat{A}_{\gamma} \\
\hat{D}_{21} \hat{R} \hat{B}_1^* & \hat{D}_{21} \hat{R} \hat{D}_{21}^* & \hat{C}_Y 
\end{bmatrix},
\]

where $\hat{A}_Y \doteq \hat{A} + \hat{B}_1 \hat{R} \hat{D}_{11}^* \hat{C}_1$ and $\hat{C}_Y \doteq \hat{C}_2 + \hat{D}_{21} \hat{R} \hat{D}_{11}^* \hat{C}_1$. Then, we have:

**Proposition 5.** Whenever $\| \hat{D}_{11} \| < 1$, $M_{23}$ is nonsingular,

\[
\begin{bmatrix}
-M_{21} & I \\
M_{12} & M_{22} \\
M_{31} & M_{32} & M_{33}
\end{bmatrix}
= 0,
\]

and

\[
\begin{bmatrix}
M_{11} - M_{12}M_{22}^{-1}M_{21} & M_{13}M_{22}^{-1} \\
-M_{23}^{-1}M_{21} & M_{23}^{-1}
\end{bmatrix} = \Sigma' \begin{bmatrix}
I & C_{14}C_{14} \\
0 & I
\end{bmatrix}.
\]

Proof. Assume that $\| \hat{D}_{11} \| < 1$ and let

\[
\hat{O} \doteq \begin{bmatrix}
\hat{A} & \hat{B}_1 \\
\hat{C}_2 & \hat{D}_{21}
\end{bmatrix}.
\]
It is seen that $M = \tilde{\Omega}^\ast \ast \tilde{\Omega}$ and the star product is well defined. Hence, application of Proposition 1 with $B_\alpha = I$, $B_\beta = B_1$, $C_\alpha = [I \ C_2', C_\beta = C_{1c}$, and $C_h = C_{1d}$ yields:

$$M = \begin{pmatrix} J'_0 & 0 & 0 \\ 0 & C_\alpha & 0 \\ 0 & 0 & J'_h \end{pmatrix} \begin{pmatrix} A & -B_1B'_1 \\ C_{1c}C_{1c} & -A' \\ 0 & 0 & -I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J_hC'_\alpha & 0 \\ 0 & J'_0 \end{pmatrix},$$

from which (15a) and (15b) follow immediately. Since the star product exists it follows from Proposition 2 that det$(\Sigma_{22} + C_1dC_{1d}\Sigma_{12}) \neq 0$ and

$$\begin{pmatrix} M_{11} & M_{13} \\ M_{21} & M_{23} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \Sigma_{11} \end{pmatrix} + \begin{pmatrix} I \\ -\Sigma_{12} \end{pmatrix} (\Sigma_{22} + C_1dC_{1d}\Sigma_{12})^{-1}[I \Sigma_{21} + C_1dC_{1d}\Sigma_{11}].$$

Now, (15c) follows by some tedious algebra using the facts that $M_{21}$ and $M_{13}$ are symmetric and $M_{11} = M_{23}$. The latter equality also leads to the non-singularity of $M_{23}$.

Now we are in the position to prove the main result of this subsection:

**Lemma 3.** Let $\|\tilde{D}_{11}\| < 1$, then there exists the stabilizing solution to DARE (5b) iff $(\Delta_l, \Delta_r) \in \text{dom} \ \text{Ric}_D$. Moreover, if this condition holds true, then $(Y, L'_2) = \text{Ric}_D(\Delta_l, \Delta_r)$.

**Proof.** Define the following matrices:

$$M_Y = \begin{pmatrix} M_{11} & 0 & M_{12} \\ M_{21} & -I & M_{22} \\ M_{31} & 0 & M_{32} \end{pmatrix} \quad \text{and} \quad N_Y = \begin{pmatrix} I & -M_{13} & 0 \\ 0 & -M_{23} & 0 \\ 0 & -M_{33} & 0 \end{pmatrix}. $$

Mirkin (1997) showed that whenever $\|\tilde{D}_{11}\| < 1$, the stabilizing solution to (5b) exists iff $(M_Y, N_Y) \in \text{dom} \ \text{Ric}_D$. Moreover, if the latter holds, then $(Y, L'_2) = \text{Ric}_D(M_Y, N_Y)$. Therefore, it suffices to prove that the pairs $(\Delta_l, \Delta_r)$ and $(M_Y, N_Y)$ share the same deflating subspaces. To this end, note that a deflating subspace of a matrix pair does not change by pre-multiplying it by a nonsingular matrix. On the other hand, using (15a), (15c), and then (15b) one can verify that

$$T_2T_1[ M_Y \ N_Y ] = [ \Delta_l \ \Delta_r ],$$

where

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -C_2 & 1 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 0 & 0 & I \\ I & -M_{13}M_{23}^{-1} & 0 \\ 0 & -M_{23} & 0 \end{pmatrix}.$$ 

This completes the proof. 

