Robust $H_\infty$ Control and Estimation of Retarded State-multiplicative Stochastic Systems

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Models with state MN are encountered in many areas of applications such as: nuclear fission and heat transfer, population models and immunology.

In control theory they are also encountered in GC when the scheduling parameters are corrupted by white meas. noise.

Systems with parameter uncertainties that are modelled as white noise processes in a linear setting were treated for cont. systems by Dragan & Morozan (97), Hinriechsen & Pritchard (98) and for discrete systems by Costa & Kubrusly (96), Dragan & Stoica (98).

Although systems that are modeled as linear systems with state MN may encounter time delay in the dynamics, the treatment, so far, avoided such delays.
Two approaches may be adopted. The 1st is the LKF approach (Fridman et. al 2002) and the 2nd is the input-output approach (c.f. Kao and Lincoln 04).

By the latter, the system is replaced by one without delay with norm-bounded uncertain operators.

Applying this approach, past results obtained for non-retarded systems with state MN can be used.

We treat the case where the MN appears in both the delayed and the nondelayed states.

We formulate and solve the $H_\infty$ control and estimation of retarded MN stochastic systems.
We consider the system:

\[
\begin{align*}
    dx(t) &= [A_0 x(t) + A_1 x(t - \tau(t)) + B_1 w(t) + B_2 u(t)]dt \\
    &\quad + H x(t - \tau(t)) d\zeta(t) + G x(t) d\beta(t), \quad x(\theta) = 0, \text{ over } [-h, 0], \\
    dy &= (C x + D_{21} n) dt + F x d\nu, \\
    z(t) &= C_1 x(t) + D_{12} u(t)
\end{align*}
\]  

(1a-c)

- The \( \tau(t) \) is an unknown time-delay which satisfies:

\[
0 \leq \tau(t) \leq h, \quad \dot{\tau}(t) \leq d < 1.
\]

- The zero-mean real scalar Wiener processes \( \beta(t), \nu(t) \) and \( \zeta(t) \) satisfy:

\[
\mathbb{E}\{\zeta(t)\zeta(s)\} = \min(t, s), \quad \mathbb{E}\{\beta(t)\beta(s)\} = \min(t, s),
\]

\[
\mathbb{E}\{\nu(t)\nu(s)\} = \min(t, s), \quad \mathbb{E}\{\beta(t)\zeta(s)\} = 0, \quad \mathbb{E}\{\nu(t)\zeta(s)\} = 0,
\]

and \( \mathbb{E}\{\nu(t)\beta(s)\} = 0. \)
In the robust stochastic $H_\infty$ control and estimation problems, we assume that the system parameters lie within the following polytope

$$\bar{\Omega} \triangleq \begin{bmatrix} A_0 & A_1 & B_1 & B_2 & C_1 & C_2 & D_{12} & D_{21} & H & G & F \end{bmatrix},$$

which is described by its vertices:

$$\bar{\Omega} = \text{Co}\{\bar{\Omega}_1, \bar{\Omega}_2, \ldots, \bar{\Omega}_N\},$$

where $\bar{\Omega}_i \triangleq$

$$\begin{bmatrix} A_0^{(i)} & A_1^{(i)} & B_1^{(i)} & B_2^{(i)} & C_1^{(i)} & C_2^{(i)} & D_{12}^{(i)} & D_{21}^{(i)} & H^{(i)} & G^{(i)} & F^{(i)} \end{bmatrix},$$

and where $N$ is the number of vertices. In other words:

$$\bar{\Omega} = \sum_{i=1}^{N} \bar{\Omega}_i f_i, \quad \sum_{i=1}^{N} f_i = 1, \quad f_i \geq 0.$$
We treat the following problems:

i) Robust $H_\infty$ SF control of delayed systems with state MN:

We consider the system of (1a,c) where the system matrices lie within the polytope $\bar{\Omega}$ and the following performance index:

$$J_E \triangleq \mathcal{E}\{\int_0^\infty \|z(t)\|^2 dt - \gamma^2 \int_0^\infty \|w(t)\|^2 dt\}.$$  

