Abstract. The problem of input-output stabilization in a general two-sided model matching setup is studied. As a first step, the problem is reduced to a pair of uncoupled bilateral Diophantine equations over $RH_1$. Then, recent results on bilateral Diophantine equation are exploited to obtain a numerically tractable solution given in terms of explicit state-space formulae. The resulting solvability conditions rely on two uncoupled Sylvester equations accompanied by algebraic constraints. This is in contrast to the corresponding one-sided stabilization, where no Sylvester equations are required. It is shown that imposing a mild simplifying assumption is instrumental in obtaining convenient parameterization of all stabilizing solutions, which is affine in a single $RH_1$ parameter. This demonstrates that if the aforementioned assumption is imposed, the general two-sided stabilization problem is similar to its one-sided counterpart in the sense that the constraints imposed by a stability requirement can be resolved without increasing problem complexity.

1. Introduction and problem formulation. In this paper we study the stabilization problem for the general two-sided model matching setup depicted in Fig. 1.1. The goal here is to find a stable $K$ stabilizing the system from $w$ to $e$ for some given, not necessarily stable, $G_1$, $G_2$, and $G_3$. This setup arises in a wide spectrum of control and estimation problems involving constraints on asymptotic behavior subject to non-decaying external signals.

As a simple motivating example, consider the feedforward tracking problem with a proper stable plant $P(s)$ and a controller $K(s)$. Assume that it is required to achieve asymptotically perfect tracking of any step reference signal. Clearly, this requirement is fulfilled iff the static gain $P(0)K(0) = 1$ or, equivalently, iff the transfer function $1 - P(s)K(s)$ has at least one zero at the origin. The latter condition can, in turn, be recast as the stability requirement $(1 - PK)W \in RH_1^\infty$ for any proper $W(s)$ having a pole at the origin, say $W(s) = \frac{s}{s+C}$. This corresponds to the model-matching stabilization in Fig. 1.1 for $G_1 = G_2 = W$ and $G_3 = P$.

In this example the weighting function $W$ can be thought of as the model of the non-decaying reference signal to be tracked (the step signal in our case). Then, by an appropriate choice of the weight we can formulate problems of tracking of various signals such as ramp, sine wave, etc.

Formally, the stabilization problem addressed in this paper is as follows:

**SP:** Given proper rational transfer functions $G_1(s)$, $G_2(s)$, and $G_3(s)$, find whether there is $K \in RH_1^\infty$ such that

$$T := G_1 - G_3KG_2 \in RH_1^\infty \quad (1.1)$$

and then characterize all such $K$ and all resulting $T$ when one exists.

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This problem will hereafter be referred to as the general two-sided problem. Its special cases with $G_3 = I$ and $G_3 \in RH^\infty$ will be referred to as the one-sided problem ($SP_o$) and the problem with one-sided instabilities ($SP_p$), respectively.

Note that $SP$ is not a special case of the well-understood internal stabilization problem associated with the system in Fig. 1.2(a), see [10, Appendix A], even though for $G_{22} = 0$ the system from $w$ to $e$ there reduces to the model matching, $G_{e/w} = G_{11} + G_{12}CG_{21}$. The key difference is the internal stability requirement, which renders such a problem insolvable whenever there is an unstable mode of $G$ that is not a mode of $G_{22}$. At the same time, the stabilization problem considered in this work does not involve the internal stability requirement because instabilities involved in $SP$ do not represent physical plant properties, but rather are associated with asymptotic behavior constraints. As a result, they should not be internally stabilized. On the opposite, the only way to resolve $SP$ is via unstable pole-zero cancellations, which are prohibited in any problem involving the internal stability requirement.

The $SP$ can still be related to feedback stabilization. Consider the setup depicted in Fig. 1.2(b), where $W_w$ and $W_e$ are the unstable weights that specify requirements on the asymptotic behavior. The stabilization in this setting consists of two problems: the internal stabilization of the system from $\tilde{w}$ to $\tilde{e}$, associated with “physical” instabilities, and the input/output stabilization of the system from $w$ to $e$, associated with asymptotic behavior requirements. The first problem can be resolved using the Youla–Kucera parameterization [10] and results in the parameterization of all feasible stable transfer matrices from $\tilde{w}$ to $\tilde{e}$ in the form

$$G_{\tilde{e}/\tilde{w}} = \tilde{G}_1 + \tilde{G}_3K\tilde{G}_2,$$

where $K \in RH^\infty$ but otherwise arbitrary. In this case

$$G_{e/w} = W_e\tilde{G}_1W_w + W_e\tilde{G}_3K\tilde{G}_2W_w,$$

which is exactly what we have in $SP$ for $G_1 = W_e\tilde{G}_1W_w$, $G_2 = \tilde{G}_2W_w$ and $G_3 = W_e\tilde{G}_3$. This shows that, also in feedback setting, the issues related to asymptotic behavior can be handled in the framework of the model-matching stabilization.

The one-sided problem, $SP_o$, was solved in [20, 21] using earlier ideas of [19], which also corresponds to the one-sided setup. The solution provides a necessary and sufficient solvability condition and a parameterization of all stabilizing solutions, which is affine on a single $RH^\infty$ parameter. The latter enables one to reduce a given one-sided model matching setup to a similar, in terms of the structure and complexity, setting with stable data, i.e., to sort out the stability constraints without changing the problem structure. This property of the solution might be important in various one-sided optimization problems and was exploited in [20, 21] to solve a one-sided $RH^\infty$ optimization problem with preview and asymptotic behavior constraints. Many problems of interest, however, like measured disturbance rejection
or sampled signal reconstruction, cannot be reduced to the one-sided setting. This motivates our current study, in which we aim at extending the one-sided model matching stabilization procedure of [20, 21] to the general two-sided setting.

To the best of our knowledge, SP in its present formulation has not been studied in the literature yet. Nonetheless, a large number of results dealing with the stabilization of various feedback problems with unstable weights are available. As already mentioned above, such problems can be reduced to the model matching stabilization. Thus, the analysis of problems related to SP can be traced back to at least the ‘70s, when the so called regulation problem with internal stability (RPIS) was studied by several authors. For example, in [2, 4, 5], and later on in [25, 11], solutions to different versions of RPIS, intimately related to \( \text{SP}_p \), were derived in terms of bilateral Diophantine equations (BDEs). These works, however, were restricted to problems with one-sided instabilities. They also did not provide numerically tractable solution formulae, since during that time no efficient methods for solving BDEs where available. In [15], Diophantine equations were used to solve an \( H^2 \) problem with non-proper weights, which is also an instability. In [26, 27], RPIS was treated using the geometric approach. The results obtained with this method, however, do not appear to provide a parameterization of all stabilizing solutions. In [7, 8], and later on in [3], the robust servomechanism problem, also related to a special case of \( \text{SP}_p \), was considered. Necessary and sufficient solvability conditions and a characterization of all minimal-order solutions were derived. Moreover, Davison’s approach enables one to analyze the structure of the minimal-order solution and its robustness. Still, it does not provide a complete parameterization of all solutions and is restricted to a specific problem structure. Another line of research [1, 6, 28] was based on the use of the Kronecker product formalism to reduce \( \text{SP}_p \) to an equivalent, albeit higher dimensional, \( \text{SP}_o \). This, however, blurs the structure of the problem and results in cumbersome parameterizations. Some simplifications were proposed in these papers, but these simplifications rely on non-transparent simplifying assumptions and result in solutions involving not readily tractable BDEs. Moore and Tomizuka [22] proposed a complete solution of an \( \text{SP}_p \)-related problem using a Smith-McMillan form of the denominator of \( G_2 \). The high computational cost of the Smith-McMillan form and a complicated (block-diagonal) form of the resulting free parameter still render this solution considerably more complicated and less suitable for further analysis comparing to the \( \text{SP}_o \) solution in [21]. Finally, [16, 17, 18] present a complete solution to a very general stabilization problem associated with the setting depicted on Fig. 1.2(b). Applying this result to SP yields a parameterization of all stabilizing solutions, given in terms of three independent parameters, namely, a stable but otherwise arbitrary transfer matrix and two static matrices, constrained by a set of algebraic equations. The presence of these three parameters having different kinds of constraints imposed on them complicates further use of this result. Summarizing, existing approaches result in solutions of the SP, which are not comparable to the solution of \( \text{SP}_o \). The goal of the current study is to find out whether a more convenient solution can be derived and under what conditions.

In this work we aim at mapping conceptual differences between the two-sided and one-sided stabilization problems and analyzing reasons by which the solution of SP might be considerably more complicated than that of \( \text{SP}_o \). The insight, gained from this analysis, enables us to come up with an intuitive simplifying assumption. We argue that the proposed assumption is not restrictive in most applications. At the same time, it enables us to establish a convenient parameterization of all stabilizing solutions, which is closely related to that proposed in [16, 17, 18], but is affine on a single parameter. Technically, the solution method proposed in this work relies on the relation between the model matching stabilization and BDEs. This relation has been observed in the past in the context of \( \text{SP}_p \)-related problems and in this work is established for the general two-sided case. We show that the general problem
is equivalent to the solution of two uncoupled BDEs and then exploit the recent solution of the rational BDE in [13] to derive explicit state-space formulae. Our result demonstrates that if the aforementioned simplifying assumption holds, the two-sided problem is similar to $\text{SP}_o$ in the sense that in both of them stability constraints can be sorted out without complicating the problem structure.

The paper is organized as follows. In Sections 2 and 3 the one-sided problem is briefly reviewed emphasizing conceptual differences between the two- and one-sided problems. In Section 4 the general two-sided problem is reduced to a pair of uncoupled BDEs and a simplifying assumption is introduced guaranteeing the existence of a convenient parameterization of all stabilizing solutions. Explicit state-space formulae are derived in Section 5. Section 6 is devoted to a brief illustrating example. Finally, some concluding remarks are available in Section 7. The main results of this paper are formulated in Theorems 4.3 and 4.4 in Section 4 and in Theorems 5.1 and 5.2 in Section 5.

