Linear Systems (034032)

TUTORIAL 12

1 Topics

Response to initial conditions, Modal response, Lyapunov stability.

2 Background

2.1 State-Space solution with initial conditions

Consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Its the solution in $t > 0$ is

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s$$

In the unforced (autonomous) case, e.g. under $u \equiv 0$, the state evolves according to

$$x(t) = \mathrm{e}^{At}x_0$$

2.2 Lyapunov stability: nonlinear case

An equilibrium $x_{\mathrm{eq}} \in \mathbb{R}^n$ of autonomous dynamics $\dot{x} = f(x)$ is said to be

- **stable** if for every $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0) - x_{\mathrm{eq}}\| < \delta \implies \|x(t) - x_{\mathrm{eq}}\| < \epsilon, \quad \forall t \in \mathbb{R}_+$$

- **asymptotically stable** if it is stable and there is $\delta > 0$ such that

$$\|x(0) - x_{\mathrm{eq}}\| < \delta \implies \lim_{t\to\infty}\|x(t) - x_{\mathrm{eq}}\| = 0$$

The **region of attraction** of an asymptotically stable equilibrium is the set of initial conditions $x(0)$ that generate states x converging to x_{eq}. If the region of attraction is the whole $\mathbb{R}^n$, then the equilibrium is said to be **globally asymptotically stable**.

2.3 Lyapunov stability: linear case

Theorem 1. *An equilibrium of the autonomous dynamics $\dot{x} = Ax$ is*

- *stable iff* $\mathrm{spec}(A) \in \{s \in \mathbb{C} \mid \mathrm{Re}\, s \le 0\}$ *and every imaginary eigenvalue is simple.*
- *asymptotically stable iff* $\mathrm{spec}(A) \in \{s \in \mathbb{C} \mid \mathrm{Re}\, s < 0\}$ *and those properties are global.*

2.4 Lyapunov's indirect method

Theorem 2. *Let $\dot{x} = f(x)$ for a continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}^n$, $x_{\rm eq} \in \mathbb{R}^n$ be its equilibrium, and*

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=x_{\rm eq}}$$

be the corresponding Jacobian matrix.

- *If* $\operatorname{spec}(A) \in \mathbb{C} \setminus \bar{\mathbb{C}}_0$*, then $x_{\rm eq}$ is asymptotically stable.*
- *If A has at least one eigenvalue in $\mathbb{C}_0$, then $x_{\rm eq}$ is unstable.*

If the rightmost eigenvalue of the Jacobian matrix is on the imaginary axis, then the stability conclusion is ambiguous.

2.5 Modal decomposition

If A is diagonalizable and all its eigenvalues are real, then the initial condition response can be decomposed as

$$x(t) = \sum_{i=1}^{n} \eta_i e^{\lambda_i t} \mu_i(x_0), \quad \mu_i(x_0) := \upsilon_i' x_0 \in \mathbb{R}$$

where $\lambda_i \in \mathbb{R}$ is an eigenvalue of A, $\eta_i \in \mathbb{R}^n$ is the corresponding right eigenvector, and $\upsilon_i \in \mathbb{R}^n$ is the transpose of the ith row of $\begin{bmatrix} \eta_1 & \cdots & \eta_n \end{bmatrix}^{-1}$ such that $\upsilon_i' \eta_j = \delta_{ij}$ (the Dirac delta). The signals $\boldsymbol{\eta_i}$ $\mathbf{exp}_{\boldsymbol{\lambda_i}}$ are known as **modes** of the system and scalars $\boldsymbol{\mu_i(x_0)}$ are their **degrees of excitation**.

2.6 Matlab commands

Some Matlab commands to diagonalize matrices.

- `[T,Lambda] = eig(A);` diagonalizes a square matrix `A`, returning a diagonal matrix `Lambda` comprising the eigenvalues of `A` and the corresponding square similarity transformation matrix `T`.
- `[TR,LambdaR] = cdf2rdf([T,Lambda]);` converts complex diagonal form (the output of `eig`) to a real block diagonal form.

3 Problems

Question 1. Consider the following state space equations.

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}}_{A} x(t)$$

1. Is the system Lyapunov stable?
2. Find a transformation diagonalizing the matrix A.
3. Carry out the modal decomposition of the system with respect to any initial condition $x(0)$.
4. Find the response for the following two initial conditions. Draw the responses in a phase portrait.

- $x(0) = \begin{bmatrix} 1.05 \\ -1 \end{bmatrix}$
- $x(0) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

Solution.