Our objective is to find a SF control law $u(t) = Kx(t)$ that achieves $J_E < 0$, for the worst-case $w(t) \in \tilde{L}^2_{\mathcal{F}_t}([0,\infty); \mathbb{R}^q)$ and for a prescribed $\gamma > 0$, where $\mathcal{F}_t$ denotes an increasing family of sigma algebra defined over a proper probability space.
ii) Robust $H_\infty$ estimation of delayed systems with state MN:
We consider the system of (1a-c) where $B_2 = 0$, $D_{12} = 0$ and consider the estimator of the following general form:

$$
\begin{align*}
  d\hat{x}(t) &= A_c \hat{x}(t)dt + B_c dy, \\
  \hat{z} &= C_c \hat{x}(t).
\end{align*}
$$

(2)

We denote

$$
e(t) = x(t) - \hat{x}(t), \quad \text{and} \quad \bar{z}(t) = z(t) - \hat{z}(t)
$$

and we consider the following cost function:

$$
J_F \overset{\Delta}{=} \mathcal{E}\{ \int_0^\infty \|\bar{z}(t)\|^2 dt - \gamma^2 [\int_0^\infty \|w(t)\|^2 dt + \int_0^\infty \|n(t)\|^2 dt] \}.
$$

Given $\gamma > 0$, we seek an estimate $C_c \hat{x}(t)$ of $C_1 x(t)$ over $[0, \infty)$ s.t. $J_F < 0 \ \forall \ \text{nonzero } w(t), n(t)$ where $w(t) \in \tilde{L}_2^{\mathcal{F}_t}([0, \infty); \mathcal{R}^q)$, $n(t) \in \tilde{L}_2^{\mathcal{F}_t}([0, T]; \mathcal{R}^p)$. 

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We introduce the following operators:

\[(\mathcal{D}_0 g)(t) \overset{\Delta}{=} g(t - \tau(t)), \quad (\mathcal{D}_1 g)(t) \overset{\Delta}{=} \int_{t-\tau(t)}^{t} g(s)ds\]

The induced $L_2$-norm of $\mathcal{D}_0$ is bounded by $\frac{1}{\sqrt{1-d}}$, and the induced $L_2$-norm of $\mathcal{D}_1$ is bounded by $h$ (c.f. Kao and Lincoln 04).

The system (1a) can be written as:

\[
dx(t) = [A_0 + m]x(t)dt + (A_1 - m)w_1(t)dt - mw_2(t)dt + B_1w(t)dt + B_2u(t)dt + Gx(t)d\beta(t) + Hw_1(t)d\zeta - \Gamma_\beta dt - \Gamma_\zeta dt,
\]

\[
\bar{y}(t) = [A_0 + m]x(t) + (A_1 - m)w_1(t) - mw_2(t) + B_1w(t) + B_2u(t) - \Gamma_\beta - \Gamma_\zeta,
\]

where

\[
\Gamma_\beta = m \int_{t-\tau}^{t} Gx(s)d\beta(s), \quad \Gamma_\zeta = m \int_{t-\tau}^{t} Hw_1(s)d\zeta(s),
\]

\[
w_1(t) = (\mathcal{D}_0 x)(t), \quad \text{and} \quad w_2(t) = (\mathcal{D}_1 \bar{y})(t)
\]

and where $m$ is an unknown matrix function that is to be determined.
Using the fact that $||D_0||_\infty \leq \frac{1}{\sqrt{1-d}}$ and $||D_1||_\infty \leq h$, we introduce into (3) the following 'new' variables:

$$w_1(t) = \Delta_1 x(t), \quad \text{and} \quad w_2(t) = \Delta_2 \bar{y}(t),$$

where $||\Delta_1||_\infty \leq \frac{1}{\sqrt{1-d}}$ and $||\Delta_2||_\infty \leq h$ are diagonal operators having identical scalar operators on the main diagonal.

We are now able to derive a stability condition for the retarded system.
Stability:

- We consider the system of (1a), with $B_2 = 0$ and $B_1 = 0$, and the following positive function:

$$V(t, x(t)) = x^T(t)Qx(t).$$

Taking expectation we obtain, defining the infinitesimal generator by $\mathcal{L}$:

$$\mathcal{E}\{(\mathcal{L}V)(t)\} = \mathcal{E}\{\langle Qx(t), [(A_0 + m)x(t) + (A_1 - m)\bar{\Delta}_1 x(t) - m\bar{\Delta}_2 \bar{y}(t)]\rangle\}$$

$$+ \mathcal{E}Tr\{Q(t)[Gx(t) \ Hw_1(t)]\bar{P}[Gx(t) \ Hw_1(t)]^T\},$$

where $\bar{P} \Delta \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ is the covariance matrix of the stationary augmented Wiener process vector $col\{\beta(t), \zeta(t)\}$ and where the condition for the stability of the system is $\mathcal{E}\{\mathcal{L}V\} < 0$. 
We also have the following:

\[
\text{Tr} \left\{ Q[Gx(t) \ Hw_1(t)] \bar{P}[Gx(t) \ Hw_1(t)]^T \right\} \\
= \text{Tr} \left\{ \begin{bmatrix} x^T(t)G^T \\ w_1^T(t)H^T \end{bmatrix} Q[Gx(t) \ Hw_1(t)] \bar{P} \right\} = \\
\text{Tr} \left\{ \begin{bmatrix} x^T(t)G^T QGx(t) & x^T(t)G^T QHw_1(t) \\ w_1^T(t)H^T QGx(t) & w_1^T(t)H^T QHw_1(t) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right\} \\
= x^T(t)G^T QGx(t) + w_1^T(t)H^T QHw_1(t).\]
In the attempt to establish $\mathcal{E}\{ (LV)(t) \} \leq -k \|x(t)\|^2$, for some $k > 0$ (uniformly in $t$), we obtain the following inequality:

$$2x^T Q[(A_0 + m)x(t) + (A_1 - m)w_1(t) - m\bar{\Delta}_2\bar{y}(t)] +$$

$$kx^T(t)x(t) + x^T(t)G^T QGx(t) + w_1^T(t)H^T QHw_1(t) < 0.$$ 

Adding the following term, which is nonnegative due to the diagonality of $\Delta_1$:

$$x^T(t)\left( \frac{1}{1 - d} R_1 - \bar{\Delta}_1^T R_1 \bar{\Delta}_1 \right) x(t) = x^T(t) \frac{1}{1 - d} R_1 x(t) - w_1^T(t) R_1 w_1(t),$$

and using the fact that

$$-2x^T(t) Q m \bar{\Delta}_2 \bar{y}(t) \leq h^2 \bar{y}^T(t) R_2 \bar{y}(t) + x^T(t) Qm R_2^{-1} m^T Q x(t),$$

where $R_1$ and $R_2$ are constant positive definite matrices, we obtain:

$$2x^T(t) Q[(A_0 + m)x(t) + (A_1 - m)w_1(t)] + kx^T(t)x(t) +$$

$$x^T(t) Qm R_2^{-1} m^T Q x(t) + h^2 \bar{y}^T(t) R_2 \bar{y}(t) + x^T(t) G^T QG x(t) +$$

$$x^T(t) \frac{1}{1 - d} R_1 x(t) - w_1^T(t) R_1 w_1(t) + w_1^T(t) H^T QH w_1(t) \leq 0.$$
3. Robust Stability and BRL - Cont.

- Denoting \( \zeta(t) = col\{x(t), w_1(t)\} \), we obtain that the latter holds if the following inequality is satisfied.

\[
\zeta^T(t) \begin{bmatrix}
\tilde{\Psi}_{11} & Q(A_1 - m) \\
* & -R_1 + H^T QH
\end{bmatrix} \zeta(t) + h^2 \tilde{y}^T(t) R_2 \tilde{y}(t) < 0.
\]

where:
\[
\tilde{\Psi}_{11} = Q(A_0 + m) + (A_0 + m)^T Q + \frac{1}{1-d} R_1 + G^T QG + Q m R_2^{-1} m^T Q.
\]

Applying Schur formula, denoting \( Q_m = Qm \) and taking \( R_2 = \epsilon_f Q \) where \( \epsilon_f > 0 \) is a tuning parameter, we obtain the following inequality:

\[
\Psi = \begin{bmatrix}
\tilde{\Psi}_{11} & QA_1 - Q_m & Q_m & h(A_0^T R_2 + \epsilon_f Q_m^T) \\
* & -R_1 + H^T QH & 0 & h(A_1^T R_2 - \epsilon_f Q_m^T) \\
* & * & -\epsilon_f Q & -h \epsilon_f Q_m^T \\
* & * & * & -\epsilon_f Q
\end{bmatrix} < 0
\]

where
\[
\tilde{\Psi}_{11} = QA_0 + Q_m + A_0^T Q + Q_m^T + \frac{1}{1-d} R_1 + G^T QG.
\]
Considering the fact that the system matrices lie within the polytope $\bar{\Omega}$, we obtain the following.