**Notation.** The open left and the closed right halves of the complex plain $\mathbb{C}$ are denoted by $\mathbb{C}^-$ and $\mathbb{C}^+$ respectively. For any left-invertible $A \in \mathbb{R}^{n \times m}$, the matrices $A^+ \in \mathbb{R}^{n \times n}$ and $A^\perp \in \mathbb{R}^{(n-m) \times n}$ denote a pseudo inverse of $A$ and its complement satisfying

\[
\begin{bmatrix}
A^+ \\
A^\perp
\end{bmatrix} A = \begin{bmatrix}
I \\
0
\end{bmatrix}, \quad \det \begin{bmatrix}
A^+ \\
A^\perp
\end{bmatrix} \not= 0.
\]

Similarly, if $A$ is right-invertible, $A^+ \in \mathbb{R}^{m \times n}$ and $A^\perp \in \mathbb{R}^{m \times (m-n)}$ satisfy

\[
A \begin{bmatrix}
A^+ & A^\perp
\end{bmatrix} = \begin{bmatrix}
I & 0
\end{bmatrix}, \quad \det \begin{bmatrix}
A^+ & A^\perp
\end{bmatrix} \not= 0.
\]

A proper transfer matrix $G(s)$ is said to be left (right) invertible if its left (right) inverse exists and is proper. In the rational case, the left / right invertibility of $G(s)$ is equivalent to that of $G(\infty)$. The set of rational transfer matrices having all their poles in the open left half plane is denoted by $\text{RH}^\infty$. The left and the right coprime factorizations are abbreviated as lcf and rcf, respectively. Doubly coprime factorizations for each of the transfer matrices involved in $\text{SP}$ are denoted by

\[
G_i = N_i M_i^{-1} = M_i^{-1} N_i, \quad (1.2a)
\]

\[
\begin{bmatrix}
X_i & Y_i \\
-N_i & M_i
\end{bmatrix} \begin{bmatrix}
M_i & -Y_i \\
N_i & X_i
\end{bmatrix} = \begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}, \quad (1.2b)
\]

for $i = 1, 2, 3$. The sets of all poles and zeros of a rational $G_i(s)$ are denoted by $\mathcal{P}_i$ and $\mathcal{Z}_i$, respectively. The notion of *bilateral Diophantine equation* (BDE) refers to the equation of the form $MX + YN = P$, where $M, N, P \in \text{RH}^\infty$ are given and $X, Y \in \text{RH}^\infty$ are to be found. The class of BDEs with $P = I$ is referred to as the *skew-prime* equation and the pair $X, Y \in \text{RH}^\infty$ satisfying this equation is said to be *externally skew-prime*.

**2. Preliminary: the one-sided problem.** To gain a first insight into the model matching stabilization problem, let us start with reviewing its simple one-sided version, $\text{SP}_o$, with

\[
T(s) = G_1(s) - K(s)G_2(s). \quad (2.1)
\]

Although the solution of $\text{SP}_o$ is currently well understood, the available proofs, like those in [20, 21], are not readily extendible to the general two-sided setting. We thus present below an alternative derivation, which highlights stabilization mechanisms and clarifies (see the next section) challenges that one faces in the general, two-sided case.

The first thing to notice about $\text{SP}_o$ is that (2.1) can be stabilized only by canceling all unstable poles of $G_1$ using instabilities in $KG_2$. As $K$ is constrained to be stable, only unstable
The solvability condition of Lemma 2.1 will be referred to as the containment condition because it requires that all unstable modes of $G$ are “contained” in $G$, from the right $^1$.

The particular solution of $\text{SP}_o$ provided by Lemma 2.1 is not unique and our next step is to characterize the other solutions. To this end, let $K_o$ be any given solution of $\text{SP}_o$ and consider a candidate solution of a form $K = K_o + K_\Delta$, for some stable transfer matrix $K_\Delta$.

Substituting this expression into (2.1) yields
\[
T = G_1 - (K_o + K_\Delta)G_2 = (G_1 - K_oG_2) - K_\Delta G_2.
\]

Since $G_1 - K_oG_2 \in RH^\infty$, the only source of instability in the expression above is $G_2$ in the last term. This suggests that once $K_\Delta$ cancels all instabilities of $G_2$, i.e., once it can be written as $K_\Delta = QM_2$ for some $Q \in RH^\infty$, the proposed candidate indeed solves the problem. The following lemma proves that this form of $K_\Delta$ is also necessary for stabilizing $T$, thus providing a complete parameterization of all solutions to $\text{SP}_o$ given a particular one:

**Lemma 2.2.** Assume that $\text{SP}_o$ is solvable and let $K_o$ be its particular solution. Then all solutions of this problem can be characterized as
\[
K = K_o + Q\tilde{M}_2,
\]
where $Q \in RH^\infty$ but otherwise arbitrary.

*Proof.* Define $T_o := G_1 - K_oG_2 \in RH^\infty$. Substituting (2.2) into (2.1) yields
\[
T = G_1 - (K_o + Q\tilde{M}_2)\tilde{M}_2^{-1}\tilde{N}_2 = T_o - Q\tilde{N}_2 \in RH^\infty,
\]
i.e., any $K$ given by (2.2) indeed solves the problem. To complete the proof we need to show that the proposed parameterization covers all solutions. To this end, let $K_a \in RH^\infty$ be a solution of $\text{SP}_o$ and define $K_\Delta := K_a - K_o \in RH^\infty$ and
\[
T_\Delta := G_1 - K_aG_2 - T_o = K_\Delta G_2 \in RH^\infty
\]

Then $T_\Delta\tilde{Y}_2 = K_\Delta\tilde{M}_2^{-1}\tilde{N}_2\tilde{Y}_2 = K_\Delta\tilde{M}_2^{-1}(I - \tilde{M}_2\tilde{X}_2)$, which can be rewritten as $K_\Delta = (T_\Delta\tilde{Y}_2 + K_\Delta\tilde{X}_2)\tilde{M}_2$. Thus, $K_a$ is of the form (2.2) with $Q = T_\Delta\tilde{Y}_2 + K_\Delta\tilde{X}_2 \in RH^\infty$, which completes the proof.  

$^1$See [23] for details on the “containment” notion.
The lemma above shows that the set of all stabilizing solutions of the one-sided problem can be characterized by an affine parameterization in a single stable but otherwise arbitrary parameter. Combining this with the result of Lemma 2.1 leads to the complete solution of \( \text{SP}_o \), formulated below.

**Theorem 2.3.** \( \text{SP}_o \) is solvable iff \( G_1M_2 \in RH^\infty \). If this condition holds, then all solutions of \( \text{SP}_o \) and all corresponding stabilized \( T 's \) can be characterized as

\[
K = G_1M_2Y_2 + Q\hat{M}_2 \quad \text{and} \quad T = G_1M_2X_2 - Q\hat{N}_2,
\]

respectively, where \( Q \in RH^\infty \) but otherwise arbitrary.

Note that the dependence of feasible stable \( T 's \) on the parameter \( Q \) described by (2.3) has the same structure as the original dependence of \( T \) on \( K \) in (2.1). Moreover, it can be shown [20, 12] that the state-space realization of \( [G_1M_2X_2]_N \) associated with the resulting stabilized model-matching setup can always be obtained of the same order as that of the original composite system \( [G_1] \). Thus, Theorem 2.3 enables us to reformulate the original one-sided model-matching problem as a similar problem with stable data. This is a key property of the \( \text{SP}_o \) solution, which plays an important role in various one-sided optimization problems with asymptotic behavior constraints, see [20, 21]. Our main goal in this work is to extend the result of Theorem 2.3 to the general two-sided setting.

3. **Why is the extension to the two-sided problem nontrivial.** Having revised the one-sided problem, our next step is to understand why there is no immediate extension of its solution to the general two-sided setting. Most of the discussion in this section will be restricted to the problem with one-sided instabilities, \( \text{SP}_p \). This is a simplified version of \( \text{SP} \), which, at the same time, is sufficiently rich to capture the essence of the problem.

At the first sight \( \text{SP}_p \) looks very similar to \( \text{SP}_o \). Indeed, because \( G_3 \) in this problem is stable, it does not affect the containment requirement and one would expect that the problem can be resolved by the same arguments as \( \text{SP}_o \). This, however, is wrong. The introduction of \( G_3 \neq I \) into the problem substantially complicates it as explained below.

It is intuitively clear that, similarly to the one-sided problem, \( \text{SP}_p \) is solvable only if all instabilities of \( G_1 \) are contained in \( G_2 \). Indeed, (1.1) can be stabilized only by canceling all unstable poles of \( G_1 \) using instabilities of \( G_3KG_2 \). Since both \( K \) and \( G_3 \) are stable, only unstable poles of \( G_2 \) can be exploited toward this end. The following result can be formulated:

**Proposition 3.1.** \( \text{SP}_p \) is solvable only if \( G_1M_2 \in RH^\infty \).

*Proof.* Assume that the problem is solvable, i.e., there is \( K_p \in RH^\infty \) such that \( T_p = G_1G_3K_pG_2 \in RH^\infty \) as well. Post-multiplying this equation by \( M_2 \), we obtain that \( T_pM_2 = G_1M_2 - G_3K_pN_2 \) and, therefore, \( G_1M_2 = T_pM_2 + G_3K_pN_2 \in RH^\infty \), which is what we need.

The analogy with the \( \text{SP}_o \) solution (Lemma 2.1) is not complete though. Due to the presence of \( G_3 \), the containment condition in Proposition 3.1 is no longer sufficient for solvability. Just think of the problem in which \( G_3(s) = 0 \). In this case, \( G_3 \) completely blocks the lower branch on Fig. 1.1, rendering any problem with unstable \( G_1 \) unsolvable even if the containment condition holds. To understand mechanisms by which \( G_3 \) can interfere the stabilization, we need to consider some geometric properties of rational transfer matrices. In particular, we are going to use the notions of normal and singular null spaces, for which the reader is referred to Appendix A.

Let \( s_p \in \mathbb{C}^+ \) be an unstable pole of \( G_1 \) and let \( P^\text{out}_1 \) be its output direction. According to Proposition A.2, given a constant vector \( v \) of an appropriate dimension, \( v'G_1 \) has a pole at \( s_p \) iff \( v \in P^\text{out}_1 \). Denote the output null-space of \( G_3 \) at \( s_p \) by \( N^\text{out}_3 = \text{Ker} \ G_3(s_p)' \). We say that \( G_3 \) blocks access to an unstable pole \( s_p \) of \( G_1 \) if \( P^\text{out}_1 \not\subset N^\text{out}_3 \) or, equivalently, if there is \( v_h \in N^\text{out}_3 \), which is not orthogonal to \( P^\text{out}_1 \). To see the meaning of this definition, note that...
once such $v_b$ exists, $v'_bG_1$ has a pole at $p$ and, at the same time, $v'_bG_3(s_p) = 0$. As a result, pre-multiplying (1.1) by $v'_b$ and considering the resulting equality at $s \rightarrow s_p$ yields

$$
\lim_{s \rightarrow s_p} (v'_bT(s)) = \lim_{s \rightarrow s_p} \left(v'_bG_4(s)\right) - \lim_{s \rightarrow s_p} \left(v'_bG_3(s)K(s)G_2(s)\right).
$$

This shows that blocking occurs when the output null-space of $G_3(s_p)$ tends to prevent some instability of $G_1$ to be reached by $K$ and thus impedes the stabilization of $T$. For a better understanding of the nature of blocking, let us take a closer look at the output null-space of $G_3$ at the point $s_p$. Two different types of blocking can be recognized.

