Achievable $H^\infty$ Performance in Sampled-Data Smoothing: Beyond the $\|\mathbf{D}_1\|$-Barrier

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Abstract

The lifting technique is a powerful tool for handling the periodically time-varying nature of sampled-data systems. Yet all known solutions of sampled-data $H^\infty$ problems are limited to the case when the feedthrough part of the lifted system, $\mathbf{D}_1 : L^2[0,h] \mapsto L^2[0,h]$, satisfies $\|\mathbf{D}_1\| < \gamma$, where $\gamma$ is the required $H^\infty$ performance level. While this condition is always necessary in feedback control, it might be restrictive in signal processing applications, where some amount of delay or latency between measurement and estimation can be tolerated. In this paper the sampled-data $H^\infty$ fixed-lag smoothing problem with a smoothing lag of one sampling period is studied. The problem corresponds to the a-posteriori filtering problem in the lifted domain and is probably the simplest problem for which a smaller than $\|\mathbf{D}_1\|$ $H^\infty$ performance level is achievable. The necessary and sufficient solvability conditions derived in the paper are compatible with those for the sampled-data filtering problem. This result extends the scope of applicability of the lifting technique and paves the way to the application of sampled-data methods in digital signal processing.

Key words: Sampled-data systems; $H^\infty$ smoothing; Lifting technique

1 Introduction

Since the early 90’s, much attention has been paid to the control and estimation of continuous-time systems using sampled measurements, see the book [3] and the references therein. This class of systems is referred to as sampled-data systems. An important break-through in the treatment of sampled-data systems was the introduction of “lifting,” an operation that reduces periodically time-varying sampled-data problems to equivalent
time-invariant, albeit inherently infinite-dimensional, discrete ones [20, 17, 16, 2]. The lifting technique enables unified and intuitively clear solutions to a wide spectrum of sampled-data (and other periodic) problems.

The key step in the use of the lifting technique is the reduction of infinite-dimensional $H^\infty$ problems in the lifted domain to equivalent finite-dimensional discrete problems. Several different approaches to perform this step are currently available [2, 6, 18, 13]. Roughly, all of them are based on the assumption that the $(1,1)$ sub-block of the feedthrough part of the lifted generalized plant, say $\check{D}_1 : L^2[0, h] \mapsto L^2[0, h]$, satisfies $\|\check{D}_1\| < \gamma$ (here $\gamma$ is the required $H^\infty$ level). When the sampled-data controller/estimator is causal this assumption is not restrictive. Indeed, such a controller/estimator is always strictly proper in the lifted domain [12] and therefore the $\gamma$-suboptimal $H^\infty$ sampled-data problem is solvable only if $\|\check{D}_1\| < \gamma$.

The assumption $\|\check{D}_1\| < \gamma$ does become restrictive when non-causal (smoothed) solutions are allowed. This is the case, for example, in signal processing applications, where some amount of delay or latency between measurement and estimation can be tolerated, see [8, 7] for the application of the sampled-data $H^\infty$ techniques to some problems in digital signal processing. Such problems are reduced to the standard $H^\infty$ control/estimation problem in the lifted domain with delayed regulated signal (or, equivalently, with a non-causal controller/estimator). When $\gamma > \|\check{D}_1\|$, the latter problem can be solved exactly by the reduction to an equivalent discrete $H^\infty$ problem. Yet the available approaches cannot be applied if further reduction of $\gamma$ is required. To circumvent this difficulty, Khargonekar and Yamamoto [8] used the rational approximation of the continuous-time delay, whereas Ishii et al. [7] used the fast sampling approximation of $\check{D}_1$. These approximations, however, might considerably increase the dimension of the problem. It is therefore important to develop new approaches, which can cope with the case of $\gamma \leq \|\check{D}_1\|$, see in this respect the discussion in [21].

The purpose of this paper is to demonstrate, that the machinery introduced in [9] enables one to treat the case of interest in a straightforward manner with no need for approximations. To this end, the sampled-data fixed-lag smoothing problem with the smoothing lag equal to the sampling period $h$ is considered. In the lifted domain this problem corresponds to the a-posteriori filtering problem. This appears to be the simplest problem for which the $H^\infty$ performance level smaller than $\|\check{D}_1\|$ can be achieved. The problem is solved in two steps. First, the formal solution to the lifted a-posteriori filtering problem is just written down in terms of the operator-valued parameters of the lifted system. Second, the time-domain solvability conditions are “peeled-off” from their lifted counterparts. To this end the representation of the lifted parameters introduced in [9] plays a central role and enables the derivation of the solvability conditions to the sampled-data smoothing compatible with those for the sampled-data filtering [15, 10]. In the later case the solvability conditions involve the existence of the solution to a differential Riccati equation over one sampling interval $[0, h]$. As shown in the paper, the solvability conditions for the smoothing problem are based upon the same differential Riccati equation, yet the smoothing problem may be solvable even when this equation does have a finite escape
point in $[0,h]$.