1. To check stability x A, we must find the eigenvalues.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 \\ -3 & \lambda - 2 \end{bmatrix} = \lambda^2 - 2\lambda - 3 = 0$$

   Thus, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$. One of the eigenvalues is in $\mathbb{C}_0$ therefore the system is unstable.

2. Not to find diagonalization we must also find the eigenvectors, we have

$$\lambda_1 = 3: \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} \eta_{11} \\ \eta_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \eta_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$\lambda_2 = -1: \begin{bmatrix} -1 & -1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} \eta_{21} \\ \eta_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \eta_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

   Thus, the transformation matrix is

$$T = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

   and the matrix A can be diagonalized as

$$A = T \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{\Lambda_A} T^{-1}$$

3. The response to any initial condition $x(0)$ is given by

$$x(t) = \mathrm{e}^{At} x(0)$$

   where e^{At} is the matrix exponential. We can compute the matrix exponential as follows.

$$\begin{aligned} \mathrm{e}^{At} &= T \mathrm{e}^{\Lambda_A t} T^{-1} \\ &= T \begin{bmatrix} \mathrm{e}^{3t} & 0 \\ 0 & \mathrm{e}^{-t} \end{bmatrix} T^{-1} \end{aligned}$$

   Therefore,

$$x(t) = T \begin{bmatrix} \mathrm{e}^{3t} & 0 \\ 0 & \mathrm{e}^{-t} \end{bmatrix} T^{-1} x(0) = T \begin{bmatrix} \mathrm{e}^{3t} & 0 \\ 0 & \mathrm{e}^{-t} \end{bmatrix} \mu = T \begin{bmatrix} \mu_1 \mathrm{e}^{3t} \\ \mu_2 \mathrm{e}^{-t} \end{bmatrix},$$

   with $\mu = T^{-1} x(0)$. Given that $T = \begin{bmatrix} \eta_1 & \eta_2 \end{bmatrix}$, we can write

$$x(t) = \mu_1 \mathrm{e}^{3t} \eta_1 + \mu_2 \mathrm{e}^{-t} \eta_2.$$

4. We can now find the response for the two initial conditions. First, we must invert the matrix T.

$$T^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

This gives us the ability to calculate μ.

$$\mu = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} x_1(0) + x_2(0) \\ 3x_1(0) - x_2(0) \end{bmatrix}$$

- $x(0) = \begin{bmatrix} 1.05 \\ -1 \end{bmatrix}$. This initial condition gives

$$\mu = \frac{1}{4}\begin{bmatrix} 1.05 - 1 \\ 3.15 + 1 \end{bmatrix} = \begin{bmatrix} 0.0125 \\ 1.0375 \end{bmatrix}.$$

Therefore, the response is

$$x(t) = 0.0125e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1.0375e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.0125e^{3t} + 1.0375e^{-t} \\ 0.0375e^{3t} - 1.0375e^{-t} \end{bmatrix}.$$

- $x(0) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. This initial condition gives

$$\mu = \frac{1}{4}\begin{bmatrix} -2 + 2 \\ -6 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

Therefore, the response is

$$x(t) = -2e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2e^{-t} \\ 2e^{-t} \end{bmatrix}.$$

The phase portrait for both initial conditions is shown in Fig. 1.

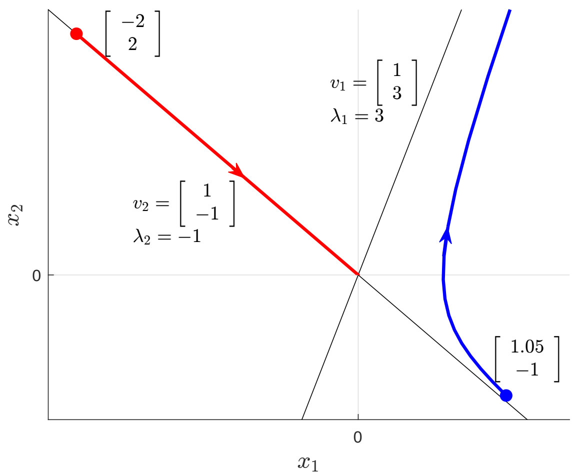

Fig. 1: Phase portrait for both initial conditions.

Question 2. Given the autonomous dynamics

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} x(t)$$

Is their equilibria Lyapunov stable? Carry out the modal decompositions of the responses.