The first one—we call it the geometric blocking—appears if the stabilization is impeded by the normal null-space of $G_3$ denoted by $\mathcal{N}^\text{out}_3$, namely, if $\mathcal{P}^\text{out}_1 \not\subseteq \mathcal{N}^\text{out}_3$. An important property of the geometric blocking is that it inevitably renders the stabilization problem unsolvable. To see this, assume that geometric blocking occurs and choose $v_b \in \mathcal{N}^\text{out}_3$ such that $v_b \not\in \mathcal{P}^\text{out}_1$. According to Proposition A.6, there always exist $G_3^+$ and $G_3^-$ such that

$$\hat{T}_3 G_3 = \begin{bmatrix} G_3^+ & G_3^- \\ 0 & 0 \end{bmatrix}, \quad \text{where } \hat{T}_3 = \begin{bmatrix} G_3^+ \\ G_3^- \end{bmatrix} \text{ is bi-stable and } \hat{G}_3 \text{ is invertible.}
$$

By Proposition A.7. $v_b \in \text{Im}(G_3^+(s_p))$. This, together with the fact that $v'_bG_1(s)$ has a pole at $s_p \in \mathbb{C}^+$, implies that $G_3^+G_1$ has a pole at this point too and, as a result, is unstable. At this point, pre-multiplying $T$ by a bi-stable $\hat{T}_3$ yields an equivalent problem of stabilizing

$$\hat{T}_3 T = \begin{bmatrix} G_3^+ & G_3^- \\ G_3^+ & G_3^- \end{bmatrix} - \begin{bmatrix} \hat{G}_3 \\ 0 \end{bmatrix} KG_2,$$

which is unsolvable because its unstable lower block does not depend on the design parameter.

To illustrate the notion of geometric blocking, consider the problem of stabilizing

$$T(s) = \begin{bmatrix} 1 \\ \frac{1}{s} \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{2}{s+1} \\ \frac{1}{s+2} \\ 1 \end{bmatrix} K \begin{bmatrix} \frac{1}{s+2} \\ 1 \end{bmatrix}. \quad (3.1)
$$

In this problem $G_3$ contains a single unstable pole $s_p = 0$. Its output direction and the output null-space of $G_3(0)$ are given by

$$\mathcal{P}^\text{out}_1 = \text{span} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad \mathcal{N}^\text{out}_3 = \text{span} \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right).$$

Obviously, $\mathcal{P}^\text{out}_1 \not\subseteq \mathcal{N}^\text{out}_3$, which indicates that $G_3$ blocks access to the unstable pole. Moreover, since $s_p = 0$ is not a transmission zero of $G_3$, we have that $\mathcal{N}^\text{out}_3 = \mathcal{N}^\text{out}_3$. This implies that $\mathcal{P}^\text{out}_1 \not\subseteq \mathcal{N}^\text{out}_3$, showing that blocking in this example is geometric. Thus, although the considered problem satisfies the containment condition, the presence of blocking renders it not stabilizable. This can be easily verified by pre-multiplying (3.1) by the following bistable transfer function:

$$\hat{T} = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{s+2} \end{bmatrix},$$

which leads to an equivalent problem of stabilizing

$$\hat{T} T = \begin{bmatrix} \frac{1}{s+1} & 0 \\ -\frac{1}{s+2} & -1 \end{bmatrix} - \begin{bmatrix} \frac{s+2}{s+1} \\ \frac{2}{s+2} \end{bmatrix} K \begin{bmatrix} \frac{1}{s+2} \\ 1 \end{bmatrix}. \quad (3.2)$$
Clearly, the second row in the expression above is independent of $K$ and cannot be stabilized. This implies that the considered problem is unsolvable. Moreover, it can be easily inferred from (3.2) that (in)solvability of the problem does not depend on $G_2$. In other words, due to the presence of geometric blocking, any two-sided problem with $G_1$ and $G_3$ as in (3.1) is not stabilizable.

The second situation—we call it the analytical blocking—appears if $\mathcal{P}^{\text{out}}_1 \perp \mathcal{N}^{\text{out}}_3$ while $\mathcal{P}^{\text{out}}_1 \perp \mathcal{N}^{\text{out}}_3$. It corresponds to the case when the normal output null-space of $G_3(s_p)$ does not interfere stabilization and the blocking occurs due to a zero of $G_3$ at $s_p$. As an illustrative example, consider the problem of stabilizing

$$T = \begin{bmatrix} \frac{i}{s^2} & 0 & 0 \\ \frac{i}{s} & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{i}{s} & 1 \\ \frac{i}{s^2} & \frac{i}{s(s+1)} \\ 0 & 1 \end{bmatrix} K \begin{bmatrix} \frac{i}{s} & 1 \end{bmatrix}. \quad (3.3)$$

In this problem $G_1$ has a single unstable pole at $s = 0$, its output direction is $\mathcal{P}^{\text{out}}_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. The output null-space of $G_3$ at $s = 0$ is the entire $\mathbb{R}^2$ and the normal output null-space of $G_3$ at this point is given by $\mathcal{N}^{\text{out}}_3 = \text{Im}(G_3^+(0)) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. It is readily seen that $\mathcal{P}^{\text{out}}_1 \perp \mathcal{N}^{\text{out}}_3$ and $\mathcal{P}^{\text{out}}_1 \perp \mathcal{N}^{\text{out}}_3$, indicating the existence of analytical blocking. To see this, just note that the $(1, 1)$ element of $T$, $\frac{i}{s} - K$, is clearly not stabilizable.

An important difference between the geometric and analytical blockings, however, is that the latter does not necessarily prevent stabilization. Stabilization problems with analytical blocking might still be solvable if $G_2$ has enough extra poles at $s = s_p$, so that the zeros of $G_3$ responsible for analytical blocking of this pole can be canceled. To illustrate this, replace $G_2$ in (3.3) with

$$T = \begin{bmatrix} \frac{i}{s^2} & 0 & 0 \\ \frac{i}{s} & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{i}{s} & 1 \\ \frac{i}{s^2} + \frac{i}{s(s+1)} \\ 0 & 1 \end{bmatrix} K \begin{bmatrix} \frac{i}{s} & 1 \end{bmatrix}. \quad (3.4)$$

Now the problem is solvable (e.g., by $K = 1$) even though an analytical blocking occurs. The reason for this is that $G_2$ contains two integrators instead of one. The first integrator is needed to satisfy the containment condition, while the second one “unlocks” the blocking.

The discussion above gives an intuitive explanation of the fact that in two-sided problems the containment condition is not sufficient for solvability. This, however, is not the only difference between the two- and one-sided settings. Let us proceed to the characterization of all stabilizing solutions. Following the reasoning of the one-sided solution and taking into account the stability of $G_3$, it might appear natural to consider a candidate parameterization of the form

$$K = K_p + Q \tilde{M}_2, \quad Q \in RH^\infty \quad (3.5)$$

where $K_p$ is a particular solution of $\text{SP}_p$. The following result can be formulated:

**Proposition 3.2.** Assume that $\text{SP}_p$ is solvable and let $K_p$ be its particular solution. Then any $K$ given by (3.5) is a solution of $\text{SP}_p$. 

Proof. Since $K_p$ is a particular solution of the problem, $T_p := G_1 - G_3 K_p G_2 \in R H^\infty$. Substituting (3.5) into (1.1) yields

$$T = G_1 - G_3 (K_p + Q \tilde{M}_2) \tilde{M}_2^{-1} \tilde{N}_2 - G_3 Q \tilde{N}_2 \in R H^\infty,$$

which completes the proof.

An important difference between the result of Proposition 3.2 and the corresponding result for the $\text{SP}_p$ solution (Lemma 2.2) is that the parameterization provided by (3.5) is not necessarily complete. Namely, it does not necessarily cover all stabilizing solutions of $\text{SP}_p$. To understand the reason of this, note that in the one-sided problem the completeness of the affine parameterization (2.2) essentially relies on the fact that

$$KG_2 \in R H^\infty \Rightarrow \exists Q \in R H^\infty \text{ such that } K = Q \tilde{M}_2,$$

or, equivalently, on the fact that the only way to make $KG_2$ stable is by canceling all instabilities of $G_2$ by zeros of $K$. This is no longer true in the $\text{SP}_p$ case, i.e.,

$$G_3 KG_2 \in R H^\infty \Rightarrow \exists Q \in R H^\infty \text{ such that } K = Q \tilde{M}_2$$

for a general $G_3 \in R H^\infty$. Indeed, it might happen that not all unstable poles of $G_2$ need to be canceled by $K$ because some of them can be canceled by zeros of $G_3$. This can be illustrated by the problem of stabilizing

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{s}{s+1} \end{bmatrix} K \begin{bmatrix} 1 \\ \frac{s}{s+1} \end{bmatrix}. \tag{3.7}$$

Because $G_1$ in this problem is stable, the trivial particular solution $K_p(s) = 0$ can be chosen. Substituting this $K_p$ into (3.5) and choosing $\tilde{M}_2 = \frac{1}{s+1}$, we end up with the parametrization $K = Q \frac{s}{s+1}$. This effectively parametrizes all $K$’s canceling the integrator in $G_2$. This cancellation, however, is not necessary for the stabilization because the integrator in $G_2$ is anyway canceled by the zero of $G_3$ at the origin. Consequently, the parameterization above does not cover all stabilizing solutions. This can be clearly seen by noting that

$$T = \begin{bmatrix} 1 & 0 \\ K \frac{s}{s+1} & K \frac{s}{s+1} \end{bmatrix}$$

is stable for every $K \in R H^\infty$.

