H^∞ Control of Systems with Multiple I/O Delays via Decomposition to Adobe Problems

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***Abstract*— In this paper the standard (four-block) H^∞ control problem for systems with multiple i/o delays in the feedback loop is studied. The central idea is to see the multiple delay operator as a special series connection of elementary delay operators, called the *adobe delay* operators. The adobe delay case is solved and thereby the general case is solved as a nested set of solutions to adobe delay problems.**

***Index Terms*— Time-delay systems, dead-time compensation, H^∞ control.**

I. INTRODUCTION

Input/output time delays arise naturally in numerous control applications, both from physical delays in processes and control interfaces and from the use of delays to model complicated high-frequency dynamics. Optimal control of time-delay systems has been an active research area since the late 60's, first in the H^2 (LQG) [1], [2] and then in the H^∞ [3], [4] settings.

Time-delay systems can in principle be treated in the framework of a general theory of infinite-dimensional systems, both in the time [5] and in the frequency [3] domains. These approaches, however, result in rather abstract results (i.e., in terms of operator Riccati equations), from which it may not be clear what the structures of solvability conditions and controllers are and how (if) they can be computed and implemented. This motivated researchers to seek for more problem-oriented approaches that exploit the special structure of the delay operator, see the review paper [4] and the references therein.

Although substantial progress has been made in this direction during the last two decades, the vast majority of the results (in both H^2 and H^∞ settings) is still limited to systems with a *single* delay. On the other hand, in MIMO systems different input/output channels can have different delays, so that multiple delay results are of great importance. Earlier treatments of multiple-delay systems either produced quite complicated solutions [2], [3] or were heavily based on the simplifying assumption that the delay operator commutes with the plant [6] which limits the scope of their applicability. An exception to this is a recent work by Kojima and Ishijima [7], who derive explicit H^∞ solution for the case when the disturbance and/or control inputs are delayed. Yet in [7] only input delays are considered and it is assumed that the controller has access to the full plant state.

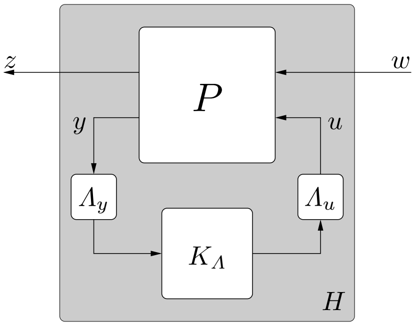

Fig. 1. Original 4-block problem formulation

In this paper the H^∞ control of systems with input/output delays is studied. The setup that we shall address is depicted in Fig. 1, where P is a given finite-dimensional plant, K_Λ is a controller to be determined, and Λ_u and Λ_y are given delay operators. When $\Lambda_u = \mathrm{e}^{-h_u s} I$ and $\Lambda_y = \mathrm{e}^{-h_y s} I$, such a setup corresponds to the single-delay problem. In our case the delay operators are more general diagonal matrices (see Section II for details). This enables to deal with different delays in different control and measurement channels.

The central idea of this paper is to split the multiple-delay problem to a nested sequence of simpler problems which we call *adobe problems*. The adobe problem is a problem with a single delay in *a part of* input or output channels. We sometimes distinguish *adobe input delay* and *adobe output delay* problems. These are apparently the simplest nontrivial generalizations of the single delay case. We show that both input and output adobe delay problems can be solved in a unified fashion using the approach developed in [8] (though with some nontrivial modifications). The solutions to the adobe problems are then tailored to constitute the solution to the original problem.

The advantage of the proposed approach is twofold. First, the split of the problem to elementary adobe problems (apart from the fact that this allows us to find the solution) clarifies how additional delays in certain channels affect the performance. This might be used to analyze the cost of delay in each channel and to judge a relative delay sensitivity of different channels. Second, the approach results in a transparent structure of the optimal controller. The latter consists of a finite-dimensional system with a feedback/feedforward part that, though infinite dimensional, can be easily implemented owing to the fact that its components may be chosen to be

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This research was supported by THE ISRAEL SCIENCE FOUNDATION (grant No. 106/01).

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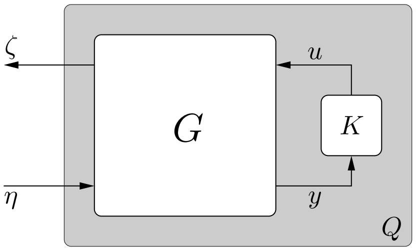


Fig. 2. A scattering representation

FIR. This structure is reminiscent of that of the single-delay H^∞ dead-time compensators proposed in [9], [10], though the presence of feedforward interchannel interconnections is unique to the multiple delay situation.

It is worth stressing in this respect that there appears to be no natural generalization of single-delay Smith predictor (dead-time compensator) schemes to the case of multiple delays, see, e.g., the discussion in [11]. We believe that a byproduct of our solution might be a suggestion of a possible form of the multiple delay dead-time compensator.

The paper is organized as follows. In Section II the multiple-delay H^∞ problem is formulated. Section III is devoted to the reformulation of the original 4-block problem as an equivalent 1-block problem having a special structure. In Section IV the adobe-delay problem is formulated and solved. Then, in Section V we show how the multiple-delay problem is solved by the decomposition to a sequence of adobe problems. An illustrative example is studied in Section VI. The paper also includes two appendices. In Appendix A the solution of the delay-free H^∞ problem is revised and in Appendix B some technicalities and proofs are collected.

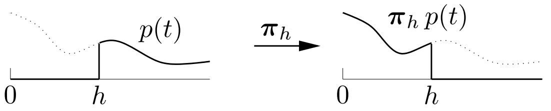
Notation: Throughout the paper we use scattering representations such as shown in Fig. 2. The arrows here can be confusing: what is meant in this figure is that $\left[\begin{smallmatrix}\zeta\\ \eta\end{smallmatrix}\right] = G\left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]$ and $u = Ky$. If the dimensions of η and y are the same, then each K generically defines a unique transfer matrix Q from η to ζ, denoted as $Q = C_r(G, K)$. It is easy to verify that

$$C_r(G,K) = (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1}.$$

Once in a while we use the conventional lower linear fractional transformations (LFT's). For example the LFT $\mathcal{F}_l(P, \Lambda_u K_\Lambda \Lambda_y)$ means by definition the mapping from w to z in the system of Fig. 1.

We say that $K(s)$ is *proper* if $\sup_{\operatorname{Re} s>\rho} \|K(s)\| < \infty$ for some large enough $\rho \in \mathbb{R}$. As shown in [12], an LTI system has a causal implementation iff its transfer matrix is proper. If $G(\infty) = I$ then properness of K implies properness of $Q := C_r(G, K)$, and since $K = C_r(G^{-1}, Q)$ we then in fact have that the mapping is causally invertible.

Borrowing from [10] we define the *completion* operator $\boldsymbol{\pi}_h$, which "analytically completes" the impulse response of an h-delay system to a delay-free system. Informally:



The completion operator for delayed systems of the form $\mathrm{e}^{-hs}P = \mathrm{e}^{-hs}C(sI - A)^{-1}B$ is defined formally as

$$\boldsymbol{\pi}_h(\mathrm{e}^{-hs}P) = \left[\begin{array}{c|c} A & B \\ \hline C\,\mathrm{e}^{-Ah} & 0 \end{array}\right] - \mathrm{e}^{-hs}\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

($h > 0$). For finite dimensional P, the sum of $\mathrm{e}^{-hs}P$ and its completion $\boldsymbol{\pi}_h(\mathrm{e}^{-hs}P)$ is again finite dimensional.

A mapping $Q \in H^\infty$ is γ*-contractive* if $\|Q\|_\infty < \gamma$ (when $\gamma = 1$ we simply say *contractive*). A transfer matrix Q is *bistable* if $Q, Q^{-1} \in H^\infty$. The number of entries of a vector-valued signal w is denoted as n_w, for example $u(t) \in \mathbb{R}^{n_u}$.

II. PROBLEM FORMULATION

As mentioned in the introduction, we study the feedback setup in Fig. 1. We assume that the plant P there has the realization

$$P(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \tag{1}$$

and that the following standard assumptions hold:

$\boldsymbol{\mathcal{A}}_1$: (C_2, A, B_2) is stabilizable and detectable;
$\boldsymbol{\mathcal{A}}_2$: $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank $\forall \omega \in \mathbb{R} \cup \infty$;
$\boldsymbol{\mathcal{A}}_3$: $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank $\forall \omega \in \mathbb{R} \cup \infty$.

Assumptions $\boldsymbol{\mathcal{A}}_2$ and $\boldsymbol{\mathcal{A}}_3$ guarantee that $D_{12}'D_{12} > 0$ and $D_{21}D_{21}' > 0$, respectively. Note also that we do not assume that D_{11} and D_{22} are zero as these assumptions hardly simplify the results to come and, moreover, in delay systems nonzero D_{22} might appear naturally.

The delay elements are assumed to be of the diagonal form

$$\Lambda_u(s) = \begin{bmatrix} \mathrm{e}^{-h_{u,q}s} I_{m_q} & & & \\ & \ddots & & \\ & & \mathrm{e}^{-h_{u,1}s} I_{m_1} & \\ & & & I_{m_0} \end{bmatrix} \tag{2a}$$

with $0 < h_{u,1} < \cdots < h_{u,q}$ ($\sum m_i = n_u$) and

$$\Lambda_y(s) = \begin{bmatrix} I_{p_0} & & & \\ & \mathrm{e}^{-h_{y,1}s} I_{p_1} & & \\ & & \ddots & \\ & & & \mathrm{e}^{-h_{y,r}s} I_{p_r} \end{bmatrix} \tag{2b}$$

with $0 < h_{y,1} < \cdots < h_{y,r}$ ($\sum p_i = n_y$). In other words, we assume that there are q different input delay channels, r different output delay channels, and, possibly, two delay-free channels; $m_0 = 0$ ($p_0 = 0$) implies that there is no delay-free input (output) channel. Moreover, all delay channels are assumed ordered (from large to small in Λ_u and from small to large in Λ_y). These assumptions can be made without loss of generality (otherwise a simple channel permutation is to be applied).