Solution. These are linear dynamics, so the stability of their equilibria is completely determined by the eigenvalues of A. Because the matrix A is triangular, its eigenvalues are on the main diagonal and therefore

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -2.$$

The eigenvalues are in the open left half plane, therefore every equilibrium of this unforced dynamics is asymptotically stable. Note that the eigenvalues are simple, so that the matrix is diagonalizable.

To carry out the modal decomposition we need the corresponding right eigenvectors. They must satisfy $(\lambda_i I - A)\eta_i = 0$. If $\lambda = \lambda_1$, then this equality reads

$$\left(-1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}\right)\begin{bmatrix} \eta_{11} \\ \eta_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \eta_{11} \\ \eta_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ \eta_{12} - \eta_{11} \end{bmatrix} = 0 \quad \Longrightarrow \quad \eta_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

as a possible choice. If $\lambda = \lambda_2$, then this equality reads

$$\left(-2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}\right)\begin{bmatrix} \eta_{21} \\ \eta_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \eta_{21} \\ \eta_{22} \end{bmatrix} = -\begin{bmatrix} \eta_{21} \\ \eta_{21} \end{bmatrix} = 0 \quad \Longrightarrow \quad \eta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The similarity transform diagonalizing the A matrix is then

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \upsilon_1' \\ \upsilon_2' \end{bmatrix}$$

Hence, the response to initial conditions $x_0 = \left[\begin{smallmatrix} x_{01} \\ x_{02} \end{smallmatrix}\right]$ is

$$\begin{aligned} x(t) &= \eta_1 e^{\lambda_1 t}\upsilon_1' x_0 + \eta_2 e^{\lambda_2 t}\upsilon_2' x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} x_{01} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2t}(x_{02} - x_{01}) = \begin{bmatrix} e^{-t}x_{10} \\ (e^{-t} - e^{-2t})x_{10} + e^{-2t}x_{20} \end{bmatrix} \end{aligned}$$

Graphically, that can be seen in Fig. 2 for various x_0.

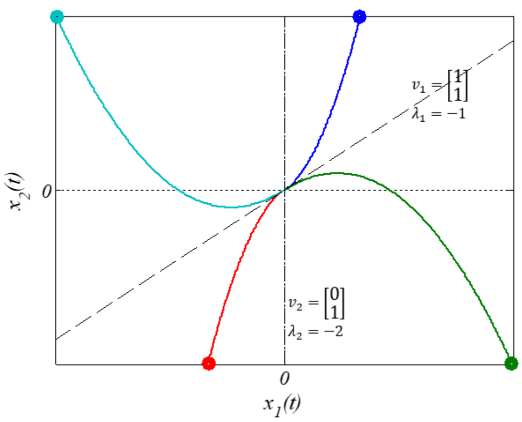

Fig. 2: Phase portrait for initial conditions.

There are 2 modes in this system, λ_1 and λ_2, with λ_2 faster than λ_1. Therefore, the exponent $e^{\lambda_2 t} = e^{-2t}$ tends to 0 faster than the exponent $e^{\lambda_1 t} = e^{-t}$. In general, we say that a mode corresponding to λ_i is faster than that corresponding to λ_j if

$$|\text{Re}\,\lambda_i| > |\text{Re}\,\lambda_j|.$$

The geometric meaning of this condition is that given a system in which all modes are asymptotically stable and an initial condition $x_0 \neq 0$ and $x_0 \neq \alpha\eta_i$ for some $\alpha \in \mathbb{R}$, the trajectory of the state vector will be a combination of each of the convergent modes, initially parallel to the eigenvector belonging to the fast mode, then the trajectory will align towards the slow mode and in the end will move towards the origin parallel to the eigenvector which corresponds to the slow mode. As $t \to \infty$, the state vector tends to the origin, which is the only equilibrium point of every asymptotically stable system. The trajectory of the state vector can never cross its eigenvectors. ∇

Question 3. Consider the following systems:

1. $\dot{x}(t) = \begin{bmatrix} 2 & 8 \\ -8 & 2 \end{bmatrix} x(t)$
2. $\dot{x}(t) = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix} x(t)$
3. $\dot{x}(t) = \begin{bmatrix} -2 & 8 \\ -8 & -2 \end{bmatrix} x(t)$

For each of them determine whether it is stable and carry out the modal decompositions of the responses.

Solution. In our problem all "A" matrices are in the form

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}.$$

Therefore, their matrix exponents are in the form

$$e^{At} = \exp\left(\left(\sigma \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\right) t\right) = \exp(\sigma I t) \exp\left(\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t\right) = e^{\sigma t} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}$$

and we get "spirals" in the state plane, where

- if $\sigma > 0$, then the spiral is diverging,
- if $\sigma = 0$, then the "spiral" is the unit circle,
- if $\sigma < 0$, then the spiral is converging.