Continuing the discussion in the previous paragraph, it can be shown that the set of all solutions of $\text{SP}_p$ cannot in general be characterized by any affine parameterization given in terms of a single stable but otherwise arbitrary parameter. To see this, it is sufficient to consider the problem of stabilizing

$$T(s) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & \frac{s}{s+1} \end{bmatrix} K(s) \begin{bmatrix} \frac{s+1}{s} & 0 \\ \frac{s}{s+1} K_{11}(s) & K_{12}(s) \\ \frac{s}{s+1} K_{21}(s) & K_{22}(s) \end{bmatrix}. \tag{3.7}$$

Clearly, $K$ is a stabilizing solution of this problem iff its $(2, 2)$ component has a zero at the origin. It can be verified, see [13, Example 2] for a similar problem, that such $K$’s cannot be characterized by $A(s) + Q(s)C(s)$ for some given $A, C \in R H^\infty$ and a stable parameter $Q(s)$ without imposing additional constraints on the latter.

Hitherto, all our arguments were restricted to problems with one-sided instabilities, $\text{SP}_p$. In the following few paragraphs we will briefly extend them to the general problem to gain some further insight into $\text{SP}$. First, let us address the containment issue and its relation to the
solvability of SP. Remember, that the only way to stabilize T is to cancel all unstable poles of \(G_1\) using instabilities of \(G_1K G_2\). In the general two-sided setting, this does not require that all unstable poles of \(G_1\) are contained in \(G_2\) because some of them might be contributed by \(G_3\). This observation leads to the following result, which extends the containment condition of Proposition 3.1 to the general setting by showing that SP is solvable only if all instabilities of \(G_1\) are contained either in \(G_2\) from the right or in \(G_3\) from the left.

**Proposition 3.3.** SP is solvable only if \(M_2 G_1 M_2 \in RH^\infty\).

Proof. Let the problem be solvable, i.e., a \(T_1 \in RH^\infty\) such that \(T_1 = G_1 - G_3 K_2 G_2 \in RH^\infty\) exists. Post- and pre-multiplying this equation by \(M_2\) and \(M_3\), respectively, we obtain that \(M_3 T_1 M_2 = M_3 G_1 M_2 - N_3 K_1 N_2\). This yields \(M_3 G_1 M_2 - M_3 T_1 M_2 + N_3 K_1 N_2 \in RH^\infty\) and thus completes the proof.

Similarly to \(SP_p\), the containment condition of Proposition 3.3 is not sufficient for the solvability of \(SP\) due to the possibility of blocking. This time, however, the blocking may occur due to the null spaces associated with both \(G_2\) and \(G_3\), see [12, §3.5.1] for more details. Using the same arguments as in the case of one-sided instabilities, it can be shown that in the general two-sided problem an affine parameterization of the form

\[
K = K_p + M_3 Q M_2, \quad Q \in RH^\infty
\]  

(3.8)

may not be complete.

At this point, the main differences between \(SP_p\) and \(SP\) can be perceived. In problems with one-sided instabilities the roles played by \(G_2\) and \(G_3\) are clearly separated: the former is responsible for the containment, while the latter is capable of blocking. In general problems, however, each of these transfer functions can facilitate stabilization by containing unstable poles of \(G_1\) and, yet, impede stabilization by blocking instabilities contained in the other.

Summarizing this section, we may list three aspects of two-sided problems that were not relevant in the one-sided case:

- geometric blocking by \(G_2\) and \(G_3\);
- analytical blocking by \(G_2\) and \(G_3\);
- incompleteness of the affine parameterization (3.8).

These properties render the extension of the one-sided solution to \(SP\) nontrivial and constitute the main challenge throughout this work. In the next section we will show that these issues can be formalized by bilateral Diophantine equations, which later on will be used as a tool for finding an explicit state-space solution of the two-sided problem.

**4. Solution in terms of bilateral Diophantine equations.** As we already mentioned, BDEs play an important role in the current work. To understand relations between BDEs and the two-sided model matching stabilization better, let us first focus on a very special case of \(SP_p\) in which \(G_1\) and \(G_2\) coincide and are invertible and minimum-phase. Namely, consider the problem of stabilizing

\[
T = M^{-1} - G_3 K M^{-1},
\]  

(4.1)

where \(M, G_3 \in RH^\infty\). This simplified setup will hereafter be referred to as \(SP_s\). It often appears in tracking problems, in which the reference model equals identity. A problem of this type was considered, for example, in [25].

It is easy to see that \(SP_s\) always satisfies the containment condition. Moreover, the one-sided version of this problem (obtained by assuming \(G_3 = I\)) is a trivial \(SP_o\), all solutions of which can be parameterized as \(K = I + QM\). This suggests that by considering \(SP_s\) we focus on the distinctive aspects of the two-sided problem outlined in the previous section. The following result indicates an immediate relation between \(SP_s\) and the notion of skew-primeness.
**Proposition 4.1.** \( \mathbf{SP}_s \) is solvable iff the pair \((G_3, M)\) is externally skew-prime, i.e., there are \( S, W \in RH^\infty \) satisfying the BDE

\[
G_3 S + WM = I. \tag{4.2}
\]

If this condition holds, then all stabilizing solutions of \( \mathbf{SP}_s \) and all resulting stable \( T \)'s can be characterized as all \( S \) and \( W \) that satisfy (4.2), respectively.

**Proof.** Follows immediately by post-multiplying (4.1) by \( M \).

Our next step will be to extend the result of Proposition 4.1 to the more general problem with one-sided instabilities. The following lemma shows that this can be done using a BDE of a more general form, the right hand side of which is different from the identity matrix. In principle, the result of the lemma below is known for at least three decades, see [4], and is just restated using the current notations. In addition, an alternative proof is presented, which uses somewhat simpler arguments than those in [4].

**Lemma 4.2.** \( \mathbf{SP}_p \) is solvable iff

1. \( G_1 M_2 \in RH^\infty \),
2. there are \( U, V \in RH^\infty \) such that

\[
G_3 U + V \tilde{M}_2 = G_1 M_2 Y_2. \tag{4.3}
\]

If these conditions hold, then all solutions of \( \mathbf{SP}_p \) and all resulting stable \( T \)'s can be characterized as

\[
K = U \quad \text{and} \quad T = G_1 M_2 X_2 + V \tilde{N}_2,
\]

where \( U \) and \( V \) are arbitrary solutions of BDE (4.3).

**Proof.** To prove sufficiency, substitute \( K = U \) into (1.1) to obtain

\[
T = G_1 - G_3 U G_2 = G_1 - (G_1 M_2 Y_2 - V \tilde{M}_2)G_2 = G_1 M_2 X_2 + V \tilde{N}_2 \in RH^\infty.
\]

The necessity of the first condition was established in Proposition 3.1. To prove the necessity of the second condition, let \( K_p \in RH^\infty \) be a solution of \( \mathbf{SP}_p \) and define \( T_p := G_1 - G_3 K_p G_2 \in RH^\infty \). Post-multiplying the last equality by \( \tilde{Y}_2 \tilde{M}_2 \) yields

\[
T_p \tilde{Y}_2 \tilde{M}_2 = G_1 \tilde{Y}_2 M_2 - G_3 K_p + G_3 K_p \tilde{X}_2 \tilde{M}_2.
\]

This, taking into account that \( \tilde{Y}_2 \tilde{M}_2 = M_2 Y_2 \) (see (1.2b)), leads to

\[
G_3 K_p + (T_p \tilde{Y}_2 - G_3 K_p \tilde{X}_2) \tilde{M}_2 = G_1 M_2 Y_2,
\]

which is of the form (4.3) with \( U = K_p \in RH^\infty \) and \( V = T_p \tilde{Y}_2 - G_3 K_p \tilde{X}_2 \in RH^\infty \). This proves both the necessity of the second condition and the completeness of the provided parameterizations.

The lemma above can be considered as a preliminary step for the solution of the general two-sided problem, which is formulated below and can be considered as one of the main results of this section.

**Theorem 4.3.** \( \mathbf{SP} \) is solvable iff

1. \( \tilde{M}_3 G_2 M_2 \in RH^\infty \),
2. There are \( U_2, V_2 \in RH^\infty \) satisfying the BDE

\[
N_3 U_2 + V_2 \tilde{M}_2 = \tilde{X}_3 \tilde{M}_3 G_1 M_2 Y_2. \tag{4.4}
\]
3. There are $U_3, V_3 \in RH^\infty$ satisfying the BDE

$$U_3 \tilde{N}_2 + M_3 V_2 = \tilde{Y}_3 \tilde{M}_3 G_1 M_2 X_2.$$  

(4.5)

If these conditions hold, then all solutions of SP and all resulting stable $T$’s can be characterized as

$$K = \tilde{Y}_3 \tilde{M}_3 G_1 M_2 Y_2 + M_3 U_2 + U_3 \tilde{M}_2,$$  

(4.6)

$$T = \tilde{X}_3 \tilde{M}_3 G_1 M_2 X_2 + V_2 \tilde{N}_2 + N_3 V_3,$$  

(4.7)

where $U_2, V_2, U_3$ and $V_3$ are arbitrary solutions of the BDEs (4.4) and (4.5).

Proof. To prove sufficiency, substitute the expression for stabilizing solutions given by (4.6) into the original problem (1.1). This yields

$$T = G_1 - G_3 \tilde{Y}_3 \tilde{M}_3 G_1 M_2 Y_2 G_2 + N_3 U_2 G_2 + G_3 U_3 \tilde{N}_2.$$  

(4.8)

Using BDEs (4.4) and (4.5), we have that

$$N_3 U_2 = \tilde{X}_3 \tilde{M}_3 G_1 M_2 Y_2 - V_2 \tilde{M}_2$$  

and, consequently,

$$N_3 U_2 G_2 = \tilde{X}_3 \tilde{M}_3 G_1 M_2 Y_2 G_2 - V_2 \tilde{N}_2$$  

and

$$G_3 U_3 \tilde{N}_2 = G_3 \tilde{Y}_3 \tilde{M}_3 G_1 M_2 X_2 - N_3 V_3.$$  

(4.9)

which, in turn, can be used to verify that

$$G_1 - G_3 \tilde{Y}_3 \tilde{M}_3 G_1 M_2 Y_2 G_2 - \tilde{X}_3 \tilde{M}_3 G_1 M_2 Y_2 G_2 - G_3 \tilde{Y}_3 \tilde{M}_3 G_1 M_2 X_2$$  

$$= \tilde{X}_3 \tilde{M}_3 G_1 M_2 X_2$$

At this point, the substitution of the last two displayed formulae into (4.8) yields

$$T = \tilde{X}_3 \tilde{M}_3 G_1 M_2 X_2 + V_2 \tilde{N}_2 + N_3 V_3 \in RH^\infty.$$  

This proves that any $K$ given by (4.6) constitutes a stabilizing solution and verifies the expression for feasible stable $T$’s given in (4.7).