The problem studied in this paper is formulated as follows:

SHP: Given the system in Fig. 1 with the generalized plant P as in (1) satisfying $\boldsymbol{\mathcal{A}}_{1\text{–}3}$ and the delays Λ_u and Λ_y

> as in (2). Determine whether there exists a proper K_Λ so that $K := \Lambda_u K_\Lambda \Lambda_y$ internally stabilizes the system and guarantees that
>
> $$\|\mathcal{F}_l(P, \Lambda_u K_\Lambda \Lambda_y)\|_\infty < \gamma \tag{3}$$
>
> for a given $\gamma > 0$, and then characterize all such K_Λ if one exists.

This problem is a nontrivial generalization of the single-delay H^∞ problem extensively studied in the control literature for the last two decades [4].

III. EQUIVALENT ONE-BLOCK REFORMULATION

It is clear that the problem is solvable only if so is its delay-free counterpart (delays just impose additional constraints on the controller). Following [10] and [8], we exploit this fact to reduce the *four-block H^∞* problem with multiple delays to an equivalent *one-block H^∞* problem with multiple delays. To this end we first need the standard solution, i.e., the solution for the situation when there are no delays.

A. Review of the standard delay-free solution

The solution to the standard delay-free H^∞ problem is currently well understood [13], [14], so we only present here its features that are relevant for our development. For more details the reader is referred to Appendix A. Also, hereafter we assume without loss of generality that γ is such that the delay-free version of the **SHP** is solvable.

We start with some nomenclature related to the H^∞ solution. Let $X \geq 0$ and $Y \geq 0$ be the stabilizing solutions to the standard H^∞ Riccati equations; $Z := (I - \gamma^{-2} Y X)^{-1}$ (well-defined by the solvability assumption); let F_1 and F_2 be the H^∞ gains associated with the state-feedback problem and L_1 and L_2 the corresponding filtering gains; let $A_F := A + B_1 F_1 + B_2 F_2$ and $A_L := A + L_1 C_1 + L_2 C_2$ be the (stable) “closed-loop” matrices associated with state-feedback and filtering, respectively. Introduce also the following transfer matrix,

$$G_\infty(s) := D_\infty \left[\begin{array}{c|c} A_L & B_\infty \\ \hline C_\infty Z & I \end{array}\right], \tag{4}$$

where

$$B_\infty := \begin{bmatrix} -(B_2 + L_1 D_{12} + L_2 D_{22}) & L_2 \end{bmatrix},$$
$$C_\infty := \begin{bmatrix} F_2 \\ C_2 + D_{21} F_1 + D_{22} F_2 \end{bmatrix},$$

and D_∞ is a nonsingular matrix obtained by the J-factorization of a matrix constructed from the feedthrough term of $P(s)$. It can be shown [14] that

$$G_\infty(s)^{-1} = \left[\begin{array}{c|c} A_F & -Z B_\infty \\ \hline C_\infty & I \end{array}\right] D_\infty^{-1},$$

so that G_∞ is bistable.

With these definitions, the standard solution then goes to show that the **SHP** is equivalent to finding a (proper) K for which

$$Q := \mathcal{C}_r(G_\infty, K) \tag{5}$$

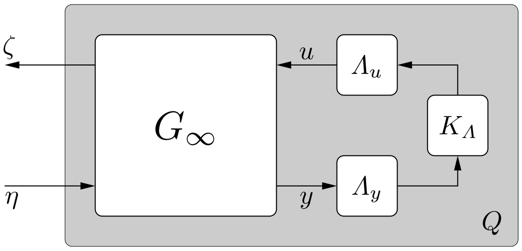


Fig. 3. An equivalent one-block problem

is contractive. In the delay-free case this settles the problem completely because the mapping $K \to Q$ is invertible,

$$K = \mathcal{C}_r(G_\infty^{-1}, Q), \tag{6}$$

and K is proper for almost every contractive Q. This yields the well known parameterization of all solutions K to the **SHP**: simply take any contractive Q and the resulting K does the job.

B. Including the delays

In our situation K equals $K = \Lambda_u K_\Lambda \Lambda_y$ and we cannot invert the mapping (5) because the resulting K in (6) might simply cancel the delays in $\Lambda_u K_\Lambda \Lambda_y$ resulting in a non-proper controller K_Λ.

Still we may begin the analysis with the simplified problem of finding contractive Q as in (5) so that the **SHP** is recast as the (one-block) problem of finding a proper K_Λ guaranteeing that the mapping $\eta \to \zeta$ in Fig. 3 is contractive, i.e., that $\|\mathcal{C}_r(G_\infty, \Lambda_u K_\Lambda \Lambda_y)\|_\infty < 1$. This reduction has a couple of advantages over a direct treatment of the **SHP**. First, it separates the delay-free problem from the delay problem thereby clarifying what part of the problem may be contributed purely to the delays. Moreover, it is useful to adopt chain-scattering representations rather than the more common LFT’s since it reveals some extra structure. For example, the fact that G_∞ is bistable simplifies the further analysis considerably. Furthermore it allows us to consider the input and output delays on an equal footing. To see this, let us define the joint delay operator $\Lambda = \operatorname{diag}\{\Lambda_u, \Lambda_y^{-1}\}$ (mind the *inverse* Λ_y^{-1}). Then

$$\mathcal{C}_r(G_\infty, \Lambda_u K_\Lambda \Lambda_y) = \mathcal{C}_r(G_\infty \Lambda, K_\Lambda), \tag{7}$$

see Fig. 4. The so defined joint delay operator generally has advance elements (negative delays). Yet this is not an obstacle as Λ may be multiplied by a scalar operator α without affecting the mapping $\mathcal{C}_r(G_\infty \Lambda \alpha, K)$. We choose α to be the maximal delay term $\mathrm{e}^{-h_{y,r} s}$ in Λ_y, which results in

$$\begin{aligned} \Lambda &:= \mathrm{e}^{-h_{y,r} s} \operatorname{diag}\{\Lambda_u, \Lambda_y^{-1}\} \\ &= \begin{bmatrix} \mathrm{e}^{-h_{q+r} s} I_{n_{q+r}} & & & \\ & \ddots & & \\ & & \mathrm{e}^{-h_1 s} I_{n_1} & \\ & & & I_{n_0} \end{bmatrix} \end{aligned} \tag{8}$$

(note that the input and output delay-free channels are united). Here $h_{q+r} > \cdots > h_1 > 0$ so that $h_{q+r} = h_{u,q} + h_{y,r}$ is the

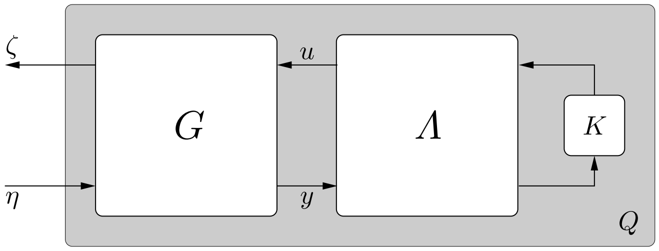


Fig. 4. Input and output delays combined into one block

maximal delay between any two channels u_i and y_j in the system in Fig. 1. Note also that

$$n_0 = \begin{cases} p_r & \text{if } p_r \neq 0 \text{ (equivalently, } p_0 \neq n_y) \\ n_y + m_0 & \text{otherwise} \end{cases}$$

so that $n_0 \neq 0$.

It will be useful to perform another simplification at this stage: to replace G_∞ with a transfer matrix having the identity feedthrough term. Toward this end, some preliminary discussion is needed. Note that the **SHP** is solvable only if so is its finite-horizon version at any interval $[0, \tau]$. Therefore, the **SHP** must also be solvable at $[0, \tau]$ for $\tau \to 0$. In the delay-free case the latter is equivalent to the existence of a matrix D_K so that $\mathcal{F}_l(P(\infty), D_K)$ is γ-contractive (in fact, this is what condition $\boldsymbol{\mathcal{C}_1}$ in Appendix A says). Yet delayed loops do not participate in such a finite-horizon problem (they are open on $[0, \tau]$ whenever τ is small enough). Hence, the **SHP** is solvable only if there exists a matrix $D_0 \in \mathbb{R}^{m_0 \times p_0}$ so that

$$\| \mathcal{F}_l(P(\infty), E_u D_0 E_y') \| < \gamma, \tag{9}$$

where the matrices

$$E_u := \begin{bmatrix} 0 \\ I_{m_0} \end{bmatrix} \in \mathbb{R}^{n_u \times m_0} \quad \text{and} \quad E_y := \begin{bmatrix} I_{p_0} \\ 0 \end{bmatrix} \in \mathbb{R}^{n_y \times p_0}$$

are the directions of the delay-free input and output channels, respectively. When there are no delay-free loops in the system (i.e., when either p_0 or m_0 is zero), condition (9) reduces to the γ-contractiveness of D_{11}. Also, if algebraic loops are ruled out by imposing the assumption $E_y' D_{22} E_u = 0$, then (9) becomes $\|D_{11} + D_{12} E_u D_0 E_y' D_{21}\| < \gamma$.

Now, let us rewrite the right-hand side of (7) as follows:

$$C_r\big(G_\infty \Lambda, K_\Lambda\big) = C_r\big(G_\infty D_\infty^{-1} \Lambda, C_r(\Lambda^{-1} D_\infty \Lambda, K_\Lambda)\big).$$

The transfer matrix

$$G(s) := G_\infty(s) D_\infty^{-1} = \left[\begin{array}{c|c} A_L & B_\infty D_\infty^{-1} \\ \hline D_\infty C_\infty Z & I \end{array} \right] \tag{10}$$

has the identity feedthrough term, as required. On the other hand, by Lemma 1.1, D_∞ can always be chosen in the form

$$D_\infty = V \begin{bmatrix} I & -E_u D_0 E_y' \\ 0 & I \end{bmatrix}, \quad V \text{ is lower triangular.} \tag{11}$$

It can be verified that

$$\begin{bmatrix} I & -E_u D_0 E_y' \\ 0 & I \end{bmatrix} \Lambda = \Lambda \begin{bmatrix} I & -E_u D_0 E_y' \\ 0 & I \end{bmatrix}$$

and also that the transfer matrix $\Lambda^{-1} V \Lambda$ is bistable (as V is lower triangular and the delays in the diagonal Λ are ordered descendantly). Thus, if D_∞ is as in (11), then the transfer matrix

$$D_\Lambda(s) := \Lambda(s)^{-1} D_\infty \Lambda(s) \tag{12}$$

is bistable and the mapping $K_\Lambda \to K = C_r(D_\Lambda, K_\Lambda)$ is causally invertible.