Consider now each case separately.

- In the first case $\sigma = 2$ and $\omega = 8$ and the eigenvalues of A are

$$\lambda_{1,2} = 2 \pm 8j.$$

They are in the open right-half plane, so the system is unstable. Therefore, we get the response shown in Fig. 3(a), which indeed shows a diverging spiral. This is because the value of $e^{\sigma t} = e^{2t}$ diverges.

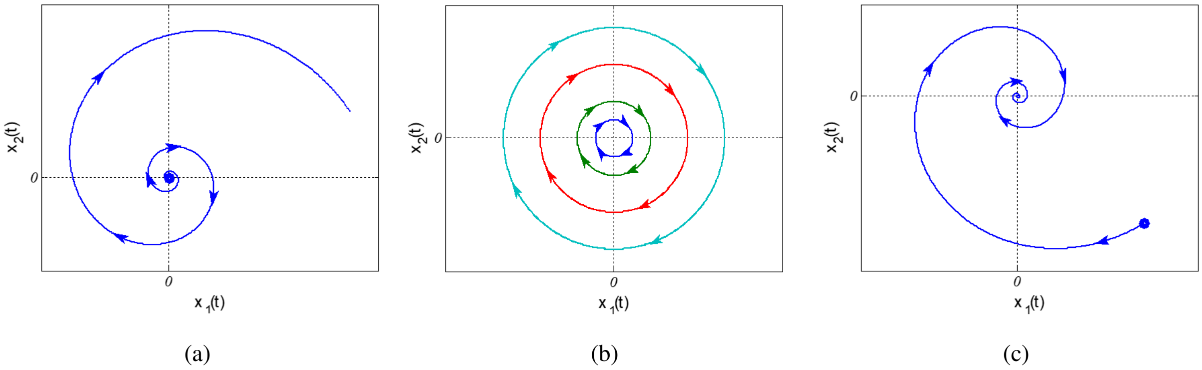

Fig. 3: Phase portraits for unforced motions.

- In the second case $\sigma = 0$ and $\omega = 8$ and the eigenvalues of A are

$$\lambda_{1,2} = \pm 8\mathrm{j}.$$

These are simple eigenvalues on the imaginary axis, so the system is stable (but not asymptotically stable). Because $\mathrm{e}^{\sigma t} = 1$, the response keeps $\|x(t)\|$ constant for all t and we have circles, see Fig. 3(b). You can find the direction of movement by selecting some point, for example: $x_0 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ then calculating the derivative

$$\dot{x}(0^+) = Ax_0 = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -8 \end{bmatrix}$$

Hence, the derivative of the state vector at x_1 faces downwards and the direction of movement will be clockwise.

- In the third case $\sigma = -2$ and $\omega = 8$ and the eigenvalues of A are

$$\lambda_{1,2} = -2 \pm 8\mathrm{j}.$$

Therefore, the system is asymptotically stable and we get a converging spiral shown in Fig. 3(c).

That's all . . . ▽

Question 4. Given the free system in figure 4,

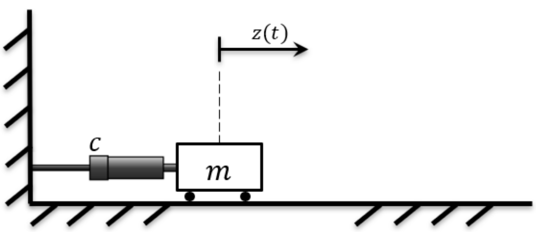

Fig. 4: Mass damper system.

The system is a mass damper system without a spring. The parameter values are $m = 1,\ c = 2$. Also, the state variables are defined

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}$$

Is the system stable? Find the response of the system to the initial conditions $x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ via the modes.

Solution.

1. It can be seen that the equation of motion of the system is

$$m\ddot{z}(t) + c\dot{z}(t) = 0$$

   We will substitute the numerical values and get

$$\ddot{z}(t) + 2\dot{z}(t) = 0$$

   Defining the state variables

$$\begin{aligned} x_1(t) &= z(t) \\ x_2(t) &= \dot{z}(t) \end{aligned}$$

   We take the derivative of the state variables and get

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -2x_2(t) \end{aligned}$$

   and therefore the realization of the equation of state will be

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad , \quad \begin{bmatrix} x_1\,(0) \\ x_2\,(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2. The eigenvalues of A can be found,

$$\lambda_1 = 0 \ \ , \ \ \lambda_2 = -2$$

   The system is stable, but not asymptotically stable.