Our next step is to prove the necessity of the solvability conditions. The necessity of the first one was proved in Proposition 3.3. To prove the necessity of the second condition, assume that the problem is solvable, i.e., that there is $K \in RH^\infty$ such that $T = G_1 - G_3 K G_2 \in RH^\infty$. This, in turn, implies that $\tilde{M}_3 T = \tilde{M}_3 G_1 - \tilde{N}_3 K G_2 \in RH^\infty$, i.e., that $K$ solves SP in which $G_1$ and $G_3$ are replaced with $\tilde{M}_3 G_1$ and $\tilde{N}_3$, respectively. According to Lemma 4.2, this implies that there exist $U_2, V_2 \in RH^\infty$ satisfying the BDE

$$\tilde{N}_3 U_2 + \tilde{V}_2 M_2 = \tilde{M}_3 G_1 M_2 Y_2.$$  

Pre-multiplying this equation by $\tilde{X}_3$ yields that BDE (4.4) holds for $U_2 = X_3 \tilde{U}_2$ and $V_2 = \tilde{X}_3 V_2$ and, hence, the second solvability condition is satisfied. Following similar arguments, we can see that $TM_2 = G_1 M_2 - G_3 K N_2 \in RH^\infty$, implying that there exist $U_3, V_3 \in RH^\infty$ satisfying the BDE

$$\tilde{U}_3 N_2 + M_3 \tilde{V}_2 = \tilde{Y}_3 M_3 G_1 M_2.$$  

(4.9)

Post-multiplying this equation by $X_2$ yields that BDE (4.5) holds for $U_3 = \tilde{U}_3 \tilde{X}_2$ and $V_3 = \tilde{V}_3 X_2$ and, hence, the third solvability condition is satisfied.
To complete the proof we now need to show that any stabilizing solution of SP can be written by (4.6). To this end, assume that the problem is solvable and let \( K^p \) be a particular solution satisfying \( T^p = G_1 - G_3 K^p G_2 \in RH^\infty \). Then, according to the discussion above and the result of Lemma 4.2, there are particular solutions of BDE (4.9), \( \tilde{U}_p^p \) and \( \tilde{V}_p^p \), such that

\[
K^p = \tilde{U}_p^p.
\]

Moreover, it follows by similar arguments that there is a particular solution of BDE (4.5), \( U_p^2 \), such that

\[
U_p^2 = \tilde{U}_p^p \tilde{X}_2 = K^p \tilde{X}_2.
\]  

(4.10)

Remember that our aim is to show that \( K^p \) can be represented as

\[
K^p = \tilde{Y}_3 \tilde{M}_3 G_1 M_2 Y_2 + M_3 U_2 + U_3 \tilde{M}_2
\]

for some \( U_2 \) and \( U_3 \) solving BDEs (4.4) and (4.5). Let us choose \( U_3 = U_p^p \), which is a particular solution of BDE (4.5) associated with \( K^p \). The goal now is to show that the resulting \( U_2 \), namely,

\[
U_2 = M_2^{-1}(K^p - \tilde{Y}_3 \tilde{M}_3 G_1 M_2 Y_2 - U_p^p \tilde{M}_2)
\]  

(4.11)

satisfies BDE (4.4). As the first step, we show that \( U_2 \) above is stable. Clearly, \( M_3 U_2 \in RH^\infty \) and we need to show only that \( N_3 U_2 \in RH^\infty \). Pre-multiplying (4.11) by \( N_3 \) yields

\[
N_3 U_2 = G_3 K^p - N_3 Y_3 \tilde{M}_3 G_1 M_2 Y_2 - G_3 U_p^p \tilde{M}_2.
\]

By (4.10), this implies that

\[
N_3 U_2 = G_3 K^p - N_3 Y_3 \tilde{M}_3 G_1 M_2 Y_2 - G_3 K^p \tilde{X}_2 \tilde{M}_2
\]

\[
= G_3 K^p N_2 Y_2 - N_3 Y_3 \tilde{M}_3 G_1 M_2 Y_2 = G_3 K^p G_2 M_2 Y_2 - N_3 Y_3 \tilde{M}_3 G_1 M_2 Y_2
\]

\[
= (G_1 - T^p) M_2 Y_2 - N_3 Y_3 \tilde{M}_3 G_1 M_2 Y_2
\]

\[
= -T^p M_2 Y_2 + \tilde{X}_3 \tilde{M}_3 G_1 M_2 Y_2 \in RH^\infty.
\]

Finally, we need to show that \( U_2 \) satisfies (4.4). To this end, it suffices to show that \( V_2 \) defined by this equation is stable, namely, we need to show that \( V_2 = (\tilde{X}_3 \tilde{M}_3 G_1 M_2 Y_2 - N_3 U_2) M_2^{-1} \) is stable for \( U_2 \) given by (4.11). Using the previously derived expression of \( N_3 U_2 \), we have:

\[
V_2 = (\tilde{X}_3 \tilde{M}_3 G_1 M_2 Y_2 + T^p M_2 Y_2 - \tilde{X}_3 \tilde{M}_3 G_1 M_2 Y_2) M_2^{-1} = T^p \tilde{Y}_2 \in RH^\infty,
\]

which completes the proof.

\[ \square \]

Remark 4.1. Although the scope of this work is limited to the rational case, the result of Theorem 4.3 is more general. In fact, it can be seen from the proof above that this result can be formulated for any \( G_1, G_2 \) and \( G_3 \) admitting doubly coprime factorizations. The result is then valid modulo the replacement of the space of real-rational transfer functions \( RH^\infty \) with \( H^\infty \).

The solvability conditions of Theorem 4.3 have clear intuitive interpretations. The first condition resolves the containment issue and is a natural extension of the solvability condition of Theorem 2.3. The last two conditions then rule out geometric and harming analytical blocking. The set of all stabilizing solutions of SP in Theorem 4.3 is then characterized in terms of all solutions of the BDEs. According to the discussion in Section 3, this set might not admit a convenient parameterization in terms of a single \( RH^\infty \) parameter. In the next subsection we will show that this difficulty can be circumvented by introducing a mild simplifying assumption, which is satisfied in many practical problems.
4.1. Simplified parameterizations. As discussed in Section 3, there are several potential complications in the general two-sided problem with respect to $\text{SP}_o$, which render the solution of the former more complicated. Some of this complications, however, do not show up generically. The reason is that in the vast majority of nonsingular problems unstable poles of $G_2$ do not coincide with zeros of $G_1$ and, similarly, unstable poles of $G_3$ do not coincide with zeros of $G_2$. Indeed, unstable poles in our problem typically originate from unstable weights with imaginary axis instabilities. Yet the presence of imaginary axis transmission zeros in $G_3$ or $G_2$ would render virtually any optimization problem based on these $G_3$ and $G_2$ singular. In fact, it is a standard assumption that neither $G_2$ nor $G_3$ has purely imaginary zeros. Motivated by this, we may assume that

$$\mathcal{A}_1: \exists_3 \cap \Psi_2 \subset C^- \text{ and } \exists_2 \cap \Psi_3 \subset C^-.$$  

We argue that this assumption is reasonable and causes virtually no loss of generality, while greatly simplifying the problem. To understand the impact of $\mathcal{A}_1$ on the problem, let us return to the list of three differences between $\text{SP}$ and $\text{SP}_o$, presented at the end of Section 3.

We begin with the second difference, which is concerned with potential analytical blockings. These analytical blockings may occur only if there is a coincidence between an unstable pole of $G_1$ and a zero of either $G_2$ or $G_3$. At the same time, the problem is solvable only if the containment condition is satisfied, i.e., only if all unstable poles of $G_1$ are contained either in $G_2$ or $G_3$. This observation, together with $\mathcal{A}_1$, imply that if the containment condition is satisfied, the coincidence between unstable poles of $G_1$ and zeros of $G_2$ and $G_3$ is impossible and no analytical blocking may occur. This renders the second issue of the list irrelevant.

It is even of greater importance to notice that, according to the discussion in Section 3, an affine parameterization (3.8) is incomplete only if there is a coincidence between unstable poles and zeros of $G_2$ and $G_3$. Obviously, $\mathcal{A}_1$ rules out this possibility, rendering also the third difference from the list irrelevant. One way to see this is by noting that $\mathcal{A}_1$ implies that the pairs $(N_3, M_2)$ and $(N_2, M_3)$ have no common unstable transmission zeros. Consequently, the BDEs (4.4) and (4.5) satisfy the condition of [13, Corollary 1] and admit affine parameterizations in terms of a single $RH^\infty$ parameter. This, in turn, results in simplified parameterizations of all $K$'s and $T$'s given by (4.6) and (4.7) and leads to the following result.

**Theorem 4.4.** Let $\mathcal{A}_1$ hold. Then $\text{SP}$ is solvable iff the conditions of Theorem 4.3 are satisfied and all resulting stable $T$'s can be characterized as

$$K = \tilde{Y}_3 \tilde{M}_3 G_1 M_2 Y_2 + M_3 U_2^p + U_3^p \tilde{M}_2 + M_3 Q \tilde{M}_2, \quad (4.12)$$

$$T = \tilde{X}_3 \tilde{M}_3 G_1 M_2 X_2 + V_2^p \tilde{N}_2 + N_3 V_3^p - N_3 Q \tilde{N}_2. \quad (4.13)$$

where $U_2^p$ and $V_2^p$ are any particular solutions of BDE (4.4), $U_3^p$ and $V_3^p$ are any particular solutions of BDE (4.5), and $Q \in RH^\infty$ but otherwise arbitrary.