We thus end up with the following one-block H^∞ problem:

OBP: Given the system in Fig. 4 with G and Λ as in (10) and (8), respectively, determine whether there exists a proper K which guarantees that

$$\| C_r(G\Lambda, K) \|_\infty < 1, \tag{13}$$

and then characterize all such K if one exists.

The following lemma, which was actually proved above, establishes that the **SHP** can be solved in terms of the simpler **OBP**:

Lemma 3.1: The **SHP** is solvable only if so is its delay-free counterpart and there exists a matrix D_0 such that (9) holds. If these conditions hold, then the **SHP** is solvable iff the **OBP** is solvable. Moreover, a proper K solves the **OBP** iff

$$K_\Lambda := C_r(D_\Lambda^{-1}, K)$$

solves the **SHP**, where D_Λ is given by (12).

A central idea of this paper is to break down the **OBP** with its many different delays into a series of simpler problems with only a single delay in a part of its channels, problems that we call *adobe delay problems*.

IV. ADOBE DELAY PROBLEM

By *adobe* delay we mean the case that the joint delay operator is of the form

$$\Lambda = \begin{bmatrix} \mathrm{e}^{-hs} I_\mu & 0 \\ 0 & I_\rho \end{bmatrix} \tag{14}$$

for some $\mu < n_u + n_y$ and $\rho = n_u + n_y - \mu$. These adobe problems serve as building blocks from which the general **OBP** will be solved later.

Note that the dimensions (μ, ρ) need not match the dimensions of the input and output signals. In fact, the case of $\mu = n_u$ (and, consequently, $\rho = n_y$) corresponds to the single-delay problem treated in [8]. Indeed, for the single-delay problem

$$\Lambda_u = \mathrm{e}^{-h_u s} I_{n_u} \quad \text{and} \quad \Lambda_y = \mathrm{e}^{-h_y s} I_{n_y},$$

the joint delay operator Λ becomes $\Lambda = \operatorname{diag}\{\mathrm{e}^{-hs} I_{n_u}, I_{n_y}\}$ with $h = h_y + h_u$. The case $\mu \geq n_u$ can then be thought of as resulting from

$$\Lambda_u = I_{n_u} \quad \text{and} \quad \Lambda_y = \operatorname{diag}\big\{I_{n_y - \rho}, \mathrm{e}^{-h_y s} I_\rho\big\}. \tag{15}$$

We thus call the corresponding adobe problem *the adobe plant output delay* problem. Similarly, $\mu \leq n_u$ may correspond to

$$\Lambda_u = \operatorname{diag}\big\{\mathrm{e}^{-h_u s} I_\mu, I_{n_u - \mu}\big\} \quad \text{and} \quad \Lambda_y = I_{n_y}, \tag{16}$$

so we call it *the adobe plant input delay* problem. It is worth stressing that in the last two cases controller structures and interpretations are quite different (see below). On the other hand, the *formulae* in all cases above are in a sense the same.

A. The main result

Let us rewrite the realization of G from (10) as follows:

$$G(s) =: \left[\begin{array}{c|cc} A_L & B_\mu & B_\rho \\ \hline C_\mu & I_\mu & 0 \\ C_\rho & 0 & I_\rho \end{array}\right], \tag{17}$$

where the partitioning is compatible with (14). Throughout this section we denote $J := \operatorname{diag}\{I_{n_u}, -I_{n_y}\}$ and we also introduce the following two signature matrices:

$$J_\mu := \begin{bmatrix} I_\mu & 0 \end{bmatrix} J \begin{bmatrix} I_\mu \\ 0 \end{bmatrix} \quad \text{and} \quad J_\rho := \begin{bmatrix} 0 & I_\rho \end{bmatrix} J \begin{bmatrix} 0 \\ I_\rho \end{bmatrix}$$

(note that $J = \operatorname{diag}\{J_\mu, J_\rho\}$). Define the symplectic matrix function $\Sigma(t)$ as

$$\Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} := \mathrm{e}^{Ht}, \tag{18}$$

where H is the Hamiltonian matrix

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} := \begin{bmatrix} A_L - B_\rho C_\rho & -B_\rho J_\rho B_\rho' \\ -C_\mu' J_\mu C_\mu & -A_L' + C_\rho' B_\rho' \end{bmatrix} \tag{19}$$

(note that H does not depend on B_μ). To simplify the notations, we write Σ instead of $\Sigma(h)$. Then the main result of this section is as follows:

Theorem 4.1: The **OBP** with joint delay operator (14) is solvable iff $\Sigma_{22}(t)$ is nonsingular $\forall t \in [0, h]$. In that case K solves the **OBP** iff

$$K = C_r\big(\left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right]\tilde{G}^{-1}, \tilde{Q}\big)$$

where

$$\tilde{G}(s) = \left[\begin{array}{c|cc} A_L & \Sigma_{22}' B_\mu + \Sigma_{12}' C_\mu' J_\mu & B_\rho \\ \hline C_\mu \Sigma_{22}^{-\prime} & I_\mu & 0 \\ C_\rho - J_\rho B_\rho' \Sigma_{22}^{-1} \Sigma_{21} & 0 & I_\rho \end{array}\right]$$

is bistable,

$$\Pi(s) = \boldsymbol{\pi}_h \left\{ \mathrm{e}^{-hs} \left[\begin{array}{cc|c} H_{11} & H_{12} & B_\mu \\ H_{21} & H_{22} & -C_\mu' J_\mu \\ \hline C_\rho & J_\rho B_\rho' & 0 \end{array}\right] \right\}$$

is FIR, and $\|\tilde{Q}\|_\infty < 1$ but otherwise arbitrary.

The following corollary of Theorem 4.1 will be used in the sequel:

Corollary 4.2: Let the condition of Theorem 4.1 hold. Then K solves the adobe **OBP** iff

$$C_r\big(\tilde{G}, C_r\big(\left[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\right], K\big)\big)$$

is a contraction.

The proof of Theorem 4.1 follows the steps in [8], though the proofs of some of these steps are nontrivially different.

B. Outline of the proof

Details of the proof can be found in Appendix B. Here we summarize the main ideas. The proof of Theorem 4.1 is based on J-spectral factorization arguments. We are looking for a bistable W so that

$$\Phi := \Lambda^\sim G^\sim J G \Lambda = W^\sim J W \tag{20}$$

and $G\Lambda W^{-1}$ is J-lossless. It is readily seen that the infinite-dimensional part of Φ shows up only in its off-diagonal entries (we assume that Φ is partitioned according to (14)). This fact can be exploited to eliminate the irrational part from the factorization following the arguments of [8], [9]. In [8] this approach was taken to tackle the single delay case. The only difference in the construction of W between the single-delay case and a general adobe-delay case is the replacement of I_ρ and $-I_\mu$ with J_ρ and J_μ, respectively, in the final formulae.

The construction of W in [8] is heavily based on the assumption that

$$\det \Sigma_{22} \neq 0. \tag{21}$$

It is thus crucial to ensure that (21) holds. Toward this end, the single-delay proof in [8] exploits the fact that on the interval $[0, h]$ the system is open loop. This means that the problem is solvable *only if* $C_r(G, 0)$ is a contraction on the interval $[0, h]$. It turns out that the latter is equivalent to the nonsingularity of $\Sigma_{22}(t)$ for all $t \in [0, h]$, which implies (21).

In the general adobe-delay case the system is not necessarily open loop on the interval $[0, h]$, so the arguments of [8] are not readily applicable and should therefore be modified. The key point to be observed here is that the infinite-horizon H^∞ problem **OBP** is solvable *only if* so it its finite-horizon version on the interval $[0, h]$. As will be shown in §IV-C, the latter problem is solvable iff $\Sigma_{22}(t)$ is nonsingular $\forall t \in [0, h]$.

The rest of the arguments of the proof of Theorem 4.1 are fairly straightforward. With invertibility of $\Sigma_{22}(t)$ guaranteed for all $t \in [0, h]$ one can verify that W satisfying (20) is given by

$$W = \tilde{G}\left[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\right]$$

and that $G\Lambda$ has a factorization of the form

$$G\Lambda = \big(G\Lambda W^{-1}\big)\big(W\big)$$

in which the term $G\Lambda W^{-1}$ is J-lossless. This makes the **OBP** is equivalent to the problem of making

$$\tilde{Q} := C_r\big(W, K\big) = C_r\big(\tilde{G}\left[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\right], K\big)$$

contractive. This mapping *is* causally invertible (because $\lim_{s\to\infty} \tilde{G}(s)\left[\begin{smallmatrix} I & 0 \\ -\Pi(s) & I \end{smallmatrix}\right] = I$) and we end up with the formulae of Theorem 4.1.

C. Necessity: finite-horizon problem

Consider the finite-horizon version of the **OBP**. If Λ_u, Λ_y are given by (16), then delayed channels of u are zero $\forall t \in [0, h]$. Hence these channels can safely be eliminated on this finite horizon. The system of Fig. 4 then over the first h time units can be described as

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} = G_\rho \begin{bmatrix} u_\rho \\ y_\rho \end{bmatrix}, \qquad u_\rho = K_\rho y_\rho. \tag{22}$$

Here

$$G_\rho(s) := G(s)\left[\begin{smallmatrix} 0_\mu \\ I_\rho \end{smallmatrix}\right] = \left[\begin{array}{c|c} A_L & B_\rho \\ \hline C_\mu & 0 \\ C_\rho & I_\rho \end{array}\right] =: \left[\begin{array}{c|c} A_L & B_\rho \\ \hline C_g & D_\rho \end{array}\right],$$

$u_\rho : [0,h] \mapsto \mathbb{R}^{\mu-n_u}$ and $y_\rho : [0,h] \mapsto \mathbb{R}^{n_y}$. Thus, the finite-horizon version of the **OBP** with delays as in (16) is solvable only if there exists a causal K_ρ such that

$$\sup \frac{\|\zeta\|_{L^2[0,h]}}{\|\eta\|_{L^2[0,h]}} < 1, \tag{23}$$

where the supremum is taken over all η and ζ satisfying (22).