### 6.2.4 The coupling condition

Using the first row in (11b) one can get:

$$\begin{align*}
\left\| \begin{bmatrix} X^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{\Omega} & \tilde{B}_1 \\ \hat{C}_1 & \tilde{D}_{11} \end{bmatrix} \begin{bmatrix} Y^{1/2} & 0 \\ 0 & I \end{bmatrix} \right\| & = \left\| \begin{bmatrix} X^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{\Omega} & \tilde{B}_1 \\ \hat{C}_1 & \tilde{D}_{11} \end{bmatrix} \begin{bmatrix} Y^{1/2} & 0 \\ 0 & I \end{bmatrix} \right\| = \left\| \begin{bmatrix} (X + C_1dC_{1d})^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{\Omega} & \tilde{B}_1 \\ \hat{C}_1 & \tilde{D}_{11} \end{bmatrix} \begin{bmatrix} Y^{1/2} & 0 \\ 0 & I \end{bmatrix} \right\| = \left\| \begin{bmatrix} (X + C_1dC_{1d})^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{\Omega} & \tilde{B}_1 \\ \hat{C}_1 & \tilde{D}_{11} \end{bmatrix} \begin{bmatrix} Y^{1/2} & 0 \\ 0 & I \end{bmatrix} \right\|. 
\end{align*}$$
But the latter norm is exactly the one dealt with in (Mirkin et al., 1997b, §5) modulo the replacement $X \rightarrow X_d = X + C_1 d$. This leads to conditions (a)–(c) of Theorem 2.

6.2.5 Controller formulae

The natural choice for partitioning $\hat{\mathcal{K}}$ to a discrete-time controller $\tilde{\mathcal{K}}$ and generalized hold $\mathcal{H}_h$ is

$$\tilde{\mathcal{K}}(z) = z \left[ \begin{array}{c|c} \hat{A} + \hat{B} \hat{F} + ZL_2(\hat{C}_2 + \hat{D}_{22} \hat{F}) & -ZL_2 \\ \hline I & 0 \end{array} \right]$$

(including $z$ from the sampler) and

$$\tilde{\mathcal{H}}_h(z) = \hat{F}_2.$$

Taking into account the second row in (11b) we have:

$$\hat{A} + \hat{B} \hat{F} + ZL_2(\hat{C}_2 + \hat{D}_{22} \hat{F}) = (I + ZL_2 \hat{C}_2)(\hat{A} + \hat{B} \hat{F}) = (I + ZL_2 \hat{C}_2)(\Gamma_{11} + \Gamma_{12} X),$$

where Lemmas 2 and 3 are used to get the last equality. This leads to $\hat{\mathcal{K}}$. Lemma 2 yields also the formula for $\phi_{11}(\tau)$.

6.3 Proof of Lemma 1

To prove Lemma 1 some preliminary results are required.

**Proposition 6.** Let $\tilde{\mathcal{K}}$ be the $H^2$ optimal controller for $OP_{H^2}^m$, $\mathcal{Y}$ be the stabilizing solution to the DARE (5b) with $\gamma = \infty$, and $\hat{F}_2$ be the optimal state feedback “gain.” Then given an operator $\hat{U}$ such that $\hat{U} \hat{U}^* = I$,

$$\| \hat{U} \gamma_{(\hat{Y}, \hat{K})} \|^2 = \frac{1}{\bar{H}} \left( \| \hat{U} \bar{D}_1 \|^2_{HS} + \text{tr}(Q \hat{B}_1 \hat{B}_1^* + \hat{C}_1 \hat{U}^* \hat{U} \hat{C}_1 Y + (\hat{A}' Q \hat{A} - Q) Y) \right),$$

where $Q$ is the solution to the discrete Lyapunov equation

$$Q = (\hat{A} + \hat{B} \hat{F}_2)' Q (\hat{A} + \hat{B} \hat{F}_2) + (\hat{C}_1 + \hat{D}_{12} \hat{F}_2) \hat{U}^* \hat{U} (\hat{C}_1 + \hat{D}_{12} \hat{F}_2).$$

**Proof.** Denote $\hat{A}_F = \hat{A} + \hat{B}_2 \hat{F}_2$, $\hat{C}_F = \hat{C}_1 + \hat{D}_{12} \hat{F}_2$, $\hat{A}_L = \hat{A} + L_2 \hat{C}_2$, and $\hat{B}_L = \hat{B}_1 + L_2 \hat{D}_{21}$. Then the DARE (5b) can be rewritten as follows:

$$\gamma = \hat{A}_L \hat{Y} \hat{A}_L^* + \hat{B}_L \hat{B}_L^*$$

and the transfer function of the closed-loop system $\check{\mathcal{F}}_{c1} = \hat{U} \gamma_{(\hat{Y}, \hat{K})}$ is

$$\check{\mathcal{F}}_{c1}(z) = \hat{U} \left[ \begin{array}{c|c} A_F & \hat{B}_2 \hat{F}_2 \\ \hline 0 & \hat{A}_L \end{array} \right] \left[ \begin{array}{c} \hat{B}_1 \\ -\hat{B}_L \end{array} \right].$$