The paper is organized as follows. In Section 2 the problem is posed (§2.1) and its solution is formulated (§2.2). In §2.3 a simple numerical example comparing the filtering and smoothing solvability conditions is considered. The rest of the paper is devoted to the proof of that solution. In Section 3 the problem is reformulated in the lifted domain and in Section 4 the equivalence between time-domain and lifted-domain formulations is proved. Finally, some concluding remarks are provided in Section 5.

**Notation**

In order to distinguish systems in the time domain from the corresponding transfer functions, the former are denoted by script capital letters, so $G(s)$ implies the transfer function of a continuous-time LTI system $G$. The sampling operator, $S_h$, acts on continuous signals so that $\bar{\zeta} = S_h \zeta \iff \bar{\zeta}_k = \zeta(kh^-)$; the continuous-time delay operator $D_\delta$ is defined as follows: $\zeta = D_\delta \omega \iff \zeta(t) = \omega(t - \delta)$; and the backward unit shift operator, $U$, acts on discrete-time (either real valued or functional space valued) signals so that $\bar{\zeta} = U \bar{\omega} \iff \bar{\zeta}_k = \bar{\omega}_{k-1}$.

### 2 Main result

#### 2.1 Problem formulation

The problem of estimating a continuous-time signal $z$ on the basis of sampled measurements $\bar{y} = S_h y$ is considered. It is assumed that both $z$ and $y$ are driven by a “disturbance” signal $w$ as follows:

$$z = G_1 w \quad \text{and} \quad y = G_2 w,$$

where the continuous-time LTI systems $G_1$ and $G_2$ are defined in terms of their state-space realizations:

$$\begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} A & B \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}.$$

Note that the fact that $G_2(s)$ is strictly proper implies in fact that pre-filtering by an anti-aliasing filter is provided. This is necessary to guarantee the boundedness of the sampling operation [3]. $G_1(s)$ is assumed to be strictly proper to simplify the derivations and obtain more transparent results. It will also be assumed thereafter that

(A1): $(C_2, A)$ is detectable and $h$ is non-pathological with respect to $A$;

(A2): $(A, B)$ is controllable and $C_2$ is right invertible.
These assumptions guarantee that the estimation problem is nonsingular. Note that all results of this paper remain unchanged if (A2) is replaced with a milder assumption that \((A, B)\) has no uncontrollable modes on the \(j\omega\)-axis and the restriction of \(C_2\) to the controllable subspace of \((A, B)\) is right invertible. Such a replacement, however, would lead to an unnecessary complication of the proofs (in particular, the proof of Proposition 5 in Section 4), so that the more restrictive assumption (A2) is made. Moreover, since zero initial conditions are assumed, the presence of uncontrollable modes in \((A, B)\) just implies the redundancy of the problem. Indeed, uncontrollable modes remain zero and therefore need not to be estimated. This means that the controllability of \((A, B)\) is somewhat natural in the context of infinite-horizon estimation.

When the estimation of \(z(t)\) can be based on the measurements \(\bar{y}_k\) available up to the time instance \(k = \left\lfloor \frac{t}{h} \right\rfloor\), i.e., the sampled-data estimator is causal, the estimation problem for \(z\) is the standard sampled-data filtering problem extensively studied in the literature, see [15, 10]. In numerous signal processing application, however, non-causal estimation (smoothing) is permitted [1]. In other words, estimation of \(z(t)\) can be based on \(\bar{y}_k\) up to \(k = \left\lfloor \frac{t}{h} \right\rfloor + \delta\), where \(\delta \in \mathbb{Z}^+\) is called the smoothing lag. Such a problem can be equivalently formulated as the estimation of \(D_{\delta h}z\) by a causal estimator.

In this paper the problem of the achievable \(H^\infty\) performance in the sampled-data smoothing in the case of \(\delta = 1\) is studied. This problem can be formulated as follows:

**SP**: Given \(G_1\) and \(G_2\) and the sampling period \(h > 0\), determine whether there exists a causal operator (smoother) \(K\) which guarantees

\[
\|D_h G_1 - KS_h G_2\| < \gamma
\]

for a given \(\gamma > 0\) (here \(\|\cdot\|\) stands for the \(L^2\)-induced operator norm).

To the best of our knowledge, no exact solutions to **SP** is available in the literature. The problems similar to **SP** with arbitrary smoothing lags were treated in [8, 7]. Yet only approximate solutions were obtained there, see Introduction. Thus this paper presents the first treatment of **SP**, where the effect of the previewed information on the achievable \(H^\infty\) performance in sampled-data estimation is accounted for in an exact manner.