The above is a finite horizon closed loop argument. Now if the delays Λ_u, Λ_y are given by (15) then dually a finite-horizon open-loop argument applies. In this case the last ρ channels of y are delayed. Hence for y of the form $y = \left[\begin{smallmatrix} 0 \\ y_\rho \end{smallmatrix}\right]$ we do not get any response u on $[0,h]$ whatever K is (a long as it is causal). Therefore for any such y Eqn. (22) holds with u_ρ and K_ρ void. It may be verified that every y of the form $y = \left[\begin{smallmatrix} 0 \\ y_\rho \end{smallmatrix}\right] : [0,h] \mapsto \mathbb{R}^{n_y-\rho} \oplus \mathbb{R}^\rho$ is possible by proper choice of input η. Hence also in this case the finite-horizon version of the **OBP** is solvable only if (23) holds over all possible ζ, η of the form (22). The two finite-horizon necessary requirements (closed-loop and open-loop) have a joint characterization.

We start with the following technical result:

Lemma 4.3: The operator $G_\rho^* J G_\rho : L^2[0,t] \mapsto L^2[0,t]$ is singular iff $\det \Sigma_{22}(t) = 0$.

Proof: It is readily seen that $\xi_o = G_\rho \xi_i$ iff

$$\begin{cases} \dot{x} = A_L x + B_\rho \xi_i, \quad x(0) = 0, \\ \xi_o = C_g x + D_\rho \xi_i \end{cases}$$

and $\xi_o = G_\rho^* \xi_i$ iff

$$\begin{cases} \dot{p} = -A_L' p - C_g' \xi_i, \quad p(t) = 0, \\ \xi_o = B_\rho p + D_\rho' \xi_i. \end{cases}$$

Therefore, $\xi_o = G_\rho^* J G_\rho \xi_i$ iff

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A_L & 0 \\ -C_g' J C_g & -A_L' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} - \begin{bmatrix} -B_\rho \\ C_g' J D_\rho \end{bmatrix} \xi_i, \\ \xi_o = \begin{bmatrix} D_\rho' J C_g & B_\rho' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + D_\rho' J D_\rho \xi_i \end{cases}$$

with the boundary conditions

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(0) \\ p(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = 0.$$

Clearly, $G_\rho^* J G_\rho$ is nonsingular iff $(G_\rho^* J G_\rho)^{-1}$ is well-posed. The latter condition, in turn, holds iff

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \mathrm{e}^{A^\times t} \tag{24}$$

is nonsingular [15] (notice that $D_\rho' J D_\rho$ is nonsingular), where

$$A^\times := \begin{bmatrix} A_L & 0 \\ -C_g' J C_g & -A_L' \end{bmatrix} + \begin{bmatrix} -B_\rho \\ C_g' J D_\rho \end{bmatrix} (D_\rho' J D_\rho)^{-1} \begin{bmatrix} D_\rho' J C_g & B_\rho' \end{bmatrix}$$

is the "A" matrix of the realization of $(G_\rho^* J G_\rho)^{-1}$. Direct substitutions show that $A^\times = H$ as defined by (19). The result then follows by replacing $\mathrm{e}^{A^\times t}$ with $\Sigma(t)$ in (24). ■

Now we are in the position to formulate our main result:

Lemma 4.4: Let Λ be as in (14). There exists a causal K such that (23) holds only if $\det \Sigma_{22}(t) \neq 0$ for all $t \in [0,h]$.

Proof: Assume to the contrary that $\Sigma_{22}(t)$ is singular for some $t \in [0,h]$. By Lemma 4.3 this means that

$$\xi^\circ := \begin{bmatrix} u_\rho^\circ \\ y_\rho^\circ \end{bmatrix} \neq 0$$

exists such that $G_\rho^* J G_\rho \xi^\circ = 0$. Now for any such ξ° define the "worst" signals

$$\begin{bmatrix} \zeta^\circ \\ \eta^\circ \end{bmatrix} := G_\rho \xi^\circ$$

(notice that $(\zeta^\circ, \eta^\circ) \neq 0$ because D_ρ has full column rank, and that by construction, $\|\zeta^\circ\|_{L^2[0,t]} = \|\eta^\circ\|_{L^2[0,t]}$). In what follows all mappings and inner products are over $[0,t]$. Take $\eta = \eta^\circ$ as input to the system of Fig. 4. Then given any causal K the resulting closed loop signals $\left[\begin{smallmatrix} u_\rho \\ y_\rho \end{smallmatrix}\right] : [0,h] \mapsto \mathbb{R}^\rho$ are unique and they are such that

$$\begin{bmatrix} H\eta^\circ \\ \eta^\circ \end{bmatrix} = G_\rho \begin{bmatrix} u_\rho \\ y_\rho \end{bmatrix}, \qquad H := C_r(G, K).$$

Hence

$$\begin{aligned} \langle H\eta^\circ, \zeta^\circ \rangle - \langle \eta^\circ, \eta^\circ \rangle &= \left\langle \left[\begin{smallmatrix} H \\ I \end{smallmatrix}\right] \eta^\circ, J \left[\begin{smallmatrix} \zeta^\circ \\ \eta^\circ \end{smallmatrix}\right] \right\rangle \\ &= \left\langle G_\rho \left[\begin{smallmatrix} u_\rho \\ y_\rho \end{smallmatrix}\right], J G_\rho \xi^\circ \right\rangle = 0. \end{aligned}$$

This together with the fact that $\langle \zeta^\circ, \zeta^\circ \rangle = \langle \eta^\circ, \eta^\circ \rangle$ shows that

$$\langle H\eta^\circ, \zeta^\circ \rangle = \langle \eta^\circ, \eta^\circ \rangle = \langle \zeta^\circ, \zeta^\circ \rangle.$$

Cauchy-Schwartz inequality yields then that $\|H\|_{L^2[0,t]} \geq 1$ (and equality holds only if $H\eta^\circ = \zeta^\circ$, in which case $\|H\eta^\circ\|_2 = \|\eta^\circ\|_2$, hence the name "worst disturbance" for η°). The proof is complete on noting that $\|H\|_{L^2[0,t]} \leq \|H\|_{L^2[0,h]}$, $\forall t \leq h$. ■

It is worth noting that the condition of Lemma 4.4 is actually also sufficient (in fact that is a byproduct of Theorem 4.1). We, however, do not need this fact in the proof of Theorem 4.1.

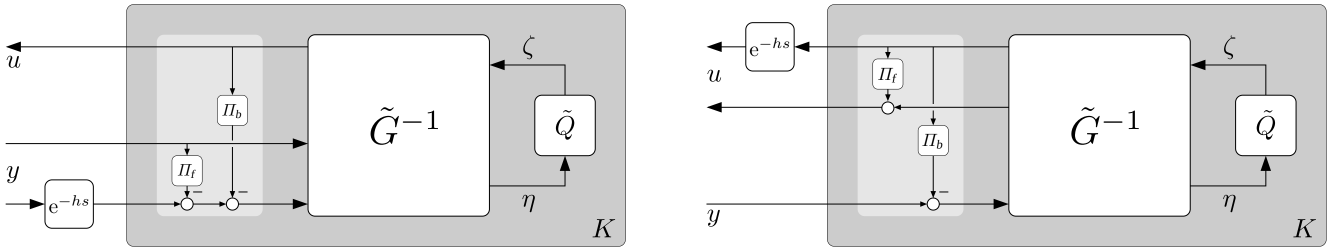
D. Controller structure

For implementation of the controller in Theorem 4.1 it is convenient to repartition $\left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right]$ compatibly with the dimensions of u and y.

In the adobe plant output delay case ($\mu \geq n_u$) we have:

$$\begin{bmatrix} I_\mu & 0 \\ \Pi & I_\rho \end{bmatrix} = \left[\begin{array}{c|cc} I_{n_u} & 0 & 0 \\ \hline 0 & I_{\mu-n_u} & 0 \\ \Pi_b & \Pi_f & I_\rho \end{array}\right].$$

The structure of the controller $K = C_r\big(\left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right] \tilde{G}^{-1}, \tilde{Q}\big)$ from Theorem 4.1 hence is as shown in Fig. 5(a). It consists of the rational (bistable) part $\tilde{G}$, a free contractive parameter $\tilde{Q}$, and two irrational stable (FIR) blocks: Π_b and Π_f. The former FIR block is in fact the internal feedback in the controller reminiscent the classical dead-time compensators (DTC) or Smith predictors. The only difference from the DTC that appears in the single-delay H^∞ control is that Π_b acts only on a part of the measurement channels, namely, on the delayed channel. On the other hand, Π_f acts as an interchannel feedforward part of the controller and has no direct counterpart in the Smith predictor literature.



(a) Adobe plant output delay (b) Adobe plant input delay

Fig. 5. Controller structure

In the adobe plant input delay case ($\mu \le n_u$) we have:

$$\begin{bmatrix} I_\mu & 0 \\ \Pi & I_\rho \end{bmatrix} = \left[\begin{array}{cc:c} I_\mu & 0 & 0 \\ \Pi_f & I_{\rho-n_y} & 0 \\ \hdashline \Pi_b & 0 & I_{n_y} \end{array}\right].$$

The structure of the controller $K = C_r\big(\big[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\big]\tilde{G}^{-1}, \tilde{Q}\big)$ now is as shown in Fig. 5(b). As in the output delay case, the DTC part of the controller contains two different FIR blocks. The first one, Π_b, acts as an internal feedback from the *delayed* control channel to the measured signal, while the second one, Π_f, acts as an interchannel feedforward from the delayed control channel to the delay-free one.

V. DECOMPOSITION

Now we are in a position to address the decomposition of the **OBP** to a series of adobe problems. We return to the general joint delay operator Λ in (8), which contains $q+r$ descendantly ordered delay blocks and for that reason we refer to it as a *$(q+r)$-delay operator*. In the future references, we denote the **OBP** with the data G and Λ as **OBP(G, Λ)**. Also, given two equally dimensioned joint delay operators Λ_α and Λ_β of the form (8), we write $\Lambda_\alpha \succ \Lambda_\beta$ (or, equivalently, $\Lambda_\beta \prec \Lambda_\alpha$) if the last (delay-free) block of Λ_α has *strictly larger* dimension than that of Λ_β.