3. Also, the eigenvectors are,

$$\eta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \ , \ \ \eta_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

   Also, the right eigenvectors (rows of the matrix T^{-1}) are

$$\upsilon_1' = \begin{bmatrix} 1 & 1/2 \end{bmatrix} \ \ , \ \ \upsilon_2' = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

4. The model solution is,

$$\begin{aligned} x(t) &= \sum_{i=1}^{2} e^{\lambda_i t} \eta_i \upsilon_i' x_0 = e^{0t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1/2 \end{bmatrix} x_0}_{\mu_1} + e^{-2t} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix} x_0}_{\mu_2} \\ &= e^{0t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-2t} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + e^{-2t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \end{aligned}$$

   It can be seen that the stable mode, $\lambda = -2$, decays along the vector $\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$ but the mode $\lambda = 0$ does not decay.

In general, when the system has modes with a real part 0, whose algebraic multiplicity is equal to the multiplicity geometrically, we will get a stable (non-asymptotic) system, i.e. a system that does not diverge from some initial condition but also does not converge.
It is important to understand that a system of this type is "stable" only for the reaction of initial conditions, and when the system will receive any input (even if it is bounded), it can diverge.

The response of our system to the given initial conditions in figure 5 will be,

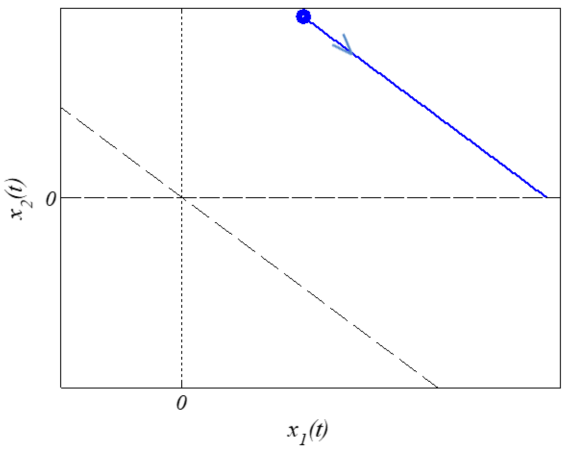

Fig. 5: Phase portrait for a single initial condition.

Also, the response of the system to different initial starting conditions in figure 6 will be,

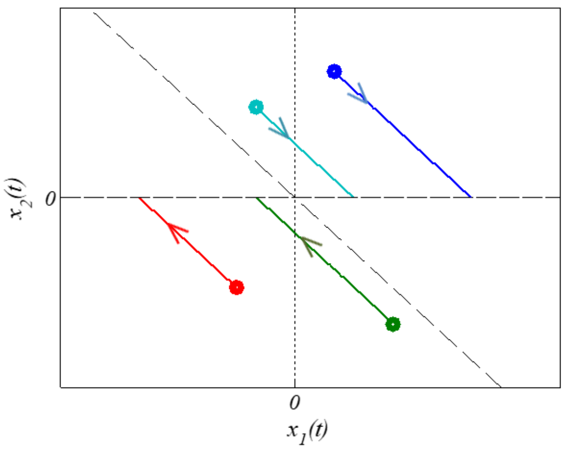

Fig. 6: Phase portrait for both many initial conditions.

As mentioned, the stable mode fades to zero but the neutral mode does not change its value.

▽

Question 5. Consider the system shown in Fig. 7.

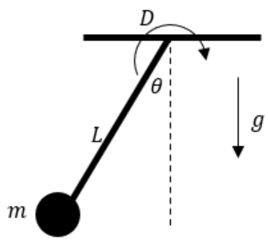

Fig. 7: Pendulum system.

It consists of a pendulum with a point mass m at the end of a massless rod of length L. On the axis of the pendulum, there is a torque due to viscous friction proportional to the speed of rotation with the friction coefficient D.

- Use the state vector
$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$
to derive the physical realization of this system.
- Find the equilibrium points and linearize the system around these points.
- Take numerical values of the system parameters as
$$m = 1\,[\text{kg}]\,, \quad L = 5\,[\text{m}]\,, \quad \text{and} \quad g = 10\left[\frac{\text{m}}{\text{s}^2}\right]$$
and determine if the nonlinear system is stable at the respective equilibrium points via Lyapunov's indirect method for two cases:
  - $D = 50\,[\text{N m s}]$ (damped system)
  - $D = 0\,[\text{N m s}]$ (undamped system)

Solution.