### 2.2 Problem solution

The solution to **SP** is based on the following symplectic matrix function:

\[
\Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} \equiv \exp \left( \begin{bmatrix} A & BB' \\ -\frac{1}{\gamma^2}C'_1C_1 & -A' \end{bmatrix} t \right).
\]

\(\left\lfloor \cdot \right\rfloor\) stands for the “floor” (round toward \(-\infty\)) operation.
To simplify the notations, we simply write $\Sigma$ instead of $\Sigma(h)$. Introduce also the matrices

$$
M_t = \begin{bmatrix}
0 & \Sigma_{11} & -\Sigma_{12} \\
C_2' & -\Sigma_{21} & \Sigma_{22} \\
0 & C_2\Sigma_{11} & -C_2\Sigma_{12}
\end{bmatrix}
$$

and

$$
M_r = \begin{bmatrix}
0 & I & 0 \\
0 & 0 & I \\
0 & 0 & 0
\end{bmatrix},
$$

which constitute [13] an extended symplectic matrix pair $(M_t, M_r)$, i.e., the associated matrix pencil $M_t - \lambda M_r$ verifies (a) $\det(M_t - \lambda M_r) \neq 0$, (b) If $\lambda \notin \{0, \infty\}$ is a generalized eigenvalue of $M_t - \lambda M_r$ of multiplicity $r$ then so is $\frac{1}{\lambda}$, and (c) if 0 is an eigenvalue of $M_t$ of multiplicity $r$ then it is an eigenvalue of $M_r$ of multiplicity at least $r + m$, where $m$ is the row dimension of $C_2$. We say that $(M_t, M_r) \in \text{dom}(\text{Ric})$ if there exist matrices $Y = Y'$ and $L$ so that

$$
\begin{bmatrix}
I & Y & L
\end{bmatrix} M_r = A_L \begin{bmatrix}
I & Y & L
\end{bmatrix} M_t
$$

for some Schur matrix $A_L$. It can be shown [19] that whenever $(M_t, M_r) \in \text{dom}(\text{Ric})$, the matrices $Y$ and $L$ as above are uniquely determined from $(M_t, M_r)$ and it is thus possible to define the function $\text{Ric} : (M_t, M_r) \rightarrow (Y, L)$. Below, this function is denoted by the following notation: $(Y, L) = \text{Ric}(M_t, M_r)$.

Let $C_1 \perp$ be any matrix such that $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ has full column rank and $C_1 \perp C_2 = 0$. Then the main result of the paper is formulated as follows:

**Theorem 1** Let (A1) and (A2) hold. Then $SP$ is solvable if, and only if,

(a) $(M_t, M_r) \in \text{dom}(\text{Ric})$;

(b) $Y \succeq 0$, where $(Y, L) = \text{Ric}(M_t, M_r)$; and

(c) $M_S(t) \doteq \begin{bmatrix}
C_2\left(\Sigma_{12}(t) + \Sigma_{11}(t)Y\right) \\
C_1 \perp \left(\Sigma_{22}(t) + \Sigma_{21}(t)Y\right)
\end{bmatrix}$ is nonsingular $\forall t \in (0, h]$.

Note that the only difference between the solution to the smoothing problem given by Theorem 1 and the solution to the sampled-data filtering problem in [10] is condition (c). In the filtering case the nonsingularity of $M_F(t) \doteq \Sigma_{22}(t) + \Sigma_{21}(t)Y$ is required instead, see also [15]. One would expect that condition (c) of Theorem 1 is milder than the latter condition. To show this, note that the nonsingularity of $M_F(t)$ is equivalent [3] to the absence of escape points in $[0, h]$ of the following differential Riccati equation:

$$
\dot{Q} = AQ + QA^t + BB^t + \frac{1}{\gamma}QC_1C_1Q, \quad Q(0) = Y.
$$

Indeed, if $M_F(t)$ is nonsingular $\forall t \in [0, h]$, then $Q(t) = (\Sigma_{12}(t) + \Sigma_{11}(t)Y)M_F^{-1}(t)$. Now, assume that $M_F(t)$ becomes singular first time at $t = t_{e_1} > 0$. Then for all $t \in (0, t_{e_1}) M_S(t)$ is nonsingular iff $\det(C_2Q(t)C_1 \perp) \neq 0$. Since $Q(t)$ is monotonically nondecreasing, the matrix $C_2Q(t)C_1 \perp > 0$ for all $(0, t_{e_1})$. This shows that condition (c) of Theorem 1 cannot fail before its filtering counterpart does.
Consider the following simple first-order system:

\[ G_1(s) = G_2(s) = \frac{a}{s+a} \]

with \( a > 0 \). Obviously, \( \gamma = 1 = \| G_1 \|_\infty \) is achievable for any \( h \) in both the filtering and smoothing settings. We therefore consider only the case \( \gamma < 1 \). In this case \( (M_l, M_r) \) always belongs to \( \text{dom}(\text{Ric}) \) with \( Y = 0 \) (since \( C_2 Y = 0 \) always) and then

\[
M_F(t) = \cos \omega t + \frac{a}{\omega} \sin \omega t \quad \text{and} \quad M_S(t) = \frac{a^2}{\omega} \sin \omega t
\]

where \( \omega = a\sqrt{\frac{a}{\gamma - 1}} \). Both these functions are depicted in Fig. 1(a) for the case of \( a = 1 \) and \( \gamma = 0.3 \). It is seen that the smoothing formulation enables one to reduce the requirement on the sampling rate to more than 2/3 of that in the filtering formulation. In general, for a given \( \gamma < 1 \) the maximal sampling period for which this \( \gamma \) is achievable in the filtering and smoothing cases are

\[
h_F = \frac{\pi}{\omega} - \frac{1}{\omega} \arctan \frac{\omega}{a} \quad \text{and} \quad h_S = \frac{\pi}{\omega},
\]

respectively. Thus, by allowing one additional measurement, \( \bar{y}_{k+1} \), to be used to estimate \( z(t), t \in [kh, (k+1)h) \), an up to 2 times relaxation of the sampling rate can be achieved.