It is readily verified that the $(q+r)$-delay operator can be decomposed as follows:

$$\Lambda = \Lambda_1 \tilde{\Lambda}, \tag{25}$$

where

$$\Lambda_1 := \begin{bmatrix} \mathrm{e}^{-h_1 s} I_{\mu_1} & 0 \\ 0 & I_{\rho_1} \end{bmatrix} \quad (\text{with } \rho_1 = n_0)$$

is the joint delay operator of the adobe problem, cf. (14), and $\tilde{\Lambda}$ is actually a $(q+r-1)$-delay operator with an (n_0+n_1)-dimensional delay-free channel (i.e., $\tilde{\Lambda} \succ \Lambda$) and the smallest delay $h_2 - h_1$. From (25) we get

$$C_r(G\Lambda, K) = C_r\big(G\Lambda_1, C_r(\tilde{\Lambda}, K)\big).$$

As the delay block $\tilde{\Lambda}$ above just imposes additional constraints on K, the **OBP(G, Λ)** is solvable *only if* so is the adobe delay problem **OBP(G, Λ_1)**. According to Corollary 4.2, the latter problem is solvable iff the condition of Theorem 4.1 holds and

$$\tilde{Q} := C_r\big(\tilde{G}\big[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\big], C_r(\tilde{\Lambda}, K)\big) \tag{26}$$

is a contraction in H^∞, where $\tilde{G}$ and Π are defined in Theorem 4.1. Now we absorb the term $\big[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\big]$ into the controller, that is, we rewrite (26) as

$$\tilde{Q} = C_r\big(\tilde{G}\tilde{\Lambda}, \tilde{K}\big), \quad \tilde{K} := C_r\big(\tilde{\Lambda}^{-1}\big[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\big]\tilde{\Lambda}, K\big).$$

The important point here is that owing to the lower triangular structure of $\big[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\big]$ and the fact that the delays in $\tilde{\Lambda}$ are ordered descendantly, the term $\tilde{\Lambda}^{-1}\big[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\big]\tilde{\Lambda}$ is bistable and has unity direct feedthrough term. Hence K is proper iff so is $\tilde{K}$. Consequently, $\tilde{Q}$ can be made contractive by choice of proper K iff there exists a proper $\tilde{K}$ so that

$$\| C_r(\tilde{G}\tilde{\Lambda}, \tilde{K}) \| < 1.$$

Yet this is just another one-block problem, **OBP($\tilde{G}, \tilde{\Lambda}$)**. Moreover, since $\tilde{\Lambda} \succ \Lambda$, the latter problem has reduced complexity comparing with the original problem **OBP(G, Λ)**. We thus just proved the following result:

Lemma 5.1: Let G be as in (17) and Λ as in (25). Then the **OBP(G, Λ)** is solvable iff the adobe problem **OBP(G, Λ_1)** and the reduced complexity **OBP($\tilde{G}, \tilde{\Lambda}$)** are both solvable. Furthermore, in that case a proper K solves **OBP(G, Λ)** iff

$$K = C_r\big(\tilde{\Lambda}^{-1}\big[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\big]\tilde{\Lambda}, \tilde{K}\big)$$

with $\tilde{K}$ a solution of the **OBP($\tilde{G}, \tilde{\Lambda}$)** (here $\tilde{G}$ and Π are as defined in Theorem 4.1).

Now, we can proceed with the $(q+r-1)$-delay operator in exactly the same manner as with the $(q+r)$-delay operator before. More precisely, let us substitute $\tilde{G} \to G$, $\tilde{\Lambda} \to \Lambda$, and $\tilde{K} \to K$. Then, repeating arguments from the beginning of this section, the solvability of the one-block problem with the $(q+r-1)$-delay operator can be shown to be equivalent to the solvability of a adobe problem with

$$\Lambda_2 := \begin{bmatrix} \mathrm{e}^{-(h_2-h_1)s} I_{\mu_2} & 0 \\ 0 & I_{\rho_2} \end{bmatrix} \quad (\rho_2 = n_0 + n_1)$$

and a one-block problem with a $(q+r-2)$-delay operator. This procedure can obviously be repeated $q+r$ times, each time resulting to an **OBP** with a "smaller" delay operator, until we end up with a one-block problem with (0)-delay operator, the solution of which consists simply of the inversion of its "G" transfer function.

The **OBP** (and therefore **SHP**) can thus be solved iteratively, in $q+r$ iterations. The ith iteration involves solving the adobe delay problem **OBP(G_i, Λ_i)**, where

$$\Lambda_i := \begin{bmatrix} \mathrm{e}^{-(h_i-h_{i-1})s} I_{\mu_i} & 0 \\ 0 & I_{\rho_i} \end{bmatrix} \quad (\rho_i = \textstyle\sum_{j=0}^{i-1} n_j) \tag{27a}$$

and (bistable) G_i is generated by the following sequence:

$$G_i = \tilde{G}_{i-1} \quad [\text{with } G_1 = G \text{ as defined by (10)}], \tag{27b}$$

where $\tilde{G}_{i-1}$ is the "$\tilde{G}$" matrix appearing in the solution of the adobe problem **OBP(**G_{i-1}, Λ_{i-1}**)**. The solutions of all iterations are then combined to constitute the solution to the original multiple delay problem. The theorem below, which is the main result of this paper, summarizes the reasoning above.

Theorem 5.2: The problem **OBP** is solvable iff so are all **OBP(**G_i, Λ_i**)**, $i = 1, \ldots, q + r$. In this case, all solutions to the former are parameterized as

$$K = C_r\big(\Pi_\Lambda G_\Lambda^{-1}, Q_\Lambda\big),$$

where $G_\Lambda := \tilde{G}_{q+r}$ is bistable and finite dimensional,

$$\Pi_\Lambda := \Lambda^{-1} \prod_{i=1}^{q+r} \Lambda_i \left[\begin{smallmatrix} I & 0 \\ \Pi_i & I \end{smallmatrix}\right]$$

is bistable, and Q_Λ is an arbitrary contraction.

Note that the steps of the iterative procedure of Theorem 5.2 can be tailored together neatly to result in a closed-form solution. Because of space limitations, this procedure will be reported separately [16], see also [17]. Here we just present the closed-form solvability condition resulting from this procedure. To this end, rewrite the transfer matrix $G(s)$ from (10) as follows:

$$G(s) = \left[\begin{array}{c|c} A_L & \hat{B} \\ \hline \hat{C} & I_{n_u+n_y} \end{array}\right] = \left[\begin{array}{c|ccc} A_L & \hat{B}_{q+r} & \ldots & \hat{B}_0 \\ \hline \hat{C}_{q+r} & I_{n_{q+r}} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{C}_0 & 0 & \ldots & I_{n_0} \end{array}\right],$$

where the partitioning is compatible with that of the joint delay operator Λ in (8). Define the following sequence of matrices:

$$\hat{H}_0 = \begin{bmatrix} A_L & 0 \\ -\hat{C}'\hat{J}\hat{C} & -A_L' \end{bmatrix},$$

$$\hat{H}_{i+1} = \hat{H}_i + \begin{bmatrix} -\hat{B}_i \\ \hat{C}_i'\hat{J}_i \end{bmatrix} \begin{bmatrix} \hat{C}_i & \hat{J}_i\hat{B}_i' \end{bmatrix}, \quad i = 0, \ldots, q + r - 1,$$

where $\hat{J} := \operatorname{diag}\{I_{n_u}, -I_{n_y}\}$ and $\hat{J}_i$ are its diagonal sub-blocks[1] partitioned compatibly with (8). Then, introduce the following matrix function, defined over $t \in [0, h_{q+r}]$:

$$\hat{\Sigma}(t) = \mathrm{e}^{\hat{H}_{i+1}(t-h_i)} \prod_{j=1}^{i} \mathrm{e}^{\hat{H}_j(h_j-h_{j-1})}, \quad t \in (h_i, h_{i+1}],$$

where we suppose that $h_0 = 0$. Then the **OBP** is solvable iff $\det \Sigma_{22}(t) \neq 0$ for all $t \in (0, h_{q+r}]$.

VI. ILLUSTRATIVE EXAMPLE

To illustrate the proposed approach, we consider the problem of signal reconstruction from delayed noisy measurements. The signal to be reconstructed, x, is assumed to have a bounded (in the L^2 sense) velocity, i.e., $x = \frac{1}{s}w_v$ for $w_v \in L^2$.

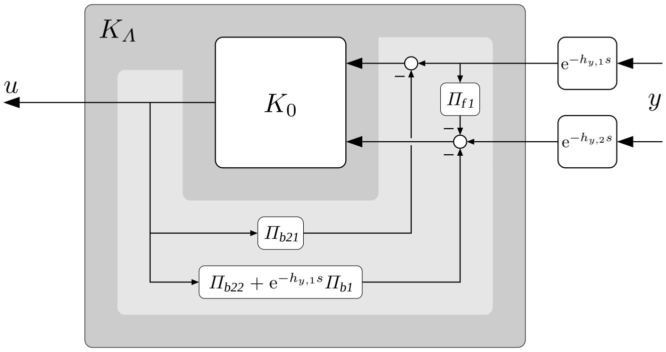


Fig. 6. Optimal estimator in the example

We assume also that there are two sensors that measure x with different delays and noise:

$$y_i(t) = x(t - h_{y,i}) + \sigma_i w_{n,i}(t - h_{y,i}), \quad i = 1, 2,$$

where σ_1 and σ_2 can be thought of as the intensities of the measurement noise $w_{n,1}$ and $w_{n,2}$, respectively, and $h_{y,1} < h_{y,2}$. The problem is do design an estimator $K_\Lambda \in H^\infty$ of x so that the error system from the inputs w_v, $w_{n,i}$ to the estimation error is stable and its H^∞-norm is smaller than γ.

This problem can be recast as the problem in Fig. 1 with

$$P(s) = \left[\begin{array}{c|ccc:c} 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & -1 \\ \hdashline 1 & 0 & \sigma_1 & 0 & 0 \\ 1 & 0 & 0 & \sigma_2 & 0 \end{array}\right],$$

$\Lambda_y = \operatorname{diag}\{\mathrm{e}^{-h_{y,1}s}, \mathrm{e}^{-h_{y,2}s}\}$, and $\Lambda_u = 1$. Note that (A, B_2) in $P(s)$ above is not stabilizable, yet we also do not require the internal stability of the system, so that our formulae apply *mutatis mutandis* (can be proved by ϵ-modification arguments).