**Theorem 1**: The exp. stability in the mean square of system (1), over $\bar{\Omega}$, is guaranteed if $\exists Q > 0, R_1 > 0, Q_m$ and scalar $\epsilon_f > 0$ that satisfy the following set of LMIs:

$$
\hat{\Psi}_i = \begin{bmatrix}
\hat{\Psi}_{11,i} & QA_i^1 - Q_m & Q_m & h\epsilon_f(A_0^i Q + Q_m^T) \\
* & -R_1 + H_i^T Q H_i & 0 & h\epsilon_f(A_1^i Q - Q_m^T) \\
* & * & -\epsilon_f Q & -h\epsilon_f Q_m^T \\
* & * & * & -\epsilon_f Q \\
\end{bmatrix} < 0,
$$

$\forall i, i = 1, 2, \ldots, N$, where

$$
\hat{\Psi}_{11,i} = QA_0^i + Q_m + A_0^i Q + Q_m^T + \frac{1}{1 - d} R_1 + G_i^T Q G_i.
$$
3. Robust Stability and BRL - Cont.

BRL:

- We consider: the system of (1a,c), where the system matrices lie in the polytope $\bar{\Omega}$ with $B_2^i = 0$, $D_{12}^i = 0$, and

$$J_B \triangleq \mathcal{E}\left\{ \int_0^\infty \|z(t)\|^2 dt - \gamma^2 \int_0^\infty \|w(t)\|^2 dt \right\}.$$ 

- Assuming that the stability condition of Th. 1 is satisfied, we seek a condition that guarantees the following:

$$\mathcal{E} \int_0^\infty [\mathcal{L}V(t, x(t)) + z^T(t)z(t) - \gamma^2 w^T(t)w(t)] dt < 0,$$

We obtain:

**Theorem 2:** Consider the system (1a,c), where the system matrices lie in $\bar{\Omega}$ with $B_2^i = 0$, $D_{12}^i = 0$. The system is exp. stable in the mean square sense and, for prescribed $\gamma > 0$ and $\epsilon_f > 0$, the requirement of $J_B < 0$ is achieved $\forall$ nonzero $w \in \mathcal{L}^2_{\mathcal{F}_t}([0, \infty); \mathcal{R}^q)$, if $\exists$ $Q > 0$, $R_1 > 0$ and $Q_m$ that satisfy the following set of LMIs:
4. Robust Stability and BRL - Cont.

\[ \Psi_i = \begin{bmatrix} \tilde{\Psi}_{11,i} & \tilde{\Psi}_{12,i} & Q_m & QB_i & \tilde{\Psi}_{15,i} \\ * & \tilde{\Psi}_{22,i} & 0 & 0 & \tilde{\Psi}_{25,i} \\ * & * & -\epsilon_f Q & 0 & -h\epsilon_f Q_m^T \\ * & * & * & -\gamma^2 I & h\epsilon_f B_i^T Q \\ * & * & * & * & -\epsilon_f Q \end{bmatrix} < 0, \]

\forall i, i = 1, 2, ..., N, where \( \tilde{\Psi}_{11,i} = \hat{\Psi}_{11,i} + C_i^T C_1 \)

\( \tilde{\Psi}_{12,i} = QA_i - Q_m, \tilde{\Psi}_{22,i} = -R_1 + H_i^T QH_i, \)

\( \tilde{\Psi}_{15,i} = h\epsilon_f (A_0^T Q + Q_m^T), \tilde{\Psi}_{25,i} = h\epsilon_f (A_1^T Q - Q_m^T). \)
4. Robust State-Feedback Control

- we address the problem of finding the SF control law \( u(t) = Kx(t) \), that stabilizes the system and achieves a prescribed level of attenuation.

- We consider the system of (1a,c) and apply the above control law, where \( A_0 \) is replaced by \( (A_0 + B_2K) \), \( C_1 \) is replaced by \( C_1 + D_{12}K \).