Alternatively, smoothing and filtering may be compared by the achievable performance for a given sampling period \( h \). To this end, let \( \gamma_F \) and \( \gamma_S \) be the achievable \( H^\infty \) performance in the filtering and smoothing cases, respectively. Fig. 1(b) shows the relative improvement of the latter with respect to the former, \( \frac{\gamma_F}{\gamma_S} - 1 \) (in %) vs. the normalized sampling period, \( ah \). One can see that as the sampling rate increases, the improvement due to the preview increases as well, approaching 100% as \( h \to 0 \). At first sight this might look counterintuitive. Indeed, for small \( h \) the additional (previewed) measurement which is
available for the optimal smoother contains practically no new information. Yet on the other hand, as \(h \to 0\) the estimation performance in both filtering and smoothing cases becomes zero, approaching the continuous-time performance (ideal estimation). So even if the unnormalized improvement due to smoothing decreases, the relative improvement increases as demonstrated in Fig. 1(b). This just shows that even infinitesimal preview is of value when the (continuous-time) ideal estimator is approached. This would not happen if the optimal continuous-time filtering performance were nonzero.

3 Reformulation and solution in the lifted domain

3.1 Lifted reformulation

The solution of \(SP\) is based on its conversion to a pure discrete-time time-invariant setting by the use of the so-called lifting technique. To this end, let \(W_h\) denote the lifting operator converting real valued signals in continuous time into functional space valued sequences, see [3] for the exact definition and properties of \(W_h\). The lifted signals \(\tilde{w} = W_h w\) and \(\tilde{z} = W_h z\) are equivalent to their continuous-time counterparts \(w\) and \(z\), respectively, in the sense that they contain the same information and their norms are equivalent. One thus can consider the lifted systems

\[
\hat{G}_1 = W_h G_1 W_h^{-1} : \tilde{w} \mapsto \tilde{z}, \\
\hat{G}_2 = S_h G_2 W_h^{-1} : \tilde{w} \mapsto \tilde{y}
\]

instead of \(G_1 : w \mapsto z\) and \(S_h G_2 : w \mapsto \tilde{y}\), respectively.

Remark 2 To improve the readability of formulae when both finite and infinite dimensional input/output spaces are involved, the following operator notation is used. A bar indicates an operator \(\bar{O}\) with both input and output spaces finite dimensional; grave accent \(-\acute{O}\), when the input space is finite dimensional and the output infinite dimensional one; acute accent \(-\grave{O}\), when the input space is infinite dimensional and the output finite dimensional one; and finally breve \(-\acute{\grave{O}}\), when both input and output spaces are infinite dimensional.

The main advantage of dealing with the lifted systems is that they are pure discrete and time invariant. The transfer functions of \(\hat{G}_1\) and \(\hat{G}_2\) are \([9, 10]\)

\[
\hat{G}_1(z) = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C}_1 & \bar{D}_1 \end{bmatrix} \quad \text{and} \quad \hat{G}_2(z) = z^{-1} \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C}_2 & \bar{D}_2 \end{bmatrix} \equiv z^{-1} \hat{G}_{2a}(z),
\]

where

\[
\begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C}_1 & \bar{D}_1 \end{bmatrix} = \begin{bmatrix} I_h & A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}
\]
and \( \begin{bmatrix} \bar{C}_2 & \bar{D}_2 \end{bmatrix} = C_2 \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} \). Here the compact block notation \( \bar{O} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is used to denote an operator \( \bar{O} : L^2[0, h] \rightarrow L^2[0, h] \) described by the following state equations:

\[
\bar{O} : \begin{cases}
\dot{x}(t) = Ax(t) + Bu(t), & x(0) = 0, \\
y(t) = Cx(t) + Du(t);
\end{cases}
\]  

(1)

The impulse operator \( \mathcal{I}_\theta \) transforms a vector \( \eta \in \mathbb{R}^n \) into a modulated \( \delta \)-impulse as \( (\mathcal{I}_\theta \eta)(t) = \delta(t - \theta)\eta \); and the sampling operator \( \mathcal{I}_\theta^* \) transforms a continuous signal \( \zeta \in C^n[0, h] \) into a vector from \( \mathbb{R}^n \) as \( \mathcal{I}_\theta^* \zeta = \zeta(\theta) \).

With the definitions above, \( \text{SP} \) can be reformulated in the lifted domain as follows:

\( \text{SP}_{\text{eq}} \): Given the systems \( \hat{G}_1(z) \) and \( \hat{G}_{2a}(z) \), determine whether there exists a proper \( \hat{K}(z) \) which guarantees

\[
\| \hat{G}_1(z) - \hat{K}(z)\hat{G}_{2a}(z) \|_\infty < \gamma
\]

for a given \( \gamma > 0 \) (here \( \| \cdot \|_\infty \) stands for the \( H^\infty \) induced norm).