It can be verified that the delay-free version of the problem is solvable iff

$$0 < \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} - \frac{1}{\gamma^2} =: \kappa^2.$$

Furthermore, the solutions of the two Riccati equations are $X = 0$ and $Y = 1/\kappa$, the transfer matrices in (12) and (10) are $D_\Lambda = \operatorname{diag}\{1, \gamma/\sigma_1, \gamma/\sigma_2\}$ and

$$G(s) = \left[\begin{array}{c|ccc} -\kappa & 1/(\gamma\kappa) & -1/(\sigma_1\kappa) & -1/(\sigma_2\kappa) \\ \hline 1/\gamma & 1 & 0 & 0 \\ 1/\sigma_1 & 0 & 1 & 0 \\ 1/\sigma_2 & 0 & 0 & 1 \end{array}\right], \tag{28}$$

respectively, and $\Lambda = \operatorname{diag}\{\mathrm{e}^{-(h_\delta+h_{y,1})s}, \mathrm{e}^{-h_\delta s}, 1\}$, where $h_\delta := h_{y,2} - h_{y,1}$ is the delay excess in the second channel (so that $h_1 = h_\delta$ and $h_2 = h_{y,1}$).

When $\sigma_2 \to \infty$, only the first sensor is usable and in that case the optimal delay-free (corresponding to $h_{y,1} = 0$) performance level is $\gamma = \sigma_1$. We now consider whether/how this performance can be recovered by *delayed* measurements with the help of the second sensor ($\sigma_2 < \infty$).

To this end, consider the application of the procedure described in Sections IV and V to (28) subject to $\gamma = \sigma_1$. Two adobe problems have to be solved: the first one with $\mu = 2$ and

---

[1] Note that for almost all i either $\hat{J}_i = -I$ ($i \leq r$) or $\hat{J}_i = I$ ($i \geq r + 2$). Potentially, only $\hat{J}_{r+1}$ might contain both negative and positive elements (if both $m_0 \neq 0$ and $p_0 \neq 0$).

$$H_X := \begin{bmatrix} A & 0 \\ -C_1'C_1 & -A' \end{bmatrix} - \begin{bmatrix} B_1 & B_2 \\ -C_1'D_{11} & -C_1'D_{12} \end{bmatrix} \begin{bmatrix} D_{11}'D_{11} - \gamma^2 I & D_{11}'D_{12} \\ D_{12}'D_{11} & D_{12}'D_{12} \end{bmatrix}^{-1} \begin{bmatrix} D_{11}'C_1 & B_1' \\ D_{12}'C_1 & B_2' \end{bmatrix}$$

and

$$H_Y := \begin{bmatrix} A' & 0 \\ -B_1B_1' & -A \end{bmatrix} - \begin{bmatrix} C_1' & C_2' \\ -B_1D_{11}' & -B_1D_{21}' \end{bmatrix} \begin{bmatrix} D_{11}D_{11}' - \gamma^2 I & D_{11}D_{21}' \\ D_{21}D_{11}' & D_{21}D_{21}' \end{bmatrix}^{-1} \begin{bmatrix} D_{11}B_1' & C_1 \\ D_{21}B_1' & C_2 \end{bmatrix}$$

$h = h_\delta$ and the second one with $\mu = 1$ and $h = h_{y,1}$. It can be verified that the first adobe problem is solvable for all h_δ (in that case $\Sigma_{22}(t) \equiv 1$). Then, applying the criterion from the end of Section V, the second adobe problem (and therefore the whole problem) is solvable iff $\cos(\frac{t}{\sigma_1}) \neq \sin(\frac{t}{\sigma_1})\frac{h_\delta+\sigma_2}{\sigma_1}$, $\forall t \in (0, h_{y,1}]$. This is clearly equivalent to

$$h_{y,1} < \bar{h}_{y,1} := \sigma_1 \arctan \frac{\sigma_1}{\sigma_2 + h_\delta}.$$

When both σ_2 and h_δ vanish, $\bar{h}_{y,1} \to \frac{\pi}{2}\sigma_1$. This is the absolute upper bound on the measurement delay for which the second sensor can help to recover the reconstruction performance achievable with the delay-free first sensor. Any worsening of the second sensor (i.e., the increase of either σ_2 or h_δ) decreases then this upper bound. An interesting observation here is that the effect of the noise intensity σ_2 on the achievable performance is exactly as that of the delay excess h_δ.

Furthermore, the optimal estimator (i.e., the one corresponding to $\gamma = \sigma_1$, $h_{y,1} = \bar{h}_{y,1}$, and $Q = 0$) can be shown to be of the form depicted in Fig. 6. Here

$$\begin{bmatrix} \Pi_{b1} & \Pi_{f1} \end{bmatrix} = -\frac{h_\delta s - 1 + \mathrm{e}^{-h_\delta s}}{\sigma_1^2 s^2} \begin{bmatrix} \frac{\sigma_2}{\sigma_1} & 1 \end{bmatrix},$$

$$\Pi_{b21} = \frac{\chi(\sigma_1^2 s - h_\delta - \sigma_2) + \mathrm{e}^{-\bar{h}_{y,1}s}}{\sigma_1^2 s^2 + 1},$$

$$\Pi_{b22} = \frac{\chi((\sigma_1^2 + h_\delta^2 + h_\delta\sigma_2)s - \sigma_2) - (h_\delta s - 1)\mathrm{e}^{-\bar{h}_{y,1}s}}{\sigma_1^2 s^2 + 1}$$

are FIR and the optimal "central controller" is the static gain:

$$K_0 = \chi \begin{bmatrix} h_\delta + \sigma_2 & \frac{\sigma_1^2}{\sigma_2} \end{bmatrix},$$

where $\chi := 1/\sqrt{\sigma_1^2 + (h_\delta + \sigma_2)^2}$. Note that the FIR transfer functions $\frac{\sigma_1}{\sigma_2}\Pi_{b1}$ and $\frac{\sigma_1}{\sigma_2}\Pi_{f1}$ are those resulting from the first adobe problem and Π_{b21} and $\frac{\sigma_1}{\sigma_2}\Pi_{b22}$ are those resulting from the second adobe problem[2].

VII. CONCLUDING REMARKS

In this paper we have derived the first complete solution to the standard H^∞ problem for systems having multiple i/o delays. The idea is to split the problem into a nested sequence of elementary problems, called *adobe delay* problems, the solutions of which can then be combined to end up with the general solution. It also turns out that when combined, the adobe problems fall into place leading to subsequent simplifications and the closed form solution. These simplifications are reported in the second part of this paper [16].

ACKNOWLEDGMENT

The authors would like to thank Banu Ataşlar for careful reading a draft version and finding errors in some formulae.

APPENDIX A
DELAY-FREE H^∞ SOLUTION

The purpose of this appendix is to present the solution to the standard delay-free H^∞ problem. The formulae are essentially from [14] with some generalizations. We assume that $\boldsymbol{\mathcal{A}}_{\mathbf{1\text{-}3}}$ hold and we define the following quantities:

$$\gamma_z := \|\big(I - D_{12}(D_{12}'D_{12})^{-1}D_{12}'\big)D_{11}\|,$$
$$\gamma_w := \|D_{11}\big(I - D_{21}'(D_{21}D_{21}')^{-1}D_{21}\big)\|.$$

Introduce also the Hamiltonian matrices H_X and H_Y as shown on the top of this page. Then γ is admissible for the delay-free version of the **SHP** iff the following conditions hold:

$\boldsymbol{\mathcal{C}}_{\mathbf{1}}$: $\max\{\gamma_z, \gamma_w\} < \gamma$;
$\boldsymbol{\mathcal{C}}_{\mathbf{2}}$: $H_X \in \operatorname{dom}\mathrm{Ric}$ and $X := \mathrm{Ric}(H_X) \geq 0$;
$\boldsymbol{\mathcal{C}}_{\mathbf{3}}$: $H_Y \in \operatorname{dom}\mathrm{Ric}$ and $Y := \mathrm{Ric}(H_Y) \geq 0$;
$\boldsymbol{\mathcal{C}}_{\mathbf{4}}$: $\rho(XY) < \gamma$.

Define now the matrices

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = -\Theta_z^{-1}\left(\begin{bmatrix} B_1' \\ B_2' \end{bmatrix} X + \begin{bmatrix} D_{11}' \\ D_{12}' \end{bmatrix} C_1\right)$$

and

$$L = \begin{bmatrix} L_1 & L_2 \end{bmatrix} = -\big(Y \begin{bmatrix} C_1' & C_2' \end{bmatrix} + B_1 \begin{bmatrix} D_{11}' & D_{21}' \end{bmatrix}\big)\Theta_w^{-1},$$

where

$$\Theta_z := \begin{bmatrix} D_{11}'D_{11} - \gamma^2 I & D_{11}'D_{12} \\ D_{12}'D_{11} & D_{12}'D_{12} \end{bmatrix},$$
$$\Theta_w := \begin{bmatrix} D_{11}D_{11}' - \gamma^2 I & D_{11}D_{21}' \\ D_{21}D_{11}' & D_{21}D_{21}' \end{bmatrix}.$$

These F and L are the full-information and the output estimation gains, respectively. If the solvability conditions above hold true, then the matrix $Z := (I - \gamma^{-2}YX)^{-1}$ is well-defined, the matrices

$$A_F := A + B_1F_1 + B_2F_2 \quad \text{and} \quad A_L := A + L_1C_1 + L_2C_2$$

are Hurwitz, and the inertia of the matrix

$$M_D := -\gamma^2 \begin{bmatrix} D_{12}' & D_{22}' \\ 0 & -I \end{bmatrix} \Theta_w^{-1} \begin{bmatrix} D_{12} & 0 \\ D_{22} & -I \end{bmatrix}$$

coincides with that of $J = \operatorname{diag}\{I_{n_u}, -I_{n_y}\}$.

With these definitions, all controllers solving the delay-free version of the **SHP** are parameterized as $C_r(G_\infty^{-1}, Q)$, where G_∞ is given by (4) with any D_∞ satisfying

$$D_\infty' J D_\infty = M_D \tag{29}$$

---

[2] The multiplier $\frac{\sigma_1}{\sigma_2}$ above results from incorporating D_Λ defined by (12).

and Q is an arbitrary contraction from H^∞ and such that $\lim_{\mathrm{Re}\, s>0, s\to\infty} C_r(D_\infty^{-1}, Q(s))$ is well-posed.