For the polytopic uncertain system we obtain:

\[
\begin{bmatrix}
Y_{11,i} & \Psi_{12,i} & Q_m & QB_1^i & Y_{15,i} & Y_{16,i} & G^T Q \\
* & \Psi_{22,i} & 0 & 0 & \Psi_{25,i} & 0 & 0 \\
* & * & -\epsilon_f Q & 0 & -h\epsilon_f Q_m^T & 0 & 0 \\
* & * & * & -\gamma^2 I & h\epsilon_f B_1^T Q & 0 & 0 \\
* & * & * & * & -\epsilon_f Q & 0 & 0 \\
* & * & * & * & * & -I & 0 \\
* & * & * & * & * & * & -Q
\end{bmatrix} < 0,
\]
∀i, i = 1, 2, ..., N, where \( Y_{11,i} = QB_i^2K + K^T B_i^2Q + QA_0^i + Qm + A_0^i Q \)

\[ + Q_m^T + \frac{1}{1 - d} R_1, \quad Y_{16,i} = (C_i^1 + D_{12}^i K)^T, \quad Y_{15,i} = h\epsilon_f ([A_0^i + B_2^i K^T]Q + Q_m^T), \]

- Multiplying the above inequality by \( \text{diag}\{Q^{-1}, Q^{-1}, Q^{-1}, I, Q^{-1}, I, Q^{-1}\} \), from both sides, denoting \( P \overset{\Delta}{=} Q^{-1}, m_p = mP, \bar{R}_p = PR_1 P \) and \( \hat{K}_p = KP \), we obtain the following inequalities:

\[
\begin{bmatrix}
Y_{11,i} & Y_{12,i} & m_p & B_i^1 & Y_{15,i} & Y_{16,i} & PG_i^T & 0 \\
* & -\bar{R}_p & 0 & 0 & Y_{25,i} & 0 & 0 & PH_i^T \\
* & * & -\epsilon_f P & 0 & -h\epsilon_f m_p^T & 0 & 0 & 0 \\
* & * & * & -\gamma^2 I & h\epsilon_f B_i^1 & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_f P & 0 & 0 & 0 \\
* & * & * & * & * & -I & 0 & 0 \\
* & * & * & * & * & * & -P & 0 \\
* & * & * & * & * & * & * & -P
\end{bmatrix} < 0,
\]
4. Robust State-Feedback Control

with

\[ \hat{Y}_{11,i} = B_2 \hat{K}_p + \hat{K}_p^T B_2^T + m_p + m_p^T + PA_0^T + A_0 P + \frac{1}{1 - d} \bar{R}_1, \]

\[ \hat{Y}_{12,i} = A_1^i P - m_p, \quad \hat{Y}_{15,i} = \epsilon f h [PA_0^T + \bar{m}_p + \hat{K}_p^T B_2^T], \]

\[ \hat{Y}_{16,i} = PC_1^i + \hat{K}_p D_{12}^T, \quad \hat{Y}_{25,i} = \epsilon f h [PA_1^T - \bar{m}_p^T]. \]

We arrive at the following:

**Theorem 3:** Consider the system (1a,c) where the system matrices lie in \( \bar{\Omega} \). For prescribed \( \gamma > 0 \) and \( \epsilon f > 0 \), there exists a robust SF gain that achieves \( J_E < 0 \) for all nonzero \( w \in \tilde{L}^2_{\mathcal{F}_t}([0, \infty); \mathcal{R}^q) \), if there exist matrices \( P > 0, \bar{R}_p, m_p \) and \( \hat{K}_p \) that satisfy the latter LMI. In the latter case the state-feedback gain is given by:

\[ K = \hat{K}_p P^{-1}. \]
4. Robust Estimation - Cont.

- We consider the system of (1a-c) with $B_2 = 0$, $D_{12} = 0$ and the general type filter of (2).