- The dynamic equation of the system is
$$mL^2\ddot{\theta} = -mgL\sin\theta - D\dot{\theta} \quad \Longrightarrow \quad \ddot{\theta} = -\frac{g}{L}\sin\theta - \frac{D}{mL^2}\dot{\theta}$$
With the chosen state vector this equation can be rewritten in the state-space form as
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{L}\sin x_1 - \frac{D}{mL^2}x_2 \end{bmatrix} = f(x)$$
This is clearly a nonlinear system. To linearize it, find its equilibrium points via the relation
$$0 = f(x) = \begin{bmatrix} x_2 \\ -\frac{g}{L}\sin x_1 - \frac{D}{mL^2}x_2 \end{bmatrix}$$
The first row yields $x_2 = 0$. Substituting this value to the second row we end up with the condition $\sin x_1 = 0$. Hence, there are infinitely many equilibria, ar
$$x_{\text{eq}} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}, \quad k \in \mathbb{Z}$$

The deviation variable is

$$x_\delta := x - x_{\text{eq}} = x - \begin{bmatrix} k\pi \\ 0 \end{bmatrix}.$$

To calculate the Jacobian matrix, note that

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos(x_1) & -\frac{D}{mL^2} \end{bmatrix}.$$

When evaluating the Jacobian at the equilibrium points, there are two possible cases. If k is even, then $\cos(x_{\text{eq}}) = 1$ (this corresponds to the down position of the pendulum). If k is odd, then $\cos(x_{\text{eq}}) = -1$ (this corresponds to the up position of the pendulum). Therefore it suffices to consider one instance of each of these cases in the analysis. Specifically, we consider two equilibrium points,

$$x_{\text{eq1}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x_{\text{eq2}} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}.$$

These cases result in the linearized "A" matrices of the form

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L} & -\frac{D}{mL^2} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ \frac{g}{L} & -\frac{D}{mL^2} \end{bmatrix}$$

respectively, which are the basis for further analysis.

1. First considering the equilibrium x_{eq1} and the parameter $D = 50$. Substituting numerical values to the first A above we end up with the linearized dynamics

$$\dot{x}_\delta = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x_\delta$$

   The eigenvalues of this A are calculated via

$$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 2 \end{vmatrix} = \lambda^2 + 2\lambda + 2 = 0 \quad \Longrightarrow \quad \lambda_{1,2} = -1 \pm \mathrm{j}$$

   The real part of these eigenvalues is negative. Hence, Theorem 2 yields that the equilibrium x_{eq1} is asymptotically stable in the nonlinear system.

2. Still around the equilibrium x_{eq1}, select now $D = 0$. In this case, the linearized dynamics in terms of deviation variables are

$$\dot{x}_\delta = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x_\delta$$

   The eigenvalues of this A are calculated via

$$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda \end{vmatrix} = \lambda^2 + 2 = 0 \quad \Longrightarrow \quad \lambda_{1,2} = \pm\sqrt{2}\mathrm{j}$$

   These eigenvalues are purely imaginary. Hence, Lyapunov's indirect method cannot be used to conclude anything about the stability of the equilibrium x_{eq1}.

3. Now consider the equilibrium x_{eq2} with $D = 50$. The dynamics here are

$$\dot{x}_\delta = \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} x_\delta$$

The eigenvalues of this A are calculated via

$$\begin{vmatrix} \lambda & -1 \\ -2 & \lambda+2 \end{vmatrix} = \lambda^2 + 2\lambda - 2 = 0 \qquad \Longrightarrow \qquad \lambda_{1,2} = -1 \pm \sqrt{3}$$

One of these eigenvalues has a positive real part, so Theorem 2 can be used to conclude that the equilibrium x_{eq2} is unstable.

4. Still around the equilibrium x_{eq2}, select now $D = 0$. In this case, the linearized dynamics in terms of deviation variables are

$$\dot{x}_\delta = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} x_\delta$$

The eigenvalues of this A are calculated via

$$\begin{vmatrix} \lambda & -1 \\ -2 & \lambda \end{vmatrix} = \lambda^2 - 2 = 0 \qquad \Longrightarrow \qquad \lambda_{1,2} = \pm\sqrt{2}$$

Like in the previous case, one of these eigenvalues had a positive real part, so the equilibrium x_{eq2} is unstable in the undamped case as well.

That's all … ∇