Note that the factorization in (29) is not unique. We exploit this freedom to bring D_∞ to a special form which is important when dealing with multiple delay systems in Section III.

A. Special form of D_∞

It is readily seen that condition $\boldsymbol{\mathcal{C}_1}$ together with the Parrott's Theorem [18, Sec. 2.11] guarantees that there exists a matrix D_K such that

$$\tilde{D}_{11} := \mathcal{F}_l\left(\left[\begin{smallmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{smallmatrix}\right], D_K\right)$$

is well-defined and $\|\tilde{D}_{11}\| < \gamma$.

Lemma 1.1: Let D_K be as above. Then D_∞ satisfying (29) can always be chosen in the form

$$D_\infty = V\left[\begin{smallmatrix} I & -D_K \\ 0 & I \end{smallmatrix}\right]$$

with a lower triangular V.

Proof: It is readily verified that

$$\Theta_w^{-1} = \begin{bmatrix} I & 0 \\ D_K'(I - D_{22}'D_K')^{-1}D_{12}' & I \end{bmatrix} \tilde{\Theta}^{-1} \times \begin{bmatrix} I & D_{12}(I - D_K D_{22})^{-1} D_K \\ 0 & I \end{bmatrix},$$

where

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} \\ \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} := \begin{bmatrix} \tilde{D}_{11}\tilde{D}_{11}' - \gamma^2 I & \tilde{D}_{11}D_{21}' \\ D_{21}\tilde{D}_{11}' & D_{21}D_{21}' \end{bmatrix}.$$

Hence,

$$M_D = -\gamma^2 \begin{bmatrix} I & 0 \\ -D_K' & I \end{bmatrix} V_\alpha' \tilde{\Theta}^{-1} V_\alpha \begin{bmatrix} I & -D_K \\ 0 & I \end{bmatrix},$$

where

$$V_\alpha := \begin{bmatrix} D_{12}(I - D_K D_{22})^{-1} & 0 \\ D_{22} & -I + D_{22}D_K \end{bmatrix}.$$

We thus only need to show that $-\gamma^2 V_\alpha' \tilde{\Theta}^{-1} V_\alpha$ can be factorized as $V'JV$.

To this end, note that since $\tilde{\Theta}_{11}$ is nonsingular, $\tilde{\Theta}$ can be factorized as follows [18, Sec. 2.3]:

$$\tilde{\Theta} = \begin{bmatrix} I & 0 \\ \tilde{\Theta}_{21}\tilde{\Theta}_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{\Theta}_{11} & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} I & \tilde{\Theta}_{11}^{-1}\tilde{\Theta}_{12} \\ 0 & I \end{bmatrix},$$

where $\Delta := \tilde{\Theta}_{22} - \tilde{\Theta}_{21}\tilde{\Theta}_{11}^{-1}\tilde{\Theta}_{12}$. Since $\tilde{\Theta}_{11} < 0$ (by construction) and the inertia of $\tilde{\Theta}$ coincides with that of Θ_w (which, in turn, coincides with the inertia of $-J$), $\Delta > 0$. Bring in two *lower triangular* Cholesky factorizations

$$V_1'V_1 = -(I - D_{22}'D_K')^{-1}D_{12}'\tilde{\Theta}_{11}^{-1}D_{12}(I - D_K D_{22})^{-1},$$
$$V_2'V_2 = (I - D_K'D_{22}')\Delta^{-1}(I - D_{22}D_K),$$

and also define

$$V_3 := \tilde{\Theta}_{21}\tilde{\Theta}_{11}^{-1}D_{12}(I - D_K D_{22})^{-1} - D_{22}.$$

Then

$$V = \gamma \begin{bmatrix} V_1 & 0 \\ V_2(I - D_{22}D_K)^{-1}V_3 & V_2 \end{bmatrix}$$

is the required factorization. ■

The proof of Lemma 1.1 is constructive, so the required V can be formed following its steps.

APPENDIX B
PROOFS AND TECHNICALITIES

In this section we present the details of the proof of Theorem 4.1, outlined in §IV-B. Having proved the invertibility of Σ_{22} in §IV-C, the rest of the developments follows the ideas of [8] with some modifications caused by the fact that the partitioning of Λ in (14) need not match the signal partitioning in Fig. 4.

A. Lower S-transformation

The derivations in this paper are substantially simplified by the use of the "lower Schur complementation" transformation $S_l(O)$ introduced in [8] (see also [14, Ch. 4], where similar transformation was introduced). The lower S-transformation is defined for a 2×2 block operator O as follows:

$$S_l(O) := \begin{bmatrix} O_{11} - O_{12}O_{22}^{-1}O_{21} & O_{12}O_{22}^{-1} \\ -O_{22}^{-1}O_{21} & O_{22}^{-1} \end{bmatrix}.$$

It is clear that the lower S-transformation is well-defined iff the lower right subblock of O is nonsingular. S-transformation can be thought of as the "swapping" of the lower part of the inputs and outputs, namely

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = O \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \iff \begin{bmatrix} \zeta_1 \\ \eta_2 \end{bmatrix} = S_l(O) \begin{bmatrix} \eta_1 \\ \zeta_2 \end{bmatrix}$$

(provided the mapping is well-defined). The relation above prompts an elegant way to perform S-transformation for systems given by their state-space realizations. Indeed, if

$$\Phi(s) = \left[\begin{array}{c|cc} A_\phi & B_{\phi 1} & B_{\phi 2} \\ \hline C_{\phi 1} & D_{\phi 1} & 0 \\ C_{\phi 2} & 0 & D_{\phi 2} \end{array}\right],$$

then the straightforward flow-tracing yields

$$S_l(\Phi(s)) = \left[\begin{array}{c|cc} A_\phi - B_{\phi 2}D_{\phi 2}^{-1}C_{\phi 2} & B_{\phi 1} & B_{\phi 2}D_{\phi 2}^{-1} \\ \hline C_{\phi 1} & D_{\phi 1} & 0 \\ -D_{\phi 2}^{-1}C_{\phi 2} & 0 & D_{\phi 2}^{-1} \end{array}\right]. \tag{30}$$

Another advantage of looking at the S-transformation of O instead of at O itself is that

$$S_l\left(\left[\begin{smallmatrix} I & \Pi_1 \\ 0 & I \end{smallmatrix}\right] O \left[\begin{smallmatrix} I & 0 \\ \Pi_2 & I \end{smallmatrix}\right]\right) = S_l(O) + \left[\begin{smallmatrix} 0 & \Pi_1 \\ -\Pi_2 & 0 \end{smallmatrix}\right]. \tag{31}$$

This relation will be used in the following subsection.

B. Proof of Theorem 4.1

This subsection is devoted to the proof of Theorem 4.1. By Lemma 4.4 solvability of the **OBP** guarantees that $\Sigma_{22}(t)$ is nonsingular for all $t \in [0, h]$. This fact will be exploited in the proof.

The proof of Theorem 4.1 is based on the J-spectral factorization arguments of [8] which, in turn, root in [9]. The main and technical part of the proof is the construction of a bistable W that satisfies $\Lambda^\sim G^\sim JG\Lambda = W^\sim JW$. Now if $\Lambda^\sim G^\sim JG\Lambda$ would have been rational then this would have been a standard problem. However, because of the delays, $\Lambda^\sim G^\sim JG\Lambda$ generally is not rational. Owing to the specific structure of Λ in (14), though, the delays enter only the off-diagonal blocks,

$$\Phi := \Lambda^\sim G^\sim J G \Lambda = \begin{bmatrix} \Psi_{11} & \mathrm{e}^{hs}\Psi_{12} \\ \mathrm{e}^{-hs}\Psi_{21} & \Psi_{22} \end{bmatrix}$$

with

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} := G^\sim J G.$$

The nonrational parts can be removed by an appropriate prefactorization as we shall now show. We contemplate a prefactorization by a bistable factor of the form $\left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right]$. The question then is: for which stable Π is

$$\begin{bmatrix} I & 0 \\ \Pi & I \end{bmatrix}^\sim \Phi \begin{bmatrix} I & 0 \\ \Pi & I \end{bmatrix}$$

rational? The answer to this question is simpler after S-transformation as that transforms the above multiplications into addition. By Eqn. (31) we have that

$$\begin{aligned} \Xi &:= S_l(\left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right]^\sim \Phi \left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right]) = S_l(\Phi) + \begin{bmatrix} 0 & \Pi^\sim \\ -\Pi & 0 \end{bmatrix} \\ &= \Lambda^\sim \overbrace{S_l(\Psi)}^{:=\Omega} \Lambda + \begin{bmatrix} 0 & \Pi^\sim \\ -\Pi & 0 \end{bmatrix} \\ &= \begin{bmatrix} \Omega_{11} & \mathrm{e}^{hs}\Omega_{12} + \Pi^\sim \\ \mathrm{e}^{-hs}\Omega_{21} - \Pi & \Omega_{22} \end{bmatrix}. \end{aligned} \tag{32}$$

In this form it is clear how we should choose Π so as to make Ξ rational: define Π as

$$\Pi(s) := -\boldsymbol{\pi}_h(\mathrm{e}^{-hs}\Omega_{21}). \tag{33}$$

Using the fact that $S_l(S_l(O)) = O$ then gives us the rational matrix we are after,

$$\tilde{\Psi} := \left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right]^\sim \Phi \left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right] = S_l(\Xi). \tag{34}$$

These manipulations have a counterpart in state space, which we shall now document. Given the realization (17) of G, a realization of Ψ is

$$\begin{aligned} \Psi &:= G^\sim J G \\ &= \left[\begin{array}{cc|cc} A_L & 0 & B_\mu & B_\rho \\ -C_\mu' J_\mu C_\mu - C_\rho' J_\rho C_\rho & -A_L' & -C_\mu' J_\mu & -C_\rho' J_\rho \\ \hline J_\mu C_\mu & B_\mu' & J_\mu & 0 \\ J_\rho C_\rho & B_\rho' & 0 & J_\rho \end{array}\right] \\ &=: \left[\begin{array}{c|cc} A_\Psi & B_{\Psi 1} & B_{\Psi 2} \\ \hline C_{\Psi 1} & J_\mu & 0 \\ C_{\Psi 2} & 0 & J_\rho \end{array}\right]. \end{aligned} \tag{35}$$