- Denoting $\xi^T(t) \overset{\Delta}{=} [x(t)^T \ ˆx(t)^T]$, $\bar{w}^T(t) \overset{\Delta}{=} [w(t)^T \ n(t)^T]$ we obtain the following augmented system:

$$
\begin{align*}
\frac{d\xi(t)}{dt} &= [\tilde{A}_0 \xi(t) + \tilde{B} \bar{w}(t)]dt + \tilde{A}_1 \xi(t - \tau(t))dt \\
&+ \tilde{H} \xi(t - \tau(t))d\zeta(t) + \tilde{G} \xi(t)d\beta(t) + \tilde{F} \xi(t)d\nu(t), \ \xi(\theta) = 0,
\end{align*}
$$

over $[-h \ 0]$, $\tilde{z}(t) = \tilde{C} \xi(t)$,

where

$$
\begin{align*}
\tilde{A}_0 &= \begin{bmatrix} A_0 & 0 \\
B_c C_2 & A_c \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 & 0 \\
0 & B_c D_{21} \end{bmatrix}, \quad \tilde{A}_1 = \begin{bmatrix} A_1 & 0 \\
0 & 0 \end{bmatrix}, \\
\tilde{H} &= \begin{bmatrix} H & 0 \\
0 & 0 \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} G & 0 \\
0 & 0 \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} 0 & 0 \\
B_c F & 0 \end{bmatrix}, \quad \tilde{C} = [C_1 - C_c].
\end{align*}
$$

Using the BRL result, applying Schur formula and using various manipulations we obtain the following result:
**Theorem 4:** Consider the system of (1a-c) with \( B_2 = 0 \) and \( D_{12} = 0 \). For a prescribed \( \gamma > 0 \) and tuning parameter \( \epsilon_f > 0 \), there exists a filter of the structure (2) that achieves \( J_F < 0 \) for all nonzero \( w \in \tilde{L}^2([0, \infty); \mathcal{R}^q) \), \( n \in \tilde{L}^2([0, \infty); \mathcal{R}^p) \), if there exist matrices \( \tilde{X} > 0 \), \( Y > 0 \), \( K_0, U, \tilde{R}_1, \tilde{M} \) and \( Z \) that satisfy the following LMI:

\[
\begin{bmatrix}
\tilde{Y}_{11} & \tilde{Y}_{12} & \tilde{M} & \tilde{Y}_{14} & \tilde{Y}_{15} & \tilde{Y}_{16} & \tilde{Y}_{17} & \tilde{Y}_{18} & 0 \\
* & -\tilde{R}_1 & 0 & 0 & \tilde{Y}_{25} & 0 & 0 & 0 & \tilde{Y}_{29} \\
* & * & -\epsilon_f \tilde{X}_Y & 0 & -\epsilon_f h\tilde{M}^T & 0 & 0 & 0 & 0 \\
* & * & * & -\gamma^2 I & -\epsilon_f h\tilde{Y}_{14}^T & 0 & 0 & 0 & 0 \\
* & * & * & * & -\epsilon_f \tilde{X}_Y & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -I & 0 & 0 & 0 \\
* & * & * & * & * & * & -\tilde{X}_Y & 0 & 0 \\
* & * & * & * & * & * & * & -\tilde{X}_Y & 0 \\
* & * & * & * & * & * & * & * & -\tilde{X}_Y \\
\end{bmatrix} < 0,
\]
In the latter case the filter parameters can be extracted using the following equations:

\[
A_c = N^{-T}K_0 \bar{X}M^{-1}, \quad B_c = N^{-T}U, \quad C_c = Z\bar{X}M^{-1}.
\]

Namely, the transfer function matrix of the filter is given by
\[
T(s) = Z(s(\bar{X} - Y) - K_0)^{-1}U.
\]

**Proof:** Following the application of the BRL result, we define \( \tilde{Q} = Q^{-1} \), and denote the following partitions:

\[
\tilde{Q} = \begin{bmatrix} X & M^T \\ M & T \end{bmatrix}, \quad Q = \begin{bmatrix} Y & N^T \\ N & W \end{bmatrix}, \quad J = \begin{bmatrix} X^{-1} & Y \\ 0 & N \end{bmatrix}
\]

We multiply the resulting inequality, from the right, by \( \hat{J} = \text{diag}\{\tilde{Q}J, \tilde{Q}J, \tilde{Q}J, I, \tilde{Q}J, I, \tilde{Q}J, \tilde{Q}J, \tilde{Q}J\} \) and by \( \hat{J}^T \), from the left, and carrying out the various multiplications with \( R_2 = \epsilon_f Q \), where \( \epsilon_f \) is a positive scalar and defining,
5. Robust Estimation - Cont.