Routine algebra gives that the controllability gramian of $\check{\mathcal{F}}_{c1}$ is

$$Q_1 = A_F Q_1 A_F' + (A' Y A' + \hat{B}_1 \hat{B}_1 - \gamma).$$
Then,
\[
\| \tilde{\mathcal{F}}_{\text{cl}} \|^2_{H^2} = \frac{1}{n} (\| \tilde{\mathcal{D}}_{11} \|^2_{HS} + \text{tr}(\tilde{\mathcal{U}}Q_1 \tilde{\mathcal{C}}_1^* + \tilde{\mathcal{U}}\tilde{\mathcal{Y}})) \\
= \frac{1}{n} (\| \tilde{\mathcal{D}}_{11} \|^2_{HS} + \text{tr}(Q_1 - Q_\alpha Q_1 \tilde{A}_f^* + \tilde{C}_1^* \tilde{\mathcal{U}}\tilde{Y})) \\
= \frac{1}{n} (\| \tilde{\mathcal{D}}_{11} \|^2_{HS} + \text{tr}(Q(\tilde{A}_Y' + \tilde{B}_1 Y' - \tilde{Y}) + \tilde{C}_1^* \tilde{\mathcal{U}}\tilde{Y})),
\]
which completes the proof. \(\square\)

The following proposition contains known results, see (Bamieh and Pearson, 1992b; Mirkin et al., 1997b) for instance, and hence is given without proof:

**Proposition 7.** Given the matrix \(\Delta\) defined by (9), then \(\Delta_{11}^{-1} = \Delta_{22}^*\) and

\[
\begin{align*}
\bar{A} &= \Delta_{22}, \\
\bar{B}_1 \bar{B}_1^* &= \Delta_{23}\Delta_{12}^*, \\
\bar{C}_1^* \bar{C}_1 &= \Delta_{12}^* \Delta_{12},
\end{align*}
\]

and

\[
\| \tilde{\mathcal{D}}_{11} \|^2_{HS} = \text{tr}(\Delta_{22} \Delta_{13}).
\]

Now we are in the position to prove Lemma 1.

**Proof of Lemma 1.** Let \(\hat{E}_2 = [\begin{smallmatrix} 0 & 1 \end{smallmatrix}]\), where the partitioning is compatible with that of \(\hat{C}_1\) and \(\hat{D}_{1t}\) in (4). Then, according to Proposition 6

\[
J_{\text{d, opt}}^2 = \| \hat{E}_2 \hat{\mathcal{F}}_t(\hat{\mathcal{P}}, \hat{\mathcal{K}}) \|^2_{tr} = \frac{1}{n} (\| \hat{\mathcal{D}}_{11d} \|^2_{HS} + \text{tr}(Q \hat{B}_1 \hat{B}_1^* + \hat{C}_1 d \hat{C}_1 d Y + (\bar{A}' Q \bar{A} - Q) Y)),
\]

where \(Q\) is the solution to the discrete Lyapunov equation

\[
Q = (\bar{A} + \hat{B}_2 \hat{F}_2)' Q (\bar{A} + \hat{B}_2 \hat{F}_2) + (\hat{C}_1 d + \hat{D}_{12d} \hat{F}_2)' (\hat{C}_1 d + \hat{D}_{12d} \hat{F}_2).
\]

Since \([\begin{smallmatrix} \hat{C}_1 d & \hat{D}_{12d} \end{smallmatrix}] = \hat{C}_1 d [\begin{smallmatrix} \bar{A} & \hat{B}_2 \end{smallmatrix}]\), it is clear that \(Q_d = Q + \hat{C}_1 d \hat{C}_1 d\) and

\[
J_{\text{d, opt}}^2 = \frac{1}{n} \text{tr}(Q_d \hat{B}_1 \hat{B}_1^* + (\bar{A}' Q_d \bar{A} - Q_d) Y + \hat{C}_1 d \hat{C}_1 d Y)
\]

from which the result for \(J_{\text{d, opt}}^2\) follows by applying Proposition 7. The result for \(J_{\text{c, opt}}^2\) then follows from the equality \(J_{\text{c, opt}}^2 = J_{\text{d, opt}}^2 - J_{\text{d, opt}}^2\). \(\square\)

### 7 Concluding remarks

In this paper the \(H^2\) and \(H^\infty\) sampled-data problems with mixed discrete/continuous specifications have been considered. The main feature of our treatment is that the hold function is not fixed beforehand, but rather is the design parameter together with the discrete-time part of the sampled-data controller. The design of the hold in the MDCS framework enables one to extend the range of the intersample tradeoff with respect to the MDCS design with the standard zero-order hold. The solution presented in Theorem 2 is actually not more involved computationally than the solution with a fixed hold.

The (sub)optimal hold functions derived in the paper are of the exponential form, like those for sampled-data problems with pure continuous-time specifications derived in (Mirkin et al.,
1997a). Unfortunately, such hold functions are not readily implementable on digital hardware. A possible solution may be to approximate the optimal hold by a piecewise-constant one. This, however, raises the question of how many divisions are needed for an accurate approximation. It is believed that a better approach in this direction is to incorporate waveform constraints into the design of the hold functions directly. This is the subject of current research.

References