**Proof.** As we have already mentioned, assumption $\mathcal{A}_1$ implies that the pairs $(N_3, \tilde{M}_2)$ and $(\tilde{N}_2, M_3)$ have no common unstable transmission zeros. Therefore, according to [13] the set of all solutions of BDE (4.4) can be given by

$$U_2 = U_2^p + Q_2 \tilde{M}_2 \quad \text{and} \quad V_2 = V_2^p + N_3 Q_2, \quad (4.13)$$

where $U_2^p$ and $V_2^p$ are some particular solutions of (4.4) and $Q_2 \in RH^\infty$ but otherwise arbitrary. Similarly, the set of all solutions of BDE (4.5) can be given by

$$U_3 = U_3^p + M_3 Q_3 \quad \text{and} \quad V_3 = V_3^p + Q_3 \tilde{N}_2.$$
where $U_s^p$ and $V_s^p$ are some particular solutions of (4.5) and $Q_3 \in RH^\infty$ but otherwise arbitrary. Substituting these parameterizations into (4.6), (4.7) yields

$$
K = \tilde{Y}_3 \tilde{M}_2 G_1 M_2 Y_2 + M_3 (U_s^p + Q_2) \tilde{M}_2 + (U_s^p + M_3 Q_3) \tilde{M}_2
$$

$$
= \tilde{Y}_3 \tilde{M}_2 G_1 M_2 Y_2 + M_3 U_s^p \tilde{M}_2 + M_3 (Q_2 + Q_3) \tilde{M}_2,$n

$$
T = \tilde{X}_3 \tilde{M}_2 G_1 M_2 X_2 + (V_s^p + N_3 Q_2) \tilde{N}_2 + N_3 (V_s^p + Q_2) \tilde{N}_2
$$

$$
= \tilde{X}_3 \tilde{M}_2 G_1 M_2 X_2 + V_s^p \tilde{N}_2 + N_3 V_s^p - N_3 (Q_2 + Q_3) \tilde{N}_2.
$$

The proof is now completed by denoting $Q = Q_2 + Q_3$. □

Theorem 4.4 can be considered as a nontrivial extension of Theorem 2.3 to the two-sided problem. It provides parameterizations of all stabilizing solutions and all stable $T$'s, which are affine in terms of a single $RH^\infty$ parameter. Consequently, the dependence of feasible stable $T$ on the parameter has the same structure as the original dependence of $T$ on $K$ in (1.1). Namely, Theorem 4.4 shows that if assumption $A_4$ holds, the SP is similar to its one-sided counterpart in a sense that also in the two-sided case stability constraints can be sorted out without altering the problem structure.

5. State-space formulae. In this section we use the result of [13] to present computationally efficient state-space version of the formulae of Theorems 4.3 and 4.4. To this end, consider the following composite system given by its minimal state-space realization

$$
G = \begin{bmatrix}
G_1 & G_3 \\
G_2 & 0
\end{bmatrix} = \begin{bmatrix}
A & B_1 & B_3 \\
C_1 & 0 & D_3 \\
C_2 & D_2 & 0
\end{bmatrix}.
$$

(5.1)

To simplify the exposition, throughout this section we assume that

$A_2$: $D_2 D_s^2 = I$.

$A_3$: $D_s^2 D_3 = I$.

These assumptions can be easily relaxed to the non-singularity of $D_2 D_s^2$ and $D_s^2 D_3$ by absorbing appropriate static matrices into $K$. The results of this section can be further extended to problems with singular $D_2$ and $D_3$, see [13, Remark 2].

Bringing in the canonical decomposition of the "A" matrix of (5.1) with respect to the second input and the second output yields an alternative representation of this system

$$
G = \begin{bmatrix}
A_3 & A_{12} & A_{13} & B_{11} & B_{12} \\
0 & A_{11} & A_{23} & B_{21} & 0 \\
0 & 0 & A_2 & B_{31} & 0 \\
C_{11} & C_{12} & C_{13} & 0 & D_3 \\
0 & 0 & C_{23} & D_2 & 0
\end{bmatrix},
$$

(5.2)

where the pair $(A_2, B_{12})$ is controllable and the pair $(A_2, C_{23})$ is observable. Without loss of generality, we may presume that the realization above has an additional structure:

$$
A_2 = \begin{bmatrix}
A_{23} & 0 \\
0 & A_{25}
\end{bmatrix}, \quad A_3 = \begin{bmatrix}
A_{33} & 0 \\
0 & A_{35}
\end{bmatrix}
$$

$$
A_{12} = \begin{bmatrix}
0 & \times
\end{bmatrix}, \quad A_{13} = \begin{bmatrix}
0 & \times \\
\times & 0
\end{bmatrix}, \quad A_{23} = \begin{bmatrix}
\times & 0
\end{bmatrix}.
$$

(5.3)

where $A_{23}$ and $A_{33}$ are Hurwitz, $A_{25}$ and $A_{35}$ are anti-stable, and "$\times$" stands for irrelevant blocks. Define also the matrices $E_2 = \begin{bmatrix} 0 \end{bmatrix}$ and $E_3 = \begin{bmatrix} 0 \end{bmatrix}$ with the dimensions such that

$$
A_2 E_2 = A_{25} E_2 \quad \text{and} \quad A_3 E_3 = E_3 A_{35}.
$$
and pick any $F_s$ and $L_s$ such that $A_3 + B_{12}F_sE_0$ and $A_2 + E_2L_sC_{23}$ are Hurwitz. Introduce the following pair of constrained Sylvester equations:

$$A_{3u}Z_2 - Z_2(A_2 - B_{31}D_3^+C_{23}) = E_0(A_{13} - B_{11}D_2^+C_{23}).$$ \hfill (5.4a)

$$E_{11}B_{11} + Z_2B_{31}) D_2^+ = 0.$$ \hfill (5.4b)

and

$$Z_3A_{2u} - (A_3 - B_{12}D_3^+C_{11})Z_3 = (A_{13} - B_{12}D_3^+C_{13})E_2.$$ \hfill (5.5a)

$$D_3^+(C_{13}E_2 + C_{11}Z_3) = 0.$$ \hfill (5.5b)

Once there exist $Z_2$ and $Z_3$ satisfying these equations, define

$$J_2 := \begin{bmatrix} E_3 & 0 & Z_2 \end{bmatrix} \quad \text{and} \quad J_3 := \begin{bmatrix} Z_3 \\ 0 \\ E_2 \end{bmatrix}.$$

where the partitioning corresponds to that of the “$A$”-matrix in (5.2). Define also

$$L_T := J_3L_s \quad \text{and} \quad F_T := F_sJ_2$$ \hfill (5.6)

and, finally,

$$L_K := \begin{bmatrix} -E_3(J_2B_1D_2^+ + Z_2E_2L_s) \\ 0 \\ E_2L_s \end{bmatrix},$$ \hfill (5.7a)

$$F_K := \begin{bmatrix} F_sE_3 & 0 & -(D_3^+C_1J_3 + F_sE_3Z_3)E_2^+ \\ \end{bmatrix}.$$ \hfill (5.7b)

where the partitioning corresponds to that in (5.2). Note that $L_K$ and $F_K$ defined above satisfy

$$J_2(L_K + B_1D_2^+) = 0 \quad \text{and} \quad (F_K + D_3^+C_1)J_3 = 0.$$ \hfill (5.8)

The main result of this section can now be formulated as follows:

**Theorem 5.1.** \textbf{SP} is solvable iff

1. $A_{11}$ is Hurwitz,
2. there exists $Z_2$ satisfying (5.4),
3. there exists $Z_3$ satisfying (5.5).

If these conditions hold, then all the stabilizing $K \in RH^\infty$ and all corresponding stabilized $T$’s can be characterized by

$$K = \tilde{K}_1 + \tilde{K}_3Q\tilde{K}_2 \quad \text{and} \quad T = \tilde{T}_1 - \tilde{T}_3Q\tilde{T}_2,$$ \hfill (5.9)

where $Q \in RH^\infty$ but otherwise arbitrary and

$$\begin{bmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{bmatrix} = \begin{bmatrix} A + B_2F_K + L_KC_2 & L_K \\ F_K \\ C_2 \end{bmatrix} \begin{bmatrix} L_K & B_2 \\ 0 & I \\ I & 0 \end{bmatrix}.$$ \hfill (5.10)

$$\begin{bmatrix} \tilde{T}_1 \\ \tilde{T}_2 \end{bmatrix} = \begin{bmatrix} A + B_2F_T + L_TC_2 & B_1 + L_TD_2 & B_2 \\ C_1 + D_3F_T \\ C_2 \end{bmatrix} \begin{bmatrix} B_1 + L_TD_2 & B_2 \\ 0 & D_3 \\ D_2 & 0 \end{bmatrix}.$$ \hfill (5.11)

with $L_T$, $F_T$, $L_K$, and $F_K$ are as defined in (5.6) and (5.7).
Proof. Follows by a direct application of the state-space solution of BDE from [13] to the result of Theorem 4.3, see [12] for details of a similar derivation.

Theorem 5.1 yields a complete solution of \( \text{SP} \) in terms of explicit state-space formulae. In particular, it reduces \( \text{SP} \) to the solution of a pair of numerically tractable constrained Sylvester equations (5.4) and (5.5). These Sylvester equations are state-space counterparts of the last two conditions of Theorem 4.3 and correspond to the blocking issues. The first condition above corresponds then to the containment condition of Theorem 4.3.

It is important to emphasize that \( Z_2 \) and \( Z_3 \) satisfying (5.4) and (5.5) are not necessarily unique. Consequently, the parameterizations in Theorem 5.1 are given in terms of three independent parameters over different domains: a stable but otherwise arbitrary transfer matrix \( Q \) and two matrices \( Z_2 \) and \( Z_3 \) restricted by the constrained Sylvester equations (5.4) and (5.5), respectively. As can be seen in the example below, the freedom existing in the choice of \( Z_2 \) and \( Z_3 \) cannot in general be absorbed into the “main parameter” \( Q \).

**EXAMPLE 5.1.** Consider the problem of stabilizing

\[
T(s) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \end{bmatrix} - \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & 1 \\ \end{bmatrix} K(s) \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{s+1} \\ \end{bmatrix}.
\]

A minimal state-space realization of \( G \) of the form (5.2) is then

\[
\begin{bmatrix} G_1(s) & G_3(s) \\ G_2(s) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{s} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{s+1} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\]

In this problem both \( A_{1i} \) and \( A_{2u} \) vanish, so that the first and third solvability conditions are irrelevant. In the verification of the second solvability condition equation, (5.4b) also vanishes, while (5.4a) reduces to

\[
0 \cdot Z_2 - Z_2 \cdot 0 = 0,
\]

which is obviously solvable. Hence, \( \text{SP} \) for this example is solvable. Because \( Z_2 \) is not constrained by the equations above, it can be chosen to be any real number, namely, \( Z_2 = p \in \mathbb{R} \). At this point, choosing \( F_K = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \) and noticing that in the considered problem \( E_2, L, J_3 \) and \( L_T \) vanish, we can construct

\[
L_K = \begin{bmatrix} 0 & -p \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad F_K = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Substituting these \( F_K \) and \( L_K \) into (5.10) yields

\[
\begin{bmatrix} \tilde{K}_1(s) & \tilde{K}_3(s) \\ \tilde{K}_2(s) & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -p & p & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.
\]
Thus, the set of all stabilizing solutions is characterized as

\[
K(s) = \begin{bmatrix}
0 & \frac{p}{1+s} \\
\frac{s}{1+s} & 0
\end{bmatrix} + \begin{bmatrix}
\frac{s}{1+s} & 0 \\
0 & 1
\end{bmatrix} Q(s) \begin{bmatrix}
1 & 0 \\
\frac{1}{1+s} & 1
\end{bmatrix}
\]

for any \( \tilde{Q} \in RH^\infty \) and \( p \in \mathbb{R} \). The presence of this \( p \) in the parametrization of all stabilizing \( K \)'s has a simple explanation. Indeed, it is readily seen that the original problem imposes interpolation constraints only on the \((1,1)\) entry of \( K \), namely, \( K_{11}(0) = 0 \). This is what the left multiplier of \( \tilde{Q} \) in the parametrization above does. This multiplier, however, constrains \( K_{12}(0) \) as well. To counteract this constraint, the \((1,2)\) element of \( K \) is then supplemented by an additional parameter, which renders \( K_{12}(0) \) arbitrary again. This clearly indicates that the degree of freedom available in \( p \) is necessary to cover all stabilizing solutions, i.e., \( p \) cannot be frozen.

The fact that \( Z_2 \) and \( Z_3 \) cannot in general be absorbed into the \( Q \) is not surprising, taking into account the discussion in Section 3. At the same time, the result of \S 4.1 suggests that these additional parameters can be omitted if the assumption \( A_1 \) holds. We thus conclude this section with an analysis of the impact of this assumption on the result of Theorem 5.1.

Let \( \tilde{L}_3 \) be any matrix such that all modes of \( A_3 - B_{13}D_2^+C_{11} \) detectable from \( D_2^+C_{11} \) are stabilized in \( A_3 - B_{13}D_2^+C_{11} - \tilde{L}_3D_2^+C_{11} \). Pre-multiplying (5.5b) by \( \tilde{L}_3 \) and adding the resulting equality to the Sylvester equation (5.5a) yields an alternative Sylvester equation

\[
Z_3A_{2a} - (A_3 - B_{12}D_2^+C_{11} - \tilde{L}_3D_2^+C_{11})Z_3 = (A_{13} - B_{12}D_2^+C_{11} - \tilde{L}_3D_2^+C_{11})E_2. \tag{5.5b'}
\]

Now, using (5.2),

\[
G_3 = \begin{bmatrix}
A_3 \\
C_{11} \\
B_{12} \\
D_3
\end{bmatrix}
\]

is a minimal state-space realization of \( G_3 \). By [13, Lem. 5] all eigenvalues of \( A_3 - B_{12}D_2^+C_{11} \) unobservable from \( D_2^+C_{11} \) and, therefore, all unstable eigenvalues of \( A_3 - B_{12}D_2^+C_{11} - \tilde{L}_3D_2^+C_{11} \) are transmission zeros of \( G_3 \). \( A_1 \) then guarantees that \( A_{2a} \) and \( A_3 - B_{12}D_2^+C_{11} - \tilde{L}_3D_2^+C_{11} \) have no common eigenvalues and, as a result, (5.5b') has a unique solution.

The second Sylvester equation can be treated in the same manner. Choose \( \tilde{L}_2 \) such that all modes of \( A_2 - \hat{B}_{31}D_2^+C_{23} \) detectable from \( C_{23}D_2^+ \) are stabilized in \( A_2 - \hat{B}_{31}D_2^+C_{23} - \hat{B}_{31}D_2^+\tilde{L}_2 \). Post-multiplying (5.4b) by \( \tilde{L}_2 \) and adding the resulting equality to the Sylvester equation (5.4a) yields an alternative Sylvester equation

\[
A_{3a}Z_2 - Z_2(A_3 - B_{31}D_2^+C_{23} - D_2^+B_{31}\tilde{L}_2) = E_1(A_{13} - B_{12}D_2^+C_{23} - D_2^+B_{11}\tilde{L}_2). \tag{5.4b'}
\]

which, similarly to (5.5b'), has a unique solution. This leads to the following result, which can be considered as the state-space counterpart of Theorem 4.4.

**Theorem 5.2.** Let \( SP \) be solvable and \( A_1 \) hold. Then \( Z_2 \) and \( Z_3 \) are uniquely defined by (5.4) and (5.5), respectively, and the parameterizations (5.9) are given in terms of a single parameter \( Q \in RH^\infty \).

It is worth mentioning that the discussion proceeding Theorem 5.2 not only proves its result, but also provides a convenient way of checking the second and third solvability conditions of Theorem 5.1. Indeed, if assumption \( A_1 \) holds, the Sylvester equations (5.5b') and (5.4b') are guaranteed to have unique solutions even if the problem is unsolvable. Therefore,
in this case the second and third solvability conditions can be validated by verifying that the
unique solutions of (5.5b) and (5.4b) satisfy the equalities (5.5a) and (5.4a), respectively.

The results of Theorems 5.1 and 5.2 conclude the main line of this work by providing a
nontrivial extension of the state-space solution of SPo, given in [20] to the two-sided setting.

5.1. Comparison with the results of [16, 17]. The formulae presented in Theorem 5.1
are related to the results derived by Liu and Mita in [16, 17] (the journal paper [18] also con-
tains a formulation of these results). It can be shown, by nontrivial algebraic manipulations
(see [12, §3.5.4] for the details), that Theorem 5.1 is a special case of [18, Lemmas 5 and
8] for G22 = 0. Note, however, that the solution presented in this paper was obtained using
a different approach. Liu and Mita address the problem directly in state space and appear
to use different arguments (some proofs in the conference papers [16, 17] are omitted). In
the current work, a frequency domain solution was derived first and then the final formulæ
were obtained using the state-space solution of BDE from [13]. Advantages of the proposed
approach and the resulting solution are as follows.

Transparency: The proposed solution procedure gives a new insight into two-sided stabil-
ization problems and their differences from the one-sided special case. In particu-
ar, it clarifies the reasons for having additional free parameters (Z2 and Z3) in the
parameterization of all stabilizing solutions and leads to an important and nearly not
restrictive assumption A1, which renders the parametrization compatible with the
Youla-Kučera parametrization. These conclusions are not readily deduced from the
solution of [16, 17].

Numerical simplicity: Sylvester equations (5.4) and (5.5) involved in the result-
ing state-space formulæ, which do not explicitly appear in [16, 17, 18], are convenient for
numerical implementation.

Extensibility: Although in this paper we discussed only the continuous-time rational case,
the frequency-domain results of Section 4 are readily extendible to discrete problems
and to a class of irrational transfer functions, admitting a coprime factorization. It is
unclear whether this applies to the solution of Liu and Mita.

It is worth emphasizing that the existence of an affine, numerically efficient, parame-
terization of all stabilizing K’s given in terms of a single RH∞ parameter may play a key
role in solving various optimization problems, see, e.g., [14]. We see in this one of the main
contributions of the present paper.

6. Illustrative example. In this section we present a simple example of feedforward
measured disturbance attenuation. It can be viewed as a motivation for the two-sided problem
studied in this paper. Consider the problem of controlling a camera installed on a rotating
platform depicted in Fig. 6.1. Assume that the angle between the camera and the platform is
regulated by an actuator, and our goal is to keep the direction of the camera constant (to the
point A) in spite of platform rotations. Denote the clockwise rotation angle of the platform
by θ and suppose that it can be measured. Suppose also that the measurement θm of θ is

![Fig. 6.1. Camera on a rotating platform](image_url)
corrupted with an additive noise $n$, i.e., that $\theta_m = \theta + n$. We assume that this measurement noise is highpass with no constant component (otherwise, there is no way to avoid an offset in the camera position) and contains a persistent harmonic component with a constant nonzero frequency $\omega_n$. Denote the counterclockwise angle between the camera and the platform by $\phi$. For the sake of simplicity, we neglect the actuator dynamics and consider $\phi$ as our control input.

The setup described above can be presented as the block diagram in Fig. 6.2, where $K$ is a feedforward controller. Here two exogenous inputs are the platform rotation angle $\theta$ and the measurement noise $n$ and two regulated outputs are the error in the camera positioning with respect to its target $e$ and the control signal $u = \phi$. Our goal here is to characterize all controllers $K$ guaranteeing the asymptotic rejection of the constant component in the platform rotation and the harmonic component in the measurement noise.

To this end, we use unstable weights in order to cast the problem as an input/output stabilization problem. The idea is presented in Fig. 6.3. One way to impose our asymptotic performance requirements would probably be to choose the following weights:

$$W_\theta = \frac{1}{s}, \quad W_n = \frac{s^2}{s^2 + \omega_n^2}, \quad W_e = 1, \quad \text{and} \quad W_u = \frac{\rho s}{s + a} \quad (6.1)$$

for any $a > 0$ and $\rho > 0$. These $W_\theta$ and $W_n$ may be thought of as generators of $\theta$ and $n$, respectively, and the differentiator in the (bi-proper) $W_u$ is required to prevent geometric blocking. The latter has a clear explanation: a constant platform rotation cannot be counteracted with $D_0$. This choice of weighting functions would result in a $\mathbf{SP}_p$ with a fourth-order $G$. Yet one of these poles, that of $W_u$, is actually not necessary. To see this, consider an alternative choice with

$$W_\theta = 1, \quad W_n = \frac{s^2}{s^2 + \omega_n^2}, \quad W_e = \frac{1}{s}, \quad \text{and} \quad W_u = \rho. \quad (6.2)$$

These weights still do the job. Indeed, the measurement noise generator, $W_n$, enforces $K$ to contain a notch at the frequency $\omega = \omega_n$ and the integrator in $W_e$ enforces the transfer function from $\theta$ to $e$ to have a zero at the origin (because $W_\theta$ has no such zeros). An advantage of (6.2)

$$W_\theta; 0, 0; W_n$$

$$1: 0, 0; 0$$

$$1, \quad \text{and} \quad W_u = \rho. \quad (6.2)$$

These weights still do the job. Indeed, the measurement noise generator, $W_n$, enforces $K$ to contain a notch at the frequency $\omega = \omega_n$ and the integrator in $W_e$ enforces the transfer function from $\theta$ to $e$ to have a zero at the origin (because $W_\theta$ has no such zeros). An advantage of (6.2)
over (6.1) is that the former results in a third-order $G$. Indeed, with these choices we then end up with the SP with

$$G_1 = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 & \frac{s^2}{s^2 + \omega_n^2} \end{bmatrix}, \quad \text{and} \quad G_3 = \begin{bmatrix} \frac{1}{s} & -\rho \end{bmatrix},$$

for which

$$G = \begin{bmatrix} G_1 & G_3 \\ G_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\omega_n^2 & 0 & 0 & -\omega_n^2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(for the sake of simplicity, we pick $\rho = 1$ here).