\[
\tilde{X} = X^{-1}, \quad \tilde{X}_Y = \begin{bmatrix} \tilde{X} & \tilde{X} \\ \tilde{X} & Y \end{bmatrix}, \quad K_0 = N^T A_c M \tilde{X},
\]

\[
\tilde{M} = J^T \tilde{Q} m \tilde{Q} J, \quad Z = C_c M \tilde{X}, \quad \tilde{R}_1 = J^T \tilde{Q} R_1 \tilde{Q} J, \quad U = N^T B_c,
\]

we obtain the LMI of Th. 4, where

\[
\tilde{Y}_{11} = \begin{bmatrix} \tilde{X} A_0 + A_0^T \tilde{X} & \tilde{X} A_0 + A_0^T Y + C_2^T U^T + K_0^T \\ Y A_0 + U C_2 + K_0 + A_0^T \tilde{X} & YA_0 + U C_2 + A_0^T Y + C_2^T U^T \end{bmatrix} + \tilde{M} \tilde{M}^T + \frac{1}{1-d} \tilde{R}_1,
\]

\[
\tilde{Y}_{12} = \begin{bmatrix} \tilde{X} A_1 & \tilde{X} A_1 \\ Y A_1 & YA_1 \end{bmatrix} - \tilde{M},
\]

\[
\tilde{Y}_{15} = \epsilon_f h \begin{bmatrix} A_0^T \tilde{X} & A_0^T Y + C_2^T U^T + K_0^T \\ A_0^T \tilde{X} & C_2^T U^T \end{bmatrix} + \epsilon_f h \tilde{M}^T,
\]
The above result, achieved for the nominal case, can be readily extended to the robust polytopic case, similarly to the robust SF case.
6. Examples

1. Robust State-feedback Control

- We consider \((1a,c)\), with \(h = 0.6\) and \(d = 0\),

\[
A = \begin{bmatrix} 0 & 1 \\ -1 & -a_{22} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0.1 \\ -0.10 & -0.04 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},
\]

\[
G = \begin{bmatrix} 0 & 0.3 \\ -0.2 & -0.04 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0.18 \\ -0.09 & -0.15 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} -0.5 & 4 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and,} \quad D_{12} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \text{where}
\]

\(a_{22} \in [0.12 \, 0.16] \text{ in the dynamic matrix } A_0.\)

- Using Theorem 3 above we obtain for \(\epsilon_f = 0.54\), a near minimum attenuation level of \(\gamma = 7.348.\) The corresponding state-feedback gain is \(K = - \begin{bmatrix} 12.6378 & 9.5455 \end{bmatrix}.\)

- We note that the attenuation level achieved for the deterministic case (where the stochastic uncertainties are set to zero) is \(\gamma = 5.126.\)
2. Robust Estimation

- We consider (1a-c), with $B_2 = 0$, $D_{12} = 0$, $h = 0.6$, $d = 0$ and with the following matrices:

$$A_0 = \begin{bmatrix} -2 & 0 \\ 1 & -a_{22} \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -0.5 & -1 \end{bmatrix}, \quad G = H = \sqrt{0.1}I_2,$$

$$C_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 0.03 & 0.02 \end{bmatrix}, \quad \text{and} \quad B_1 = \text{col}\{1, -1\},$$

where $C_1 = [-0.5 \ 1], D_{21} = 0.01$, and where $a_{22} \in [0.1 \ 0.18]$ in the dynamic matrix $A$.

- Applying the results of Theorem 4, for the near minimum attenuation level of $\gamma = 0.707$, with $\epsilon_f = 0.5$, the following transfer function for the stationary filter is obtained: $T(s) = \frac{3.796s + 7.383}{s^2 + 13.94s + 23.57}$. 
6. Concluding remarks

- The theory of linear stochastic $H_\infty$ control of state MN systems is extended to time delayed systems, where the stochastic uncertainties are encountered in both the delayed and the non-delayed states in the model.

- The delay is assumed to be unknown and time-varying where only the bounds on its length and rate of change are given. Delay dependent analysis and synthesis methods are developed which are based on the input-output approach.

- Sufficient conditions are derived for the stability of the closed-loop system.

- Based on the latter derivation, the estimation and SF control problem are formulated and solved, for a given plant, via a single LMI.

- The LMI condition obtained is affine in the system model matrices. They can thus be readily applied to the uncertain case where the parameters of the system matrices are known to reside in a given polytope.

- Two examples are given which demonstrate the applicability of the proposed results.