Note that this realization obeys the symmetry property that

$$\begin{bmatrix} B_{\Psi 1} & B_{\Psi 2} \end{bmatrix} = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right] \begin{bmatrix} C_{\Psi 1}' & C_{\Psi 2}' \end{bmatrix}. \tag{36}$$

Next we form a realization of $\Omega := S_l(\Psi)$ which using (30) follows as

$$\Omega = S_l(\Psi) = \left[\begin{array}{c|cc} A_\Psi - B_{\Psi 2} J_\rho C_{\Psi 2} & B_{\Psi 1} & B_{\Psi 2} J_\rho \\ \hline C_{\Psi 1} & J_\mu & 0 \\ -J_\rho C_{\Psi 2} & 0 & J_\rho \end{array}\right]. \tag{37}$$

The Hamiltonian H as defined in (19) is in fact the “A-matrix” of this Ω,

$$A_\Psi - B_{\Psi 2} J_\rho C_{\Psi 2} = \begin{bmatrix} A_L - B_\rho C_\rho & -B_\rho J_\rho B_\rho' \\ -C_\mu' J C_\mu & -A_L' + C_\rho' B_\rho' \end{bmatrix} =: H.$$

Inspection of the realization of the lower-left block of Ω shows that the completion $\Pi(s) := -\boldsymbol{\pi}_h(\mathrm{e}^{-hs}\Omega_{21})$ equals

$$\Pi(s) = \boldsymbol{\pi}_h \left(\mathrm{e}^{-hs} \left[\begin{array}{c|c} H & B_{\Psi 1} \\ \hline J_\rho C_{\Psi 2} & 0 \end{array}\right] \right),$$

(which coincides with the formula for Π given in Theorem 4.1) and then $\mathrm{e}^{-hs}\Omega_{21} - \Pi$ is rational with

$$\mathrm{e}^{-hs}\Omega_{21} - \Pi = \left[\begin{array}{c|c} H & \Sigma^{-1} B_{\Psi 1} \\ \hline -J_\rho C_{\Psi 2} & 0 \end{array}\right], \tag{38}$$

in which $\Sigma := \Sigma(h)$ with $\Sigma(t) := \exp(Ht)$. Combination of (37) and (38) yields a realization of Ξ as defined in (32),

$$\Xi = \left[\begin{array}{c|cc} H & \Sigma^{-1} B_{\Psi 1} & B_{\Psi 2} J_\rho \\ \hline C_{\Psi 1} \Sigma & J_\mu & 0 \\ -J_\rho C_{\Psi 2} & 0 & J_\rho \end{array}\right]$$

so that (taking into account that $A_\Psi = H + B_{\Psi 2} J_\rho C_{\Psi 2}$)

$$\begin{aligned} \tilde{\Psi} &:= \left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right]^\sim \Phi \left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right] = S_l(\Xi) \\ &= \left[\begin{array}{c|cc} A_\Psi & \Sigma^{-1} B_{\Psi 1} & B_{\Psi 2} \\ \hline C_{\Psi 1} \Sigma & J_\mu & 0 \\ C_{\Psi 2} & 0 & J_\rho \end{array}\right] \\ &=: \left[\begin{array}{c|c} A_{\tilde{\Psi}} & B_{\tilde{\Psi}} \\ \hline C_{\tilde{\Psi}} & J \end{array}\right]. \end{aligned} \tag{39}$$

Note the strong resemblance with the realization of Ψ in (35). Since Σ is symplectic we have that $\Sigma \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right] \Sigma' = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ and hence the “B” and “C” matrices of the above realization obey the symmetry property

$$\begin{bmatrix} \Sigma^{-1} B_{\Psi 1} & B_{\Psi 2} \end{bmatrix} = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right] \begin{bmatrix} \Sigma' C_{\Psi 1}' & C_{\Psi 2}' \end{bmatrix}. \tag{40}$$

Now we are in a position to derive a (rational) J-spectral factor $\tilde{G}$ of

$$\tilde{\Psi} := \left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right]^\sim \Phi \left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right] = \tilde{G}^\sim J \tilde{G}. \tag{41}$$

To this end we exploit the resemblance of (35) and (39) and the fact Σ_{22} is invertible and that Ψ has J-spectral factor G.

The “A-matrix” of the inverse of (39) is readily seen to be the Hamiltonian

$$\begin{aligned} H_{\tilde{\Psi}} &:= A_\Psi - \Sigma^{-1} B_{\Psi 1} J_\mu C_{\Psi 1} \Sigma - B_{\Psi 2} J_\rho C_{\Psi 2} \\ &= H - \Sigma^{-1} B_{\Psi 1} J_\mu C_{\Psi 1} \Sigma \\ &= \Sigma^{-1} (H - B_{\Psi 1} J_\mu C_{\Psi 1}) \Sigma \\ &= \Sigma^{-1} \underbrace{\begin{bmatrix} A_{G^{-1}} & * \\ 0 & -A_{G^{-1}}' \end{bmatrix}}_{H_\Psi} \Sigma \end{aligned} \tag{42}$$

where $A_{G^{-1}}$ is the “A-matrix” of G^{-1} and hence is stable. (In fact the matrix H_Ψ in the middle of (42) is simply the Hamiltonian “A-matrix” of the inverse of $\Psi = G^\sim J G$.) Because of the similarity of H_Ψ and $H_{\tilde{\Psi}}$ it is immediate that the stabilizing solution of the Riccati equation

$$\begin{bmatrix} -M_{\tilde{\Psi}} & I \end{bmatrix} H_{\tilde{\Psi}} \begin{bmatrix} I \\ M_{\tilde{\Psi}} \end{bmatrix} = 0$$

is determined by the property that

$$\operatorname{Im}\begin{bmatrix} I \\ M_{\tilde{\Psi}} \end{bmatrix} = \operatorname{Im}\Sigma^{-1}\begin{bmatrix} I \\ M_{\Psi} \end{bmatrix} \tag{43}$$

where M_{Ψ} is the stabilizing solution of the Riccati equation associated with H_{Ψ},

$$\begin{bmatrix} -M_{\Psi} & I \end{bmatrix} H_{\Psi} \begin{bmatrix} I \\ M_{\Psi} \end{bmatrix} = 0.$$

Because of (42) and stability of $A_{G^{-1}}$ it is direct that $M_{\Psi} = 0$ and then (43) gives

$$M_{\tilde{\Psi}} = (\Sigma^{-1})_{21}(\Sigma^{-1})_{11}^{-1} = -\Sigma_{22}^{-1}\Sigma_{21}.$$

The second equality here exploits that $M_{\tilde{\Psi}}$ is symmetric and that Σ is symplectic so that

$$\Sigma^{-1} = \begin{bmatrix} \Sigma'_{22} & -\Sigma'_{12} \\ -\Sigma'_{21} & \Sigma'_{11} \end{bmatrix}.$$

It is now easy to verify that[3]

$$\tilde{G} = \left[\begin{array}{c|c} A_L & [\,I \;\; 0\,]B_{\tilde{\Psi}} \\ \hline JC_{\tilde{\Psi}}\left[\begin{smallmatrix} I \\ M_{\tilde{\Psi}} \end{smallmatrix}\right] & I \end{array}\right] \tag{44}$$

satisfies (41), and since $M_{\tilde{\Psi}}$ is stabilizing it follows that $\tilde{G}$ is besides stable also bistable, as required. Substituting $M_{\tilde{\Psi}} = -\Sigma_{22}^{-1}\Sigma_{21}$ in (44) results in the realization of $\tilde{G}$ as given in Theorem 4.1.

In summary: if the **OBP** is solvable then $\Phi := \Lambda^{\sim}G^{\sim}JG\Lambda$ has a J-spectral factorization

$$\Lambda^{\sim}G^{\sim}JG\Lambda = W^{\sim}JW \tag{45}$$

and W can be taken as

$$W = \tilde{G}\left[\begin{smallmatrix} I & 0 \\ -\Pi & I \end{smallmatrix}\right]. \tag{46}$$

What remains is to show that invertibility of $\Sigma_{22}(t)$ for all $t \in [0, h]$ is also *sufficient* for the **OBP** to have a solution, and that all solutions can be parameterized by an appropriate LFT. The arguments are fairly standard.

By construction of W, see (45), we have that

$$\|\, C_r(G\Lambda, K)\|_{L^{\infty}} < 1 \iff \|\tilde{Q}\|_{L^{\infty}} < 1$$

for $\tilde{Q}$ defined as

$$\tilde{Q} := C_r(W, K). \tag{47}$$

Now since $W(\infty) = I$ this $\tilde{Q}$ is proper iff K is proper, yet the set of proper operators in L^{∞} is in fact H^{∞}, [20] (see also [21, A6.26.c, A6.27]). So if K solves the **OBP** then necessarily $\tilde{Q}$ is a stable contraction. This condition on $\tilde{Q}$ is also sufficient as we shall now see. The thing to note is that

$$\Theta(h) := G\Lambda W^{-1}$$

is not only stable and J-unitary (i.e., $\Theta(h)^{\sim}J\Theta(h) = J$) but in fact J-lossless (meaning that in addition $\Theta_{22}(h)$ is bistable). Indeed, from $\Theta(h)^{\sim}J\Theta(h) = J$ it follows that $\Theta_{22}(h)^{\sim}\Theta_{22}(h) \geq I$, and as $\Theta(t)$ exists and is stable and continuous as a function of $t \in [0, h]$, and $\Theta_{22}(0) = I$ it follows that $\Theta_{22}(h)$ is bistable. It is well known that J-losslessness of $\Theta(h)$ implies that $C_r(\Theta(h), \tilde{Q})$ is a contraction if so is $\tilde{Q}$, see, e.g., [9, Thm. 6.2], hence for any contraction $\tilde{Q}$ we have that

$$C_r(G\Lambda, K) = C_r(\Theta(h)W, K) = C_r(\Theta(h), \tilde{Q})$$

is a contraction, which is what we need. The inversion of (47) yields then the parameterization of all solutions: $K = C_r(W^{-1}, \tilde{Q}) = C_r(\left[\begin{smallmatrix} I & 0 \\ \Pi & I \end{smallmatrix}\right]\tilde{G}^{-1}, \tilde{Q})$.

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[3] Generally, solutions of Riccati equations need not exist [19], however due to Lemma 4.4 we know that Σ_{22} is invertible so that $M_{\tilde{\Psi}}$ here *does* exist.

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