To solve this SP, choose

$$\begin{bmatrix} D_1^+ & D_2^+ \\ D_3^+ & D_4^+ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} D_2^+ & D_4^+ \\ D_1^+ & D_3^+ \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

and apply the result of Theorem 5.1. Obviously, the first solvability condition is satisfied, since in the considered problem $A_{11}$ is empty. The second and the third conditions are also satisfied, since (5.5) and (5.4) are solvable with

$$Z_3 = \begin{bmatrix} 0 & 0 \\ 0 & -1/\omega_n^2 \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} 0 & -1 \\ -1 & \omega_n^2 \end{bmatrix}.$$  

As this SP satisfies $\mathcal{A}_1$, $Z_2$ and $Z_3$ are uniquely defined by (5.4) and (5.5), leading to a simple parameterization of stabilizing solutions given in terms of a single $RH^\infty$ parameter. Choosing

$$L_s = \begin{bmatrix} -2\omega_n \\ 0 \end{bmatrix} \quad \text{and} \quad F_s = -1,$$

for which

$$\text{eig}(A_{2u} + L_s C_{23u}) = \{ -\omega_n, -\omega_n \} \quad \text{and} \quad \text{eig}(A_{3u} + B_{12u} L_s) = \{ -1 \},$$

and substituting them into (5.10) yields the following parameterization of stabilizing controllers:

$$K(s) = \frac{(s^2 + \omega_n^2)}{(s + 1)(s + \omega_n)^2} + \frac{s}{s + 1} Q(s) \frac{s^2 + \omega_n^2}{(s + \omega_n)^2} = \frac{s^2 + \omega_n^2}{(s + 1)(s + \omega_n)^2}(1 + s Q(s))$$

It can be verified that the resulting parameterization characterizes the set of all transfer functions with zeros at $\pm j\omega_n$ and with a unit static gain. This is not a surprise taking into account the physical meaning of the considered problem.

### 7. Concluding remarks.

In this paper the problem of input-output stabilization in a general two-sided rational model matching setup has been studied. The main contributions and conclusions are as follows:

- Conceptual differences between the general two-sided stabilization problem and its one-sided counterpart have been exposed. It has been shown that some of these differences are not relevant in most problems of interest and can be ruled out by an intuitive assumption.
A relation between the general two-sided stabilization problem and bilateral Diophantine equations has been ascertained and the results from [13] were exploited to derive a transparent and numerically feasible solution. The latter is given in terms of explicit state-space formulae and relies on two of constrained matrix Sylvester equations.

The solution derived in this work is closely related to the results of [16, 17], yet is obtained using completely different arguments and provides an alternative perspective of the considered problem.

The main advantage of the proposed approach is its transparency. This leads to the simplifying assumption mentioned above, which, in turn, enables us to end up with an affine parameterization of all stabilizing solutions in a single \( RH^\infty \) parameter. This shows that in most cases the two-sided problem is still similar to its one-sided counterpart, in a sense that the stabilization constraints there can be sorted out without changing the problem structure.

The simplified parameterization of all stabilizing solutions, derived in this work, is instrumental in the solution of various optimization problems with asymptotic behavior constraints. In particular, the results presented in this paper play an important role in [14], where the \( H^2 \) problem with preview and asymptotic behavior constraints is studied.

**Appendix A. Pole and zero directions.**

In this appendix some properties of the directions of poles and transmission zeros of rational transfer matrices are briefly overviewed. We need these properties for the discussion in Section 3. Most of the results below can be found in classical textbooks, such as [29, 24, 9], and are presented here without proofs. Some of the notions, however, such as the normal and singular null spaces of transfer matrices, are peculiar for the current work and are discussed in details.

Consider an \( m_y \times m_u \) rational transfer matrix given by its minimal state-space realization

\[
G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]  

**Definition A.1.** Given a pole of \( G(s), s_p \in \mathbb{C}, \) its output direction is defined as \( \mathcal{P}^\text{out}_{s_p} := C \operatorname{Ker}(s_p I - A). \)

The following results describes the basic property of the pole output direction, which can be considered as defining.

**Proposition A.2.** Let \( s_p \in \mathbb{C} \) be a pole of \( G(s) \) and \( y \in \mathbb{C}^{m_y} \) be some given vector. Then the transfer function \( y^\top G(s) \) has no poles at \( s_p \) iff \( y \perp \mathcal{P}^\text{out}_{s_p}. \)

The polynomial matrix

\[
R_G(s) = \begin{bmatrix} A - s I \\ C \\ D \end{bmatrix}
\]

is called the Rosenbrock system matrix and is used below to define the direction of transmission zeros.

**Definition A.3.** Given a transmission zero of \( G(s), s_z \in \mathbb{C}, \) its output direction is defined as \( \mathcal{Z}^\text{out}_{s_z} := [0 \ I] \operatorname{Ker}(R_G(s_z)). \)

An immediate property of this definition is described by the claim below and reveals the relation between the direction of a transmission zero and the null space of \( G(s_z)^\top. \)

**Proposition A.4.** Let \( s_z \in \mathbb{C} \) be a transmission zero of \( G(s) \) but not a pole of this transfer function, then \( \mathcal{Z}^\text{out}_{s_z} = \operatorname{Ker} G(s_z)^\top. \)

In the context of the current work, it is important to emphasize that the dimension of the output zero direction might be greater than the geometric multiplicity of the corresponding
zero. This is because the null space of \( G(s) \) exists not only due to the rank deficiency of \( G(s) \) at \( s = s_c \), but also due to the fact that the normal rank of \( G(s) \) may, in general, be smaller than the number of rows. To clarify this point, consider the Smith-McMillan form of \( G(s) \)

\[
Y(s)G(s)U(s) = \begin{bmatrix}
\frac{\alpha_i(s)}{\beta_i(s)} & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & \frac{\alpha_i(s)}{\beta_i(s)} & 0 \\
0 & \ldots & 0 & 0
\end{bmatrix}, \quad (A.2)
\]

where \( r \) is the normal rank of \( G(s) \), \( Y(s) \) and \( U(s) \) are unimodular polynomial matrices and for each \( i = 1, \ldots, r \) the polynomials \( \alpha_i(s) \) and \( \beta_i(s) \) have no common roots, \( \alpha_i(s) \) divides \( a_{i+1}(s) \) and \( \beta_{i+1}(s) \) divides \( \beta_i(s) \). Denote the \( i \)th row of \( Y(s) \) by \( Y_i(s) \). Let \( s_0 \in \mathbb{C} \) be an arbitrary point, which is not a pole of \( G(s) \) and denote the output null space of \( G(s) \) at this point by \( N_{\text{out}} \) := \( \ker G(s_0) \). Clearly,

\[
N_{\text{out}} := \text{span}\{Y_k(s_0), Y_{k+1}(s_0), \ldots, Y_m(s_0)\},
\]

for some \( k \leq r + 1 \) and can be represented as \( N_{\text{out}} = N_{\text{out}}^{0} \oplus N_{\text{out}}^{N} \), where

\[
N_{\text{out}}^{0} := \text{span}\{Y_k'(s_0), Y_{k+1}'(s_0), \ldots, Y_m'(s_0)\},
\]

\[
N_{\text{out}}^{N} := \text{span}\{Y_{r+1}'(s_0), Y_{r+2}'(s_0), \ldots, Y_m'(s_0)\}.
\]

The singular null space, \( N_{\text{out}}^{0} \), is nontrivial iff \( s_0 \) is a transmission zero of \( G(s) \) and the dimension of \( N_{\text{out}}^{0} \) equals the geometric multiplicity of the zero. The normal null space, \( N_{\text{out}}^{N} \), is nontrivial iff \( r < m_y \) and its dimension does not depend on \( s_0 \). The following result provides an alternative characterization of the normal output null space, which does not rely on the Smith-McMillan form. The proof of this result can be found in [12].

**PROPOSITION A.5.** Let \( s_0 \in \mathbb{C} \) be arbitrary, then \( 0 \neq \gamma \in \mathbb{C}^{m_y} \) belongs to \( N_{\text{out}}^{0} \) iff there exists a neighborhood

\[
B(s_0, \varepsilon) = \{s \in \mathbb{C} : |s - s_0| < \varepsilon\}
\]

for some \( \varepsilon > 0 \) and a continuous vector function \( f_y : B(s_0, \varepsilon) \rightarrow \mathbb{C}^{m_y} \), such that \( f_y(s_0) = \gamma \) and \( f_y(s)G(s) = 0, \forall s \in B(s_0, \varepsilon) \).

Another property of the normal output null space that we use is provided by the following two results, whose proofs can also be found in [12]:

**PROPOSITION A.6.** For any left-invertible \( G \in RH^{-\infty} \) there exist \( G^+, G^{-} \in RH^{-\infty} \) such that

\[
T(s) := \begin{bmatrix}
G^+(s) \\
G^-(s)
\end{bmatrix}
\]

is bi-stable and

\[
T(s)G(s) = \begin{bmatrix}
\hat{G}(s) \\
0
\end{bmatrix}
\]

for an invertible \( \hat{G}(s) \).

**PROPOSITION A.7.** The normal output null space of a left-invertible \( G \in RH^{-\infty} \) at any \( s_0 \in \tilde{\mathbb{C}}^+ \) can be characterized as \( N_{\text{out}}^{0} = \text{Im}(G^{-}(s_0)) \).
REFERENCES