\( \text{SP}_{\text{eq}} \) is, in principle, a standard discrete a-posteriori filtering problem, solution to which is well understood [5]. The solvability condition can thus be expressed in terms of the parameters of \( \hat{G}_1 \) and \( \hat{G}_{2a} \) in a straightforward manner. The only difference between the case treated in the literature and the one here is that \( \bar{D}_1 \neq 0 \); this, however, can easily be accounted for.

### 3.2 Lifted solution

To solve \( \text{SP}_{\text{eq}} \) by the Riccati-based approach, the following two standard assumptions have to be imposed upon the state-space realization of \( \hat{G}_{2a}(z) \):

(A3): \( (\bar{C}_2, \bar{A}) \) is detectable;

(A4): \( \begin{bmatrix} \bar{A} - e^{j\theta}I & \bar{B} \\ \bar{C}_2 & \bar{D}_2 \end{bmatrix} \) is right invertible \( \forall \theta \in [0, 2\pi] \).

It is well known [3] that (A1) implies (A3). To show that (A2) \( \Rightarrow \) (A4) note that

\[
\begin{bmatrix} \bar{A} - e^{j\theta}I & \bar{B} \\ \bar{C}_2 & \bar{D}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_2 & -e^{j\theta}I \end{bmatrix} \begin{bmatrix} \bar{A} - e^{j\theta}I & \bar{B} \\ \bar{C}_2 & 0 \end{bmatrix}.
\]

Hence, (A4) holds iff \( \begin{bmatrix} \bar{A} - e^{j\theta}I & \bar{B} \\ \bar{C}_2 & 0 \end{bmatrix} \) is right invertible \( \forall \theta \). Yet \( \bar{B}\bar{B}^* = \int_0^h e^{As}BB'e^{A's}ds \) [3], so that by (A2) it is invertible. Therefore, (A4) holds iff \( \bar{C}_2 \) has full row rank.

\[\text{It is worth stressing that } \mathcal{I}_\theta^* \text{ is not the adjoint of } \mathcal{I}_\theta. \text{ Nevertheless, we will proceed with this abuse of notation for the reasons discussed in [9].}\]
Consider now the $H^\infty$ filtering Riccati equation

$$Y = \bar{A}Y \bar{A}' + \bar{B}\bar{B}' - \hat{L}\bar{R}\hat{L}^*, \quad (2)$$

where

$$\hat{R} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{bmatrix} = \begin{bmatrix} \bar{D}_1' \bar{D}_2' \\ \bar{D}_1 \bar{D}_2 \end{bmatrix} + \begin{bmatrix} \bar{C}_1' & \bar{C}_2' \end{bmatrix} Y \begin{bmatrix} \bar{C}_1 \bar{C}_2 \end{bmatrix} - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\hat{L} = \begin{bmatrix} \hat{L}_1 & \hat{L}_2 \end{bmatrix} = -\left( \bar{B} \begin{bmatrix} \bar{D}_1' \bar{D}_2' \\ \bar{D}_1 \bar{D}_2 \end{bmatrix} + \bar{A} Y \begin{bmatrix} \bar{C}_1' & \bar{C}_2' \end{bmatrix} \right) \bar{R}^{-1}.$$

The necessary and sufficient solvability conditions of $\text{SP}_{eq}$ are then formulated as follows:

**Theorem 3** Let (A3) and (A4) hold, then $\text{SP}_{eq}$ is solvable iff there exists a solution $Y$ to (2) such that

(a) $\bar{A} + \hat{L}_1\bar{C}_1 + \hat{L}_2\bar{C}_2$ is stable (Schur);
(b) $Y = Y' \geq 0$;
(c) $\bar{R}$ and $\begin{bmatrix} -I & I \end{bmatrix}$ have the same inertia.

**PROOF.** A straightforward extension of [5, Theorem 13.3.1].

Theorem 3 yields the solution of $\text{SP}$ in terms of the infinite-dimensional parameters of the lifted problem. It therefore is not readily verifiable and has to be translated back to the time domain (“peeled-off”) and expressed in terms of finite-dimensional matrices.

Note that the difference between $\text{SP}_{eq}$ and the lifted equivalent of the sampled-data filtering problem [10] is that in the latter case $\hat{K}$ is constrained to be strictly proper rather than just proper as above. In other words, the sampled-data smoothing reduces to an a-posteriori filtering in the lifted domain, whereas the sampled-data filtering reduces to an a-priori filtering. In terms of the lifted solution, this difference affects only condition (c) of Theorem 3. More precisely, in the smoothing case this condition is equivalent to $\hat{R}_{22} > 0$ and

$$\hat{R}_{11} - \hat{R}_{12} \hat{R}_{22}^{-1} \hat{R}_{21} < 0, \quad (3)$$

while in the filtering case (3) is just replaced with

$$\hat{R}_{11} < 0. \quad (3')$$

It is clear that whenever (3’) holds so does (3), but not vice versa. The smoothing problem can thus be solvable even when the filtering problem cannot. Yet (3’) requires $\Theta = \gamma^2 I - \bar{D}_1\bar{D}_1' > 0$, while (3) requires neither $\Theta > 0$ nor even $\Theta$ to be nonsingular. This actually is the main source of difficulties in “peeling-off” the solvability conditions of Theorem 3.
4 “Peeling-off” the lifted solution

The purpose of this section is to prove the equivalence of the conditions of Theorems 1 and 3. Note that items (a) and (b) of Theorem 3 are equivalent to corresponding filtering results and therefore their peeling-off can be carried out following the line of [10]. Hence, only the proof of the equivalence between condition (c) of Theorem 1 and condition (3) will be given.

Throughout this section, we need operators $\hat{\mathcal{O}} : L^2[0, h] \mapsto L^2[0, h]$, which are described by equations like (1) but with the 2-point boundary conditions $\Omega x(0) + \Upsilon x(h) = 0$ for some square matrices $\Omega$ and $\Upsilon$. Such operators, denoted in the sequel as $\hat{\mathcal{O}} = \begin{pmatrix} \Omega & \Upsilon \\ \Omega & \Upsilon \end{pmatrix}$ and called systems with two-point boundary conditions (STPBC), are well-posed [4] iff the matrix $\Xi = \Omega + \Upsilon e^{Ah}$ is non-singular and then $y = \hat{\mathcal{O}} u$ implies that $y = Du + y_c + y_a$, where

$$y_c(t) = C e^{Ah} \Xi^{-1} \Omega \int_0^t e^{-As} Bu(s) ds,$$

$$y_a(t) = -C e^{Ah} \Xi^{-1} \Upsilon \int_t^h e^{A(h-s)} Bu(s) ds.$$  

The advantage of the STPBC representation over the conventional one based on the integral operator description stems from the fact that the manipulations over STPBC can be performed in the state space, much like the manipulations over standard finite-dimensional state-space systems, see [4]. Moreover, as shown in [9], the sampling and impulse operators fit well into the STPBC formalism. The reader is referred to the latter paper as well as to [14] for further details. We present here only the following result which will be used in the sequel:

**Proposition 4 ([9])** Let $A, B_\alpha, B_\beta, C_\alpha$, and $C_\beta$ be appropriately dimensioned matrices so that $C_\alpha B_\alpha = 0$ and $C_\beta B_\beta = 0$, then

$$\begin{pmatrix} T_h C_\alpha \\ T_0 C_\beta \end{pmatrix} \begin{pmatrix} A & I \\ \Omega = \Upsilon & 0 \end{pmatrix} \begin{pmatrix} B_\alpha T_h \\ B_\beta T_0 \end{pmatrix} = \begin{pmatrix} C_\alpha e^{Ah} \\ C_\beta \end{pmatrix} (\Omega + \Upsilon e^{Ah})^{-1} \begin{pmatrix} -\Upsilon B_\alpha & \Omega B_\beta \end{pmatrix}.$$

Now, introduce the following $L^2[0, \tau] \oplus \mathbb{R} \mapsto L^2[0, \tau] \oplus \mathbb{R}$ operator:

$$\begin{pmatrix} \hat{\mathcal{T}}_\tau C_1 \\ \hat{\mathcal{T}}_\tau C_2 \end{pmatrix} \begin{pmatrix} A & BB' \\ 0 & -A' \end{pmatrix} \begin{pmatrix} \hat{I} - Y \\ 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} C_1 \hat{\mathcal{T}}_\tau \\ C_2 \hat{\mathcal{T}}_\tau \end{pmatrix}.$$
It follows from [10, Lemma 14] that $\hat{R}_{11} = \hat{R}_h - \gamma^2 I$, $\hat{R}_{12} = \hat{R}_h$, $\hat{R}_{21} = \hat{R}_h$, and $\hat{R}_{22} = \hat{R}_h$. The following technical result plays a key role in the sequel:

**Proposition 5** Define the matrix

$$
\Pi_y = C_2'(C_2'C_2')^{-1}C_2
$$

and the operator:

$$
\bar{O}_{\tau} = \left( \begin{array}{cc} A & BB' \\ 0 & -A' \end{array} \right) - \left( \begin{array}{cc} \Pi_y & 0 \\ 0 & \Pi_y \end{array} \right) \left( \begin{array}{cc} 0 & 0 \\ 0 & -C'_2 \end{array} \right) : L^2[0, \tau] \rightarrow L^2[0, \tau].
$$

Then, whenever $Y = Y' \geq 0$, $\bar{O}_{\tau}$ is well-posed $\forall \tau > 0$ and, moreover, $\bar{R}_\tau - \bar{R}_\tau\bar{R}^{-1}_\tau \bar{R}_\tau = \bar{O}_{\tau}$.

**PROOF.** Define the matrix function

$$
\Phi(t) = \exp \left( \begin{array}{cc} A & B_2B_1' \\ 0 & -A' \end{array} t \right) = \left( \begin{array}{cc} \Phi_{11}(t) & \Phi_{12}(t) \\ 0 & \Phi_{22}(t) \end{array} \right)
$$

(below $\Phi$ stands for $\Phi(\tau)$). The STPBC $\bar{O}_{\tau}$ is well-posed iff the matrix

$$
\Xi_{\bar{O}_{\tau}} = \left[ \begin{array}{cc} I & -Y \\ 0 & 0 \end{array} \right] + \left[ \begin{array}{cc} 0 & 0 \\ \Pi_y & I - \Pi_y \end{array} \right] \Phi = \left[ \begin{array}{cc} I & -Y \\ 0 & 0 \end{array} \right] + \left[ \begin{array}{cc} 0 & 0 \\ C'_2 & (C_2'C_2)\Phi_{11} \Phi_{12} - \Phi_{22} \end{array} \right]
$$

is nonsingular. Applying the Matrix Inversion Lemma one gets:

$$
\Xi_{\bar{O}_{\tau}}^{-1} = \left[ \begin{array}{cc} I & Y\Phi_{22}^{-1} \\ 0 & \Phi_{22}^{-1} \end{array} \right] - \left[ \begin{array}{cc} Y & 0 \\ 0 & I \end{array} \right] M \left[ \begin{array}{cc} I & Y\Phi_{22}^{-1} + \Phi_{11}^{-1}\Phi_{12}\Phi_{22}^{-1} - \Phi_{11}^{-1} \end{array} \right],
$$

where

$$
M = \Phi_{22}^{-1}C'_2 \left( C_2(\Phi_{12} + \Phi_{11}Y)\Phi_{22}^{-1}C'_2 \right)^{-1} C_2\Phi_{11}.
$$

Thus, $\bar{O}_{\tau}$ is well-posed iff the matrix $C_2(\Phi_{12} + \Phi_{11}Y)\Phi_{22}^{-1}C'_2$ is invertible. Now, it can be shown [3] that $\Phi_{12}\Phi_{22}^{-1} = \int_0^\tau e^{As}BB' e^{At}ds$. Hence, taking into account (A2) and the facts that $\Phi_{22}^{-1} = \Phi_{11}'$ and $Y \geq 0$, one gets:

$$
C_2(\Phi_{12} + \Phi_{11}Y)\Phi_{22}^{-1}C_2 \geq C_2 \int_0^\tau e^{As}BB' e^{As}ds C_2 > 0,
$$

from which the first claim of the Proposition follows immediately.

Now, using (4) one gets that $\bar{R}_\tau u = y_{1c} + y_{1a}$, where

$$
y_{1c}(t) = -C_1\Phi_{11}(t) \int_0^t (\Phi_{12}(-s) - Y\Phi_{22}(-s)) C_1' u(s)ds,
$$

$$
y_{1a}(t) = C_1(\Phi_{12}(t) + \Phi_{11}(t)Y) \int_t^\tau \Phi_{22}(-s) C_1' u(s)ds.
$$
On the other hand, \( \hat{R}_\tau \hat{R}^{-1}_\tau \hat{R}_\tau u = y_2 \), where

\[
y_2(t) = -C_1 \left( \Phi_{12}(t) + \Phi_{11}(t)Y \right) M \int_0^t \left( \Phi_{12}(-s) - Y \Phi_{22}(-s) \right) C_1' u(s) ds
- C_1 \left( \Phi_{12}(t) + \Phi_{11}(t)Y \right) M \int_t^\tau \left( \Phi_{12}(-s) - Y \Phi_{22}(-s) \right) C_1' u(s) ds
= y_{2c}(t) + y_{2a}(t).
\]

Then the response of \( \hat{R}_\tau - \hat{R}_\tau \hat{R}^{-1}_\tau \hat{R}_\tau \) to \( u \) is \( y_c + y_a \), where

\[
y_c(t) = y_{1c}(t) - y_{2c}(t)
= -C_1 \left( \Phi_{11}(t)(I - YM) - \Phi_{12}(t)M \right) \int_0^t \left( \Phi_{12}(-s) - Y \Phi_{22}(-s) \right) C_1' u(s) ds
= \left[ C_1 0 \right] \Phi(t) \Xi^{-1}_{\hat{O}_\tau} \left[ \begin{array}{c} -1 \\ 0 \end{array} \right] \int_0^t \Phi(-s) \left[ \begin{array}{c} 0 \\ -1 \end{array} \right] C_1' u(s) ds
\]

and

\[
y_a(t) = y_{1a}(t) - y_{2a}(t)
= C_1 \left( \Phi_{12}(t) + \Phi_{11}(t)Y \right) \int_t^\tau \left( M \Phi_{12}(-s) + (I - MY) \Phi_{22}(-s) \right) C_1' u(s) ds
= \left[ C_1 0 \right] \Phi(t) \Xi^{-1}_{\hat{O}_\tau} \left[ \begin{array}{c} 0 \\ \Pi_y \end{array} - \Pi_y \right] \int_0^\tau \Phi(\tau - s) \left[ \begin{array}{c} 0 \\ I \end{array} \right] C_1' u(s) ds,
\]

where the equalities

\[
\left[ \begin{array}{cc} I - YM \\ -M \end{array} \right] \left[ \begin{array}{cc} I & -Y \\ 0 & 0 \end{array} \right] = \Xi^{-1}_{\hat{O}_\tau} \left[ \begin{array}{cc} 0 & 0 \\ \Pi_y & I - \Pi_y \end{array} \right]
\]

and

\[
\left[ \begin{array}{cc} I \\ Y \end{array} \right] \left[ \begin{array}{cc} M & I - MY \end{array} \right] = \Xi^{-1}_{\hat{O}_\tau} \left[ \begin{array}{cc} 0 & 0 \\ \Pi_y & I - \Pi_y \end{array} \right]
\]

(can be proved by some tedious algebra) were used. Equations (4) then yield that \( y_c + y_a \) is indeed the response of \( \hat{O}_\tau \) to \( u \), which completes the proof.

**Remark 6** Comparing Proposition 5 with Lemma 14 in [10], one can see that the only difference between the STPBC representations of \( \hat{R}_\tau \) and \( \hat{R}_\tau - \hat{R}_\tau \hat{R}^{-1}_\tau \hat{R}_\tau \) is that the former requires \( x_2(\tau) = 0 \), while the latter requires \( \Pi_y x_1(\tau) + (I - \Pi_y) x_2(\tau) = 0 \). Thus the (finite-rank) term \( \hat{R}_\tau \hat{R}^{-1}_\tau \hat{R}_\tau \) affects the result only by reshaping the final condition between two components of the state vector of \( \hat{O}_\tau \).

Proposition 5 actually establishes that \( \hat{R}_{11} - \hat{R}_{12} \hat{R}_{22}^{-1} \hat{R}_{21} = \hat{O}_h - \gamma^2 I \). This implies that (3) holds iff \( \gamma^2 I - \hat{O}_h > 0 \). It then turns out that the latter condition can be expressed in terms of the well-posedness of \( (\gamma^2 I - \hat{O}_h)^{-1} \), \( \forall \gamma > 0 \). This has an advantage that the well-posedness of (infinite-dimensional) operators, which are representable by STPBC,
is equivalent to the invertibility of finite-dimensional matrices. The following result is required toward this end:

**Proposition 7** For every $Y = Y' \geq 0 \|\hat{O}_\tau\|$ is a monotonically non-decreasing function of $\tau$ and, moreover, $\lim_{\tau \to 0} \|\hat{O}_\tau\| = 0$.

**PROOF.** Just a minor modification of [11, Proposition 3].

Now, Propositions 5 and 7 yield that (3) holds iff $\gamma^2 I - \hat{O}_\tau > 0$ for all $\tau \in (0, h]$. Yet the latter holds iff

$$
(\gamma^2 I - \hat{O}_\tau)^{-1} = \left( \begin{array}{ccc}
A & B_1B'_1 & 0 \\
\gamma^{-2}C'_1C_1 & -A' & 0 \\
-\gamma^{-2}C_1 & 0 & -\gamma^{-2}I
\end{array} \right)
$$

is well-posed for all $\tau \in (0, h]$ (since $\|\hat{O}_\tau\|$ is continuous as a function of $\tau$) which is equivalent to the non-singularity of

$$
\Xi(\gamma^2 I - \hat{O}_\tau)^{-1} = \left[ \begin{array}{c}
\Pi_y \Sigma_{11}(\tau) + (I - \Pi_y)\Sigma_{21}(\tau) \\

\Pi_y \Sigma_{12}(\tau) + (I - \Pi_y)\Sigma_{22}(\tau)
\end{array} \right].
$$

The latter, in turn, is equivalent to the invertibility of the Schur compliment of the $(1, 1)$ sub-block of $\Xi(\gamma^2 I - \hat{O}_\tau)^{-1}, \Pi_y \left( \Sigma_{12}(\tau) + \Sigma_{11}(\tau)Y \right) + (I - \Pi_y) \left( \Sigma_{22}(\tau) + \Sigma_{21}(\tau)Y \right)$, $\forall \tau \in (0, h]$, which then reduces to condition (c) of Theorem 1 by noticing that $C_2 \left[ \begin{array}{c}
\Pi_y \\
I - \Pi_y
\end{array} \right] = \left[ \begin{array}{c}
C_2 \\
0
\end{array} \right]$ and $C_2^\perp \left[ \begin{array}{c}
\Pi_y \\
I - \Pi_y
\end{array} \right] = \left[ \begin{array}{c}
0 \\
C_2^\perp
\end{array} \right]$. This completes the proof.

5 Concluding remarks

This paper has studied the sampled-data $H^\infty$ fixed-lag smoothing problem with a smoothing lag of one sampling period. This appears to be the simplest sampled-data problem for which the achievable performance level $\gamma$ can be smaller than $\|\hat{D}_1\|$, where $\hat{D}_1$ is the system feedthrough term in the lifted domain. In this paper necessary and sufficient solvability conditions have been derived, which are compatible with those in the filtering case and require neither $\gamma^2 I > \hat{D}_1\hat{D}_1^*$ nor even $\gamma^2 I - \hat{D}_1\hat{D}_1^*$ to be invertible.

The technical machinery developed in this paper is expected to be applicable to more general sampled-data smoothing problems (with an arbitrary smoothing lag) as well. The main obstacle here is the absence of transparent solvability conditions for the standard discrete-time $H^\infty$ fixed-lag smoothing problem to be used as a basis for the solution in the lifted domain. This is the subject of future research.
References


[18] H. T. Toivonen and M. F. Sågfors. The sampled-data $H\infty$ problem: a unified